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A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers

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I. Introduction

Since the invention of the alternating gradient principle and the subsequent design of the Brookhaven and CERN proton-synchrotrons based on this principle, there has been a rapid evolution of the mathematical and physical techniques applicable to charged particle optics. In this report a matrix algebra formalism will be used to develop the essential principles governing the design of charged particle beam transport systems, with a particular emphasis on the design of high-energy magnetic spectrometers. A notation introduced by John Streib[1] has been found to be useful in conveying the essential physical principles dictating the design of such beam transport systems. In particular to first order, the momentum dispersion, the momentum resolution, the particle path length, and the necessary and sufficient conditions for zero dispersion, achromaticity, and isochronicity may all be expressed as simple integrals of particular first-order trajectories (matrix elements) characterizing a system.

This formulation provides direct physical insight into the design of beam transport systems and charged particle spectrometers. An intuitive grasp of the mechanism of second-order aberrations also results from this formalism; for example, the effects of magnetic symmetry on the minimization or elimination of second-order aberrations is immediately apparent.

The equations of motion will be derived and then the matrix formalism introduced, developed, and evolved into useful theorems. Physical examples will be given to illustrate the applicability of the formalism to the design of specific spectrometers. It is hoped that the information supplied will provide the reader with the necessary tools so that he can design any beam transport system or spectrometer suited to his particular needs.

The theory has been developed to second order in a Taylor expansion about a central trajectory, characterizing the system. This seems to be adequate for most high-energy physics applications. For studying details beyond second order, we have found computer ray tracing programs to be the best technique for verification of matrix calculations, and as a means for further refinement of the optics if needed.

In the design of actual systems for high-energy beam transport applications, it has proved convenient to express the results via a multipole expansion about a central trajectory. In this expansion, the constant term proportional to the field strength at the central trajectory is the dipole term. The term proportional to the first derivative of the field (with respect to the transverse dimensions) about the central trajectory is a quadrupole term and the second derivative with respect to the transverse dimensions is a sextupole term, etc.

A considerable design simplification results at high energies if the dipole, quadrupole, and sextupole functions are physically separated such that cross-product terms among them do not appear, and if the fringing field effects are small compared to the contributions of the multipole elements comprising the system. At the risk of oversimplification, the basic function of the multipole elements may be identified in the following way: The purpose of the dipole element(s) is to bend the central trajectory of the system and disperse the beam; that is, it is the means of providing the first-order momentum dispersion for the system. The quadrupole element(s) generate the first-order imaging. The sextupole terms couple with the second-order aberrations; and a sextupole element introduced into the system is a mechanism for minimizing or eliminating a particular second-order aberration that may have been generated by dipole or quadrupole elements.

Quadrupole elements may be introduced in any one of three characteristic forms: *(1)* via an actual physical quadrupole consisting of four poles such that a first field derivative exists in the field expansion about the central trajectory; *(2)* via a rotated input or output face of a bending magnet; and (*3*) via a transverse field gradient in the dipole elements of the system. Clearly any one of these three fundamental mechanisms may be used as a means of achieving first-order imaging in a system. Of course dipole elements will tend to image in the radial bending plane independent of whether a transverse field derivative does or does not exist in the system, but imaging perpendicular to the plane of bend is not possible without the introduction of a first-field derivative.

In addition to their fundamental purpose, dipoles and quadrupoles will also introduce higher-order aberrations. If these aberrations are second order, they may be eliminated or at least modified by the introduction of sextupole elements at appropriate locations.

In regions of zero dispersion, a sextupole will couple with and modify only geometric aberrations. However, in a region where momentum dispersion is present, sextupoles will also couple with and modify chromatic aberrations.

Similar to the quadrupole, a sextupole element may be generated in one of several ways, first by incorporating an actual sextupole, that is, a six-pole magnet, into the system. However, any mechanism which introduces a second derivative of the field with respect to the transverse dimensions is, in effect, introducing a sextupole component. Thus a second-order curved surface on the entrance or exit face of a bending magnet or a second-order transverse curvature on the pole surfaces of a bending magnet is also a sextupole component.

As illustrations of systems possessing dipole, quadrupole, and sextupole elements, consider the $n = \frac{1}{2}$ double-focusing spectrometer which is widely used for low- and medium-energy physics applications. Clearly there is a dipole element resulting from the presence of a magnetic field component along the central trajectory of the spectrometer. A distributed quadrupole element exists as a consequence of the $n = \frac{1}{2}$ field gradient. In this particular case, since the transverse imaging forces are proportional to $n^{1/2}$ and the radial imaging forces are proportional to $(1 - n)^{1/2}$, the restoring forces are equal in both planes, hence the reason for the double focusing properties. In addition to the first derivative of the field $n = (r_o/B_o)(\partial B/\partial r)$, there are usually second- and higher-order transverse field derivatives present. The second derivative of the field $\beta = \frac{1}{2}(r_o^2/B_o)(\partial^2 B/\partial r^2)$ introduces a distributed sextupole along the entire length of the spectrometer. Thus to second order a typical $n = \frac{1}{2}$ spectrometer consists of a single dipole with a distributed quadrupole and sextupole superimposed along the entire length of the dipole element. Higher-order multipoles may also be present, but will be ignored in this discussion.

In the preceding example the dipole, quadrupole, and sextupole functions are integrated in the same magnet. However, in many high-energy applications it is often more economical to use separate magnetic elements for each of the multipole functions. Consider also the SLAC spectrometers which provide examples of solutions which combine the multipole functions into a single magnet as well as solutions using separate multipole elements. Three spectrometers have been designed: one for a maximum energy of 1.6 GeV/c to study large backward angle scattering processes, a second for 8 GeV/c to study intermediate forward angle production processes, and finally a 20-GeV/c spectrometer for small forward angle production. All of these instruments are to be used in conjunction with primary electron and gamma-ray energies in the range of 10–20 GeV/c.

The 1.6-GeV/c instrument (Fig. 1) is a single magnet, bending the

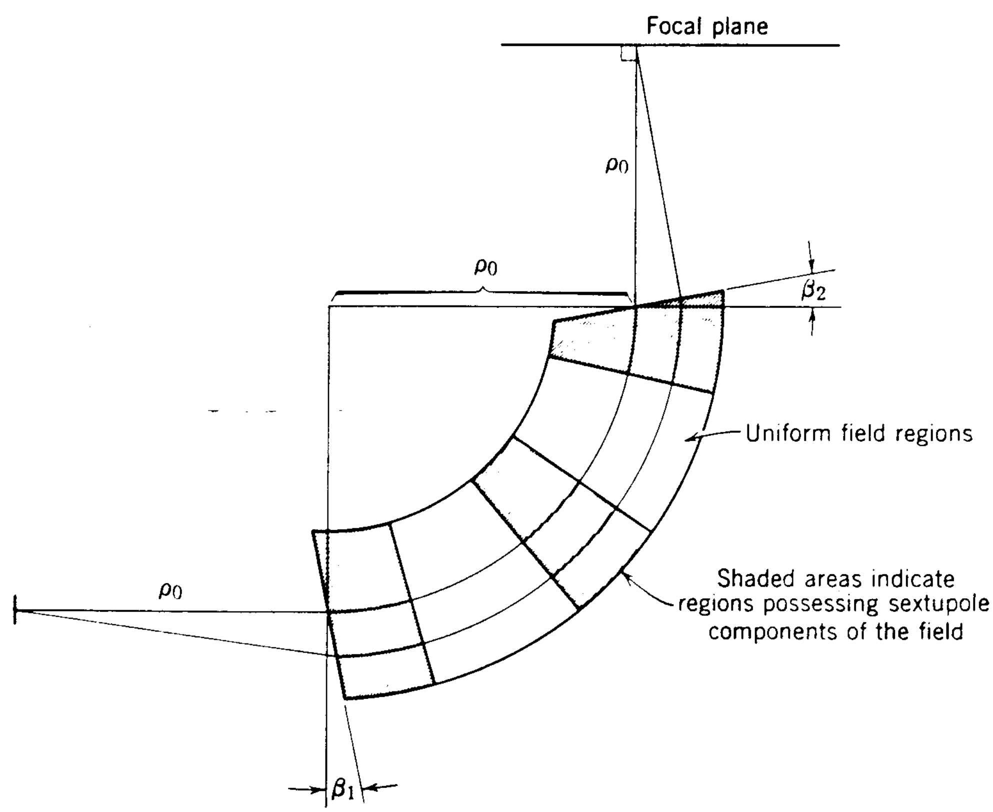

FIG. 1. 1.6-GeV/c spectrometer.

central trajectory a total of 90°, thus constituting the dipole contribution to the optics of the system. Two quadrupole elements are present in the magnet; i.e., input and output pole faces of the magnet are rotated so as to provide transverse focusing, and the 90° bend provides radial focusing via the $(1 - n)^{1/2}$ factor characteristic of any dipole magnet. The net optical result is point-to-point imaging in the plane of bend and parallel-to-point imaging in the plane transverse to the plane of bend. The solid angle and resolution requirements of the 1.6-GeV/c spectrometer are such that three sextupole components are needed to achieve the required performance. In this application, the sextupoles are generated by machining an appropriate transverse second-order curvature on the magnet pole face at three different locations along the 90° bend of the system. In summary, the 1.6-GeV/c spectrometer consists of one dipole, bending a total of 90°, two quadrupole elements, and a sextupole triplet with the quadrupole and sextupole strengths chosen to provide the first- and second-order properties demanded of the system.

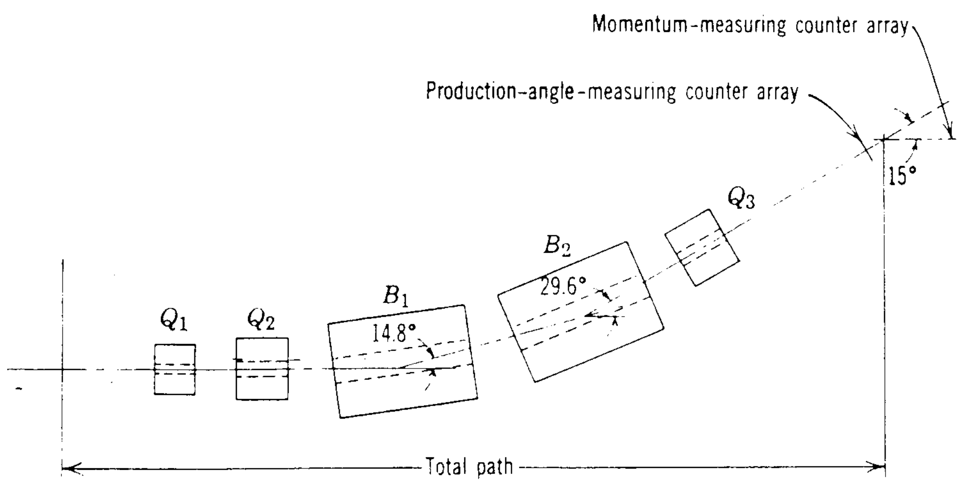

FIG. 2. Magnet arrangement, 8-GeV/c spectrometer.

Optically, the 8-GeV/c spectrometer (Fig. 2) is relatively simple. It consists of two dipoles, each bending 15°, making a total of a 30° bend, and three quadrupoles (two preceding and one following the dipole elements) to provide point-to-point imaging in the plane of bend and parallel-to-point imaging in the plane transverse to the bending plane. The solid angle and resolution requirements of the instrument are sufficiently modest that no sextupole components are needed. The penalty paid for not adding sextupole components is that the focal plane angle with respect to the optic axis at the end of the system is a relatively small angle (13.7°). With the addition of one sextupole element near the end of the system, the focal plane could have been rotated to a much larger angle. However, the 13.7° angle was acceptable for the focal plane counter array and as such it was ultimately decided to omit the additional sextupole element.

The 20-GeV/c spectrometer (Fig. 3) is a more complex design. The increased momentum requires an $\int \mathbf{B} \cdot d\mathbf{l}$ twice that of the 8-GeV/c spectrometer. The final instrument is composed of four dipole elements (bending magnets), two bending in one sense and the other two bending in the opposite sense, so the beam emanating from the instrument is parallel to the incident primary particles. The first-order imaging is achieved via four quadrupoles. The chromatic aberrations generated by the quadrupoles in this system are more serious than in the 8-GeV/c case because of an intermediate image required at the midpoint of the system. As a result, the focal plane angle with respect to the central trajectory would have been in the range of 2–4°. As a consequence, sextupoles were introduced in order to rotate the focal plane to a more satisfactory angle for the counter array. A final compromise placed the focal plane angle at 45° with respect to the optic axis of the system via the introduction of three sextupoles. Thus the 20-GeV/c spectrometer consists of four dipoles, with an intermediate crossover following the first two dipoles, a quadrupole triplet to achieve first-order imaging, and a sextupole triplet to compensate for the chromatic aberrations introduced by the quadrupoles. Optically, the 20-GeV/c spectrometer is very similar to the

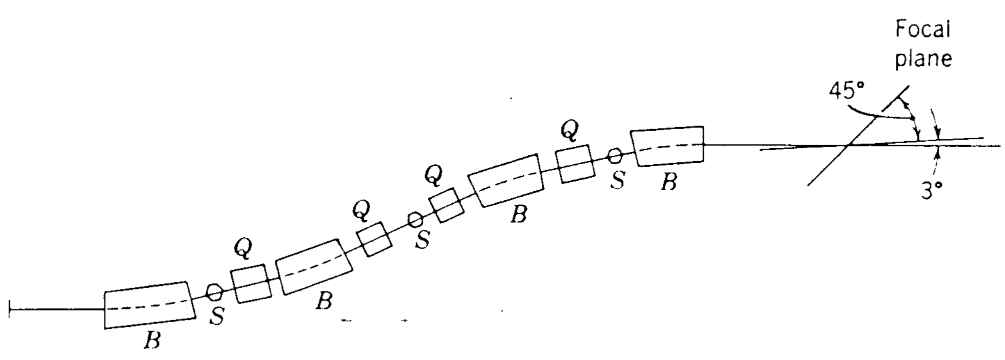

FIG. 3. 20-GeV/c spectrometer.

1.6-GeV/c spectrometer and yet physically it is radically different because of the method of introducing the various multipole components.

Having provided some representative examples of spectrometer design, we now wish to introduce and develop the theoretical tools for creating other designs.

II. A General First- and Second-Order Theory of Beam Transport Optics

The fundamental objective is to study the trajectories described by charged particles in a static magnetic field. To maintain the desired generality, only one major restriction will be imposed on the field configuration: Relative to a plane that will be designated as the magnetic midplane, the magnetic scalar potential φ shall be an odd function in the transverse coordinate y (the direction perpendicular to the midplane), i.e., $\varphi(x, y, t) = -\varphi(x, -y, t)$. This restriction greatly simplifies the calculations, and from experience in designing beam transport systems it appears that for most applications there is little, if any, advantage to be gained from a more complicated field pattern. The trajectories will be described by means of a Taylor's expansion about a particular trajectory (which lies entirely within the magnetic midplane) designated henceforth as the central trajectory. Referring to Figure 4, the coordinate t is the arc length measured along the central trajectory; and x, y, and t form a right-handed curvilinear coordinate system. The results will be valid for describing trajectories lying close to and making small angles with the central trajectory.

The basic steps in formulating the solution to the problem are as follows:

1. A general vector differential equation is derived describing the trajectory of a charged particle in an arbitrary static magnetic field which possesses "midplane symmetry."

2. A Taylor's series solution about the central trajectory is then assumed; this is substituted into the general differential equation and terms to second-order in the initial conditions are retained.

3. The first-order coefficients of the Taylor's expansion (for mono-energetic rays) satisfy homogeneous second-order differential equations characteristic of simple harmonic oscillator theory; and the first-order

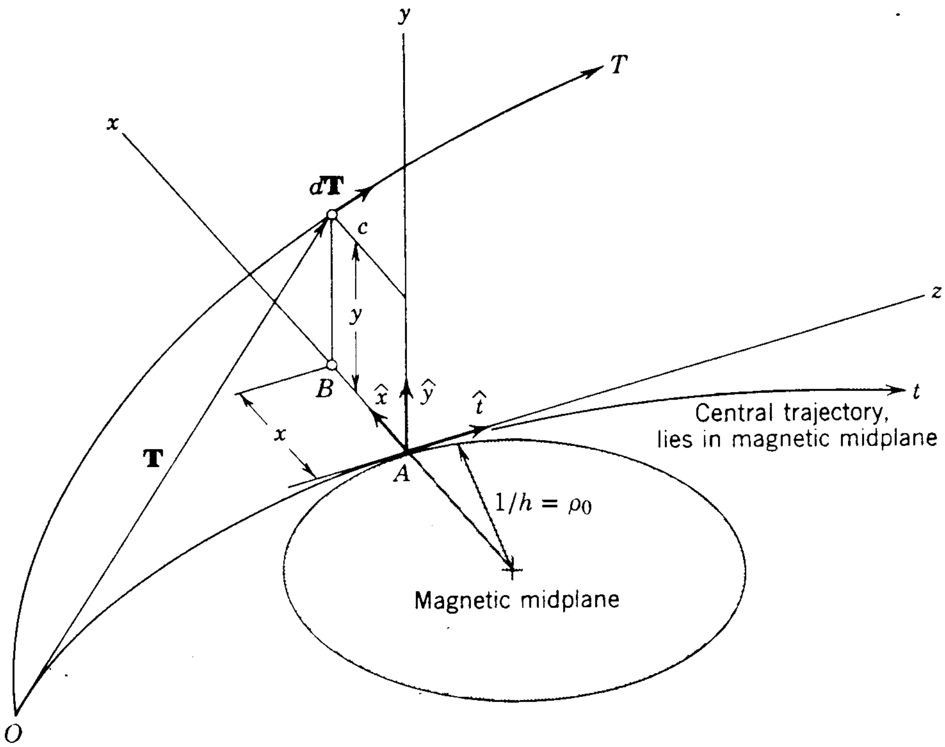

FIG. 4. Curvilinear coordinate system used in derivation of equations of motion.

dispersion and the second-order coefficients of the Taylor's series satisfy second-order differential equations having "driving terms."

4. The first-order dispersion term and the second-order coefficients are then evaluated via a Green's function integral containing the driving function of the particular coefficient being evaluated and the characteristic solutions of the homogeneous equations.

In other words, the basic mathematical solution for beam transport optics is similar to the theory of forced vibrations or to the theory of the classical harmonic oscillator with driving terms.

It is useful to express the second-order results in terms of the first-order coefficients of the Taylor's expansion. These first-order coefficients have a one-to-one correspondence with the following five characteristic first-order trajectories (matrix elements) of the system (identified by their initial conditions at $t = 0$), where prime denotes the derivative with respect to t:

1. The unit sinelike function $s_x(t)$ in the plane of bend (the magnetic midplane) where $s_x(0) = 0$; $s_x'(0) = 1$ (Fig. 5).

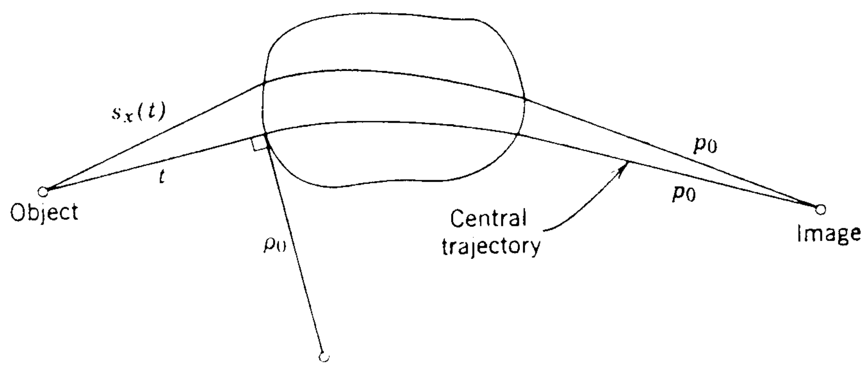

FIG. 5. Sinelike function $s_x(t)$ in magnetic midplane.

2. The unit cosine-like function $c_x(t)$ in the plane of bend where $c_x(0) = 1$; $c'_x(0) = 0$ (Fig. 6).

3. The dispersion function $d_x(t)$ in the plane of bend where $d_x(0) = 0$; $d'_x(0) = 0$ (Fig. 7).

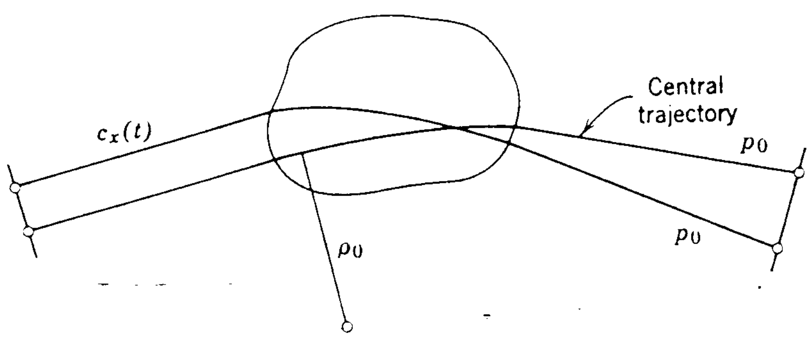

FIG. 6. Cosinelike function $c_x(t)$ in magnetic midplane.

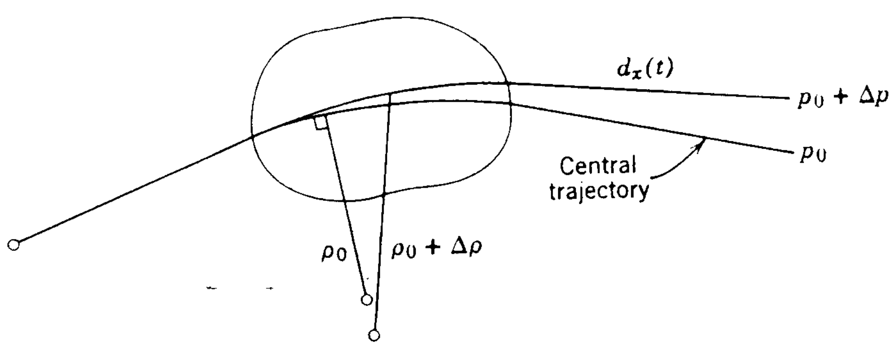

FIG. 7. Dispersion function $d_x(t)$ in magnetic midplane.

4. The unit sinelike function $s_y(t)$ in the nonbend plane where $s_y(0) = 0$; $s'_y(0) = 1$ (Fig. 8).

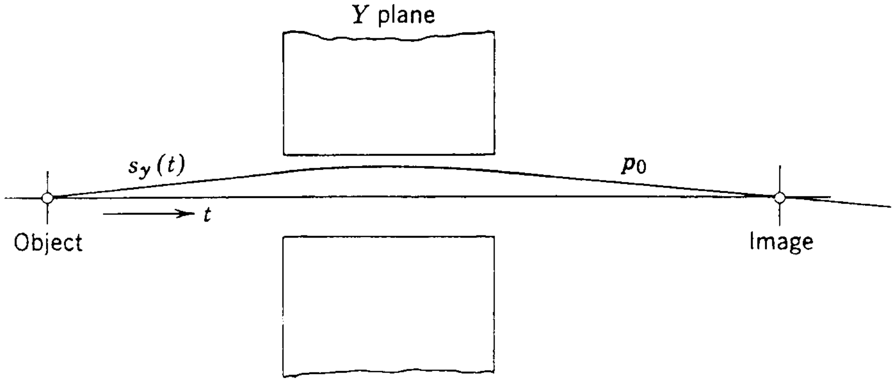

FIG. 8. Sinelike function $s_y(t)$ in nonbend (y) plane.

5. The unit cosinelike function $c_y(t)$ in the nonbend plane where $c_y(0) = 1$; $c'_y(0) = 0$ (Fig. 9).

Writing the first-order Taylor's expansion for the transverse position of an arbitrary trajectory at position t in terms of its initial conditions, the above five quantities are just the coefficients appearing in the expansion for the transverse coordinates x and y as follows:

$$x(t) = c_x(t)x_0 + s_x(t)x'_0 + d_x(t)(\Delta p/p_0)$$

and

$$y(t) = c_y(t)y_0 + s_y(t)y'_0$$

where x_0 and y_0 are the initial transverse coordinates and x'_0 and y'_0 are the initial angles (in the paraxial approximation) the arbitrary ray makes

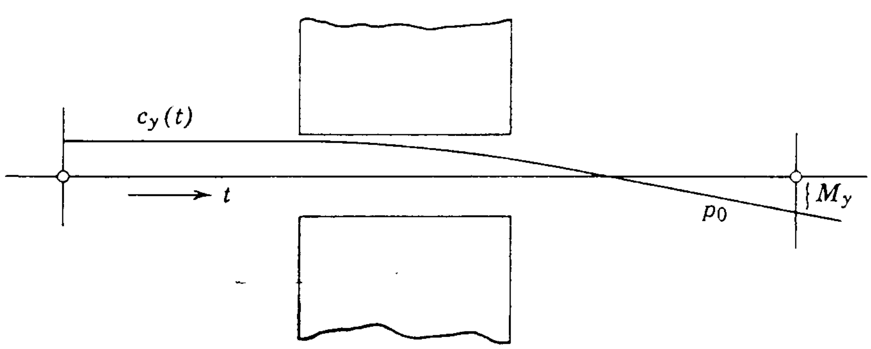

FIG. 9. Cosinelike function $c_y(t)$ in nonbend (y) plane.

with respect to the central trajectory. $\Delta p/p_0$ is the fractional momentum deviation of the ray from the central trajectory.

1. The Vector Differential Equation of Motion

We begin with the usual vector relativistic equation of motion for a charged particle in a static magnetic field equating the time rate of change of the momentum to the Lorentz force:

$$\dot{\mathbf{P}} = e(\mathbf{V} \times \mathbf{B})$$

and immediately transform this equation to one in which time has been eliminated as a variable and we are left only with spatial coordinates. The curvilinear coordinate system used is shown in Figure 4. Note that the variable t is not time but is the arc distance measured along the central trajectory. With a little algebra, the equation of motion is readily transformed to the following vector forms shown below:

Let e be the charge of the particle, $\mathbf{V}$ its speed, P its momentum magnitude, $\mathbf{T}$ its position vector, and T the distance traversed. The unit tangent vector of the trajectory is $d\mathbf{T}/dT$. Thus, the velocity and momentum of the particle are, respectively, $(d\mathbf{T}/dT)V$ and $(d\mathbf{T}/dT)P$. The vector equation of motion then becomes:

$$V\frac{d}{dT}\left(\frac{d\mathbf{T}}{dT}P\right) = eV\left(\frac{d\mathbf{T}}{dT} \times \mathbf{B}\right)$$

or

$$P\frac{d^2\mathbf{T}}{dT^2} + \frac{d\mathbf{T}}{dT}\left(\frac{dP}{dT}\right) = e\left(\frac{d\mathbf{T}}{dT} \times \mathbf{B}\right)$$

where $\mathbf{B}$ is the magnetic induction. Then, since the derivative of a unit vector is perpendicular to the unit vector, $d^2\mathbf{T}/dT^2$ is perpendicular to $d\mathbf{T}/dT$. It follows that $dP/dT = 0$; that is, P is a constant of the motion as expected from the fact that the magnetic force is always perpendicular to the velocity in a static magnetic field. The final result is:

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{e}{P}\left(\frac{d\mathbf{T}}{dT} \times \mathbf{B}\right) \tag{1}$$

2. The Coordinate System

The general right-handed curvilinear coordinate system (x, y, t) used is illustrated in Figure 4. A point O on the central trajectory is designated the origin. The direction of motion of particles on the central trajectory is designated the positive direction of the coordinate t. A point A on the central trajectory is specified by the arc length t measured along that curve from the origin O to point A. The two sides of the magnetic symmetry-plane are designated the positive and negative sides by the sign of the coordinate y. To specify an arbitrary point B which lies in the symmetry plane, we construct a line segment from that point to the central trajectory (which also lies in the symmetry plane) intersecting the latter perpendicularly at A: the point A provides one coordinate t; the second coordinate x is the length of the line segment BA, combined with a sign (+) or (−) according as an observer, on the positive side of the symmetry-plane, facing in the positive direction of the central trajectory, finds the point on the left or right side. In other words, x, y, and t form a right-handed curvilinear coordinate system. To specify a point C which lies off the symmetry-plane, we construct a line segment from the point to the plane, intersecting the latter perpendicularly at B; then B provides the two coordinates, t and x; the third coordinate y is the length of the line segment CB.

We now define three mutually perpendicular unit vectors ($\hat{x}$, $\hat{y}$, $\hat{t}$). $\hat{t}$ is tangent to the central trajectory and directed in the positive t-direction at the point A corresponding to the coordinate t; $\hat{x}$ is perpendicular to the principal trajectory at the same point, parallel to the symmetry plane, and directed in the positive x direction. $\hat{y}$ is perpendicular to the symmetry plane, and directed away from that plane on its positive side. The unit vectors ($\hat{x}$, $\hat{y}$, $\hat{t}$) constitute a right-handed system and satisfy the relations

$$\begin{aligned} \hat{x} &= \hat{y} \times \hat{t} \\ \hat{y} &= \hat{t} \times \hat{x} \\ \hat{t} &= \hat{x} \times \hat{y} \end{aligned} \tag{2}$$

The coordinate t is the primary independent variable, and we shall use the prime to indicate the operation d/dt. The unit vectors depend only on the coordinate t, and from differential vector calculus, we may write

$$\begin{aligned} \hat{x}' &= h\hat{t} \\ \hat{y}' &= 0 \\ \hat{t}' &= -h\hat{x} \end{aligned} \tag{3}$$

where $h(t) = 1/\rho_0$ is the curvature of the central trajectory at point A defined as positive as shown in Figure 4.

The equation of motion may now be rewritten in terms of the curvilinear coordinates defined above. To facilitate this, it is convenient to express $d\mathbf{T}/dT$ and $d^2\mathbf{T}/dT^2$ in the following forms:

$$\frac{d\mathbf{T}}{dT} = \frac{(d\mathbf{T}/dt)}{(dT/dt)} = \frac{\mathbf{T}'}{T'}$$

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{1}{T'}\frac{d}{dt}\left(\frac{\mathbf{T}'}{T'}\right)$$

or

$$(T')^2\frac{d^2\mathbf{T}}{dT^2} = \mathbf{T}'' - \frac{1}{2}\frac{\mathbf{T}'}{(T')^2}\frac{d}{dt}(T')^2$$

The equation of motion now takes the form

$$\mathbf{T}'' - \frac{1}{2}\frac{\mathbf{T}'}{(T')^2}\frac{d}{dt}(T')^2 = \frac{e}{P}T'(\mathbf{T}' \times \mathbf{B}) \tag{4}$$

In this coordinate system, the differential line element is given by:

$$d\mathbf{T} = \hat{x}dx + \hat{y}dy + (1 + hx)\hat{t}dt$$

and

$$(dT)^2 = d\mathbf{T}\cdot d\mathbf{T} = dx^2 + dy^2 + (1 + hx)^2dt^2$$

Differentiating these equations with respect to t, it follows that:

$$T'^2 = x'^2 + y'^2 + (1 + hx)^2$$

$$\frac{1}{2}\frac{d}{dt}(T')^2 = x'x'' + y'y'' + (1 + hx)(hx' + h'x)$$

$$\mathbf{T}' = \hat{x}x' + \hat{y}y' + (1 + hx)\hat{t}$$

and

$$\mathbf{T}'' = \hat{x}x'' + \hat{x}'x' + \hat{y}y'' + \hat{y}'y' + (1 + hx)\hat{t}' + \hat{t}(hx' + h'x)$$

Using the differential vector relations of Eq. (3), the expression for $\mathbf{T}''$ reduces to

$$\mathbf{T}'' = \hat{x}[x'' - h(1 + hx)] + \hat{y}y'' + \hat{t}[2hx' + h'x]$$

The vector equation of motion may now be separated into its component parts with the result:

$$\hat{x}\left\{[x'' - h(1 + hx)] - \frac{x'}{(T')^2}[x'x'' + y'y'' + (1 + hx)(hx' + h'x)]\right\}$$
$$+ \hat{y}\left\{y'' - \frac{y'}{(T')^2}[x'x'' + y'y'' + (1 + hx)(hx' + h'x)]\right\}$$
$$+ \hat{t}\left\{(2hx' + h'x) - \frac{(1 + hx)}{(T')^2}[x'x'' + y'y'' + (1 + hx)(hx' + h'x)]\right\}$$
$$= \frac{e}{P}T'(\mathbf{T}' \times \mathbf{B})$$
$$= \frac{e}{P}T'\{\hat{x}[y'B_t - (1 + hx)B_y] + \hat{y}[(1 + hx)B_x - x'B_t] + \hat{t}[x'B_y - y'B_x]\} \quad (5)$$

Note that in this form, no approximations have been made; the equation of motion is still valid to all orders in the variables x and y and their derivatives.

If now we retain only terms through second order in x and y and their derivatives and note that $(T')^2 = 1 + 2hx + \cdots$, the x and y components of the equation of motion become

$$x'' - h(1 + hx) - x'(hx' + h'x) = (e/P)T'[y'B_t - (1 + hx)B_y]$$
$$y'' - y'(hx' + h'x) = (e/P)T'[(1 + hx)B_x - x'B_t] \quad (6)$$

The equation of motion of the central orbit is readily obtained by setting x and y and their derivatives equal to zero. We thus obtain:

$$h = (e/P_0)B_y(0, 0, t) \quad \text{or} \quad B\rho_0 = P_0/e \quad (7)$$

This result will be useful for simplifying the final equations of motion. P_0 is the momentum of a particle on the central trajectory. Note that this equation establishes the sign convention between h, e, and B_y.

3. *Expanded Form of a Magnetic Field Having Median Plane Symmetry*

We now evolve the field components of a static magnetic field possessing median or midplane symmetry (Fig. 10). We define median

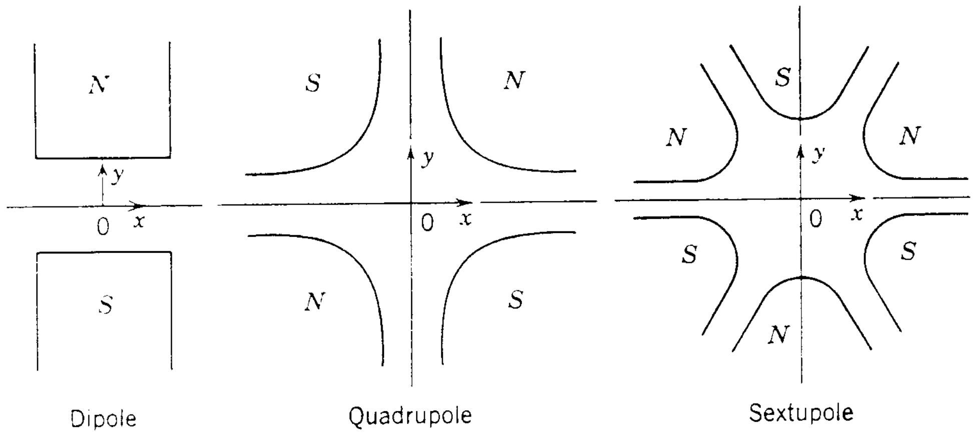

FIG. 10. Illustration of magnetic midplane for dipole, quadrupole, and sextupole elements. The magnet polarities may, of course, be reversed.

plane symmetry as follows. Relative to the plane containing the central trajectory, the magnetic scalar potential φ is an odd function in y; i.e., $\varphi(x, y, t) = -\varphi(x, -y, t)$. Stated in terms of the magnetic field components B_x, B_y, and B_t, this is equivalent to saying that:

$$B_x(x, y, t) = -B_x(x, -y, t)$$

$$B_y(x, y, t) = B_y(x, -y, t)$$

and

$$B_t(x, y, t) = -B_t(x, -y, t)$$

It follows immediately that on the midplane $B_x = B_t = 0$ and only B_y remains nonzero; in other words, on the midplane **B** is always normal to the plane. As such, any trajectory initially lying in the midplane will remain in the midplane throughout the system.

The expanded form of a magnetic field with median plane symmetry has been worked out by many people; however, a convenient and comprehensible reference is not always available. L. C. Teng[(2)] has provided us with such a reference.

For the magnetic field in vacuum, the field may be expressed in terms of a scalar potential φ by $\mathbf{B} = \nabla\varphi$.* The scalar potential will be expanded in the curvilinear coordinates about the central trajectory

* For convenience, we omit the minus sign since we are restricting the problem to static magnetic fields.

lying in the median plane $y = 0$. The curvilinear coordinates have been defined in Figure 1 where x is the outward normal distance in the median plane away from the central trajectory. y is the perpendicular distance from the median plane, t is the distance along the central trajectory, and $h = h(t)$ is the curvature of the central trajectory. As stated previously, these coordinates (x, y, and t) form a right-handed orthogonal curvilinear coordinate system.

As has been stated, the existence of the median plane requires that φ be an odd function of y, i.e., $\varphi(x, y, t) = -\varphi(x, -y, t)$. The most general expanded form of φ may, therefore, be expressed as follows:

$$\begin{aligned}\varphi(x, y, t) &= (A_{10} + A_{11}x + A_{12}(x^2/2!) + A_{13}(x^3/3!) + \cdots)y \\ &\quad + (A_{30} + A_{31}x + A_{32}(x^2/2!) + \cdots)y^3/3! + \cdots \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!}\end{aligned} \tag{8}$$

where the coefficients $A_{2m+1,n}$ are functions of t.

In this coordinate system, the differential line element dT is given by

$$dT^2 = dx^2 + dy^2 + (1 + hx)^2(dt)^2 \tag{9}$$

The Laplace equation has the form

$$\begin{aligned}\nabla^2\varphi = \frac{1}{(1+hx)}\frac{\partial}{\partial x}\left[(1+hx)\frac{\partial\varphi}{\partial x}\right] \\ + \frac{\partial^2\varphi}{\partial y^2} + \frac{1}{(1+hx)}\frac{\partial}{\partial t}\left[\frac{1}{(1+hx)}\frac{\partial\varphi}{\partial t}\right] = 0\end{aligned} \tag{10}$$

Substitution of Eq. (8) into Eq. (10) gives the following recursion formula for the coefficients:

$$\begin{aligned}-A_{2m+3,n} &= A''_{2m+1,n} + nhA''_{2m+1,n-1} - nh'A'_{2m+1,n-1} + A_{2m+1,n+2} \\ &+ (3n+1)hA_{2m+1,n+1} + n(3n-1)h^2A_{2m+1,n} + n(n-1)^2h^3A_{2m+1,n-1} \\ &+ 3nhA_{2m+3,n-1} + 3n(n-1)h^2A_{2m+3,n-2} \\ &\qquad + n(n-1)(n-2)h^3A_{2m+3,n-3}\end{aligned} \tag{11}$$

where prime means d/dt, and where it is understood that all coefficients A with one or more negative subscripts are zero. This recursion formula expresses all the coefficients in terms of the midplane field $B_y(x, 0, t)$: where

$$A_{1,n} = \left(\frac{\partial^n B_y}{\partial x^n}\right)_{\substack{x=0\\y=0}} = \text{functions of } t \tag{12}$$

Since φ is an odd function of y, on the median plane we have $B_x = B_t = 0$. The normal (in x direction) derivatives of B_y on the reference curve defines B_y over the entire median plane, hence the magnetic field $\mathbf{B}$ over the whole space. The components of the field are expressed in terms of φ explicitly by $\mathbf{B} = \nabla\varphi$ or

$$B_x = \frac{\partial\varphi}{\partial x} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n+1}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!}$$

$$B_y = \frac{\partial\varphi}{\partial y} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m}}{(2m)!}$$

$$B_t = \frac{1}{(1+hx)}\frac{\partial\varphi}{\partial t} = \frac{1}{(1+hx)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A'_{2m+1,n}\frac{x^n}{n!}\frac{y^{2m+1}}{(2m+1)!} \tag{13}$$

where B_t is not expressed in a pure power expansion form. This form can be obtained straightforwardly by expanding $1/(1 + hx)$ in a power series of hx and multiplying out the two series; however, there does not seem to be any advantage gained over the form given in Eq. (13).

The coefficients up to the sixth-degree terms in x and y are given explicitly below from Eq. (11).

$$A_{30} = -A''_{10} - A_{12} - hA_{11}$$

$$A_{31} = -A''_{11} + 2hA''_{10} + h'A'_{10} - A_{13} - hA_{12} + h_2A_{11}$$

$$\begin{aligned} A_{32} = &- A''_{12} + 4hA''_{11} + 2h'A'_{11} - 6h^2A''_{10} - 6hh'A'_{10} - A_{14} \\ &- hA_{13} + 2h^2A_{12} - 2h^3A_{11} \end{aligned}$$

$$\begin{aligned} A_{33} = &-A''_{13} + 6hA''_{12} + 3h'A'_{12} - 18h^2A''_{11} - 18hh'A'_{11} \\ &+ 24h^3A''_{10} + 36h^2h'A'_{10} - A_{15} - hA_{14} + 3h^2A_{13} \\ &- 6h^3A_{12} + 6h^4A_{11} \end{aligned} \tag{14}$$

$$\begin{aligned} A_{50} = &A''''_{10} + 2A''_{12} - 2hA''_{11} + h''A_{11} + 4h^2A''_{10} + 5hh'A'_{10} \\ &+ A_{14} + 2hA_{13} - h^2A_{12} + h^3A_{11} \end{aligned}$$

$$\begin{aligned} A_{51} = &A''''_{11} - 4hA''''_{10} - 6h'A'''_{10} - 4h''A''_{10} - h'''A'_{10} + 2A''_{13} \\ &- 6hA''_{12} - 2h'A'_{12} + h''A_{12} + 10h^2A''_{11} + 7hh'A'_{11} - 4hh''A_{11} \\ &- 3h'^2A_{11} - 16h^3A''_{10} - 29h^2h'A'_{10} + A_{15} + 2hA_{14} \\ &- 3h^2A_{13} + 3h^3A_{12} - 3h^4A_{11} \end{aligned} \tag{15}$$

In the special case when the field has cylindrical symmetry about $\hat{y}$, we can choose a circle with radius $\rho_0 = 1/h = a$ constant for the reference curve. The coefficients $A_{2m+1,n}$ in Eq. (8) and the curvature h of the reference curve are then all independent of t. Eqs. (14) and (15) are greatly simplified by putting all terms with primed quantities equal to zero.

4. *Field Expansion to Second Order Only*

If the field expansion is terminated with the second-order terms, the results may be considerably simplified. For this case, the scalar potential φ and the field $\mathbf{B} = \nabla\varphi$ become:

$$\varphi(x, y, t) = \left(A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \cdots\right)y + (A_{30} + \cdots)\frac{y^3}{3!} + \cdots$$

$$A_{1n} = \left.\frac{\partial^n B_y}{\partial x^n}\right|_{\substack{x=0\\y=0}} = \text{functions of } t \text{ only}$$

and

$$A_{30} = -[A''_{10} + hA_{11} + A_{12}]$$

where prime means the total derivative with respect to t. Then $\mathbf{B} = \nabla\varphi$ from which

$$B_x(x, y, t) = \frac{\partial\varphi}{\partial x} = A_{11}y + A_{12}xy + \cdots$$

$$B_y(x, y, t) = \frac{\partial\varphi}{\partial y} = A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \frac{1}{2!}A_{30}y^2 + \cdots$$

$$B_t(x, y, t) = \frac{1}{(1 + hx)}\frac{\partial\varphi}{\partial t} = \frac{1}{(1 + hx)}[A'_{10}y + A'_{11}xy + \cdots] \quad (16)$$

By inspection it is evident that B_x, B_y, and B_t are all expressed in terms of A_{10}, A_{11}, and A_{12} and their derivatives with respect to t. Consider then B_y on the midplane only

$$B_y(x, 0, t) = A_{10} + A_{11}x + \frac{1}{2!}A_{12}x^2 + \cdots$$

dipole quadrupole sextupole etc.

$$= B_y\Big|_{\substack{x=0\\y=0}} + \left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=0\\y=0}} x + \frac{1}{2!}\left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=0\\y=0}} x^2 + \cdots \quad (17)$$

The successive derivatives identify the terms as being dipole, quadrupole, sextupole, octupole, etc., in the expansion of the field. To eliminate the necessity of continually writing these derivatives, it is useful to express the midplane field in terms of dimensionless quantities $n(t)$, $\beta(t)$, etc., or

$$B_y(x, 0, t) = B_y(0, 0, t)[1 - nhx + \beta h^2x^2 + \gamma h^3x^3 + \cdots] \quad (18)$$

where as before $h(t) = 1/\rho_0$, and n, β, and γ are functions of t. Direct comparison of Eqs. (17) and (18) yields

$$n = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=0\\y=0}} \quad \text{and} \quad \beta = \left[\frac{1}{2!h^2B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=0\\y=0}} \quad (19)$$

We now make use of Eq. (7), the equation of motion of the central trajectory:

$$B_y(0, 0, t) = hP_0/e$$

Combining Eqs. (7) and (19), the coefficients of the field expansions become

$$A_{10} = B_y(0, 0, t) = h\left(\frac{P_0}{e}\right)$$

$$A_{11} = \left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=0\\y=0}} = -nh^2\left(\frac{P_0}{e}\right)$$

$$\frac{1}{2!}A_{12} = \left.\frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=0\\y=0}} = \beta h^3\left(\frac{P_0}{e}\right)$$

$$A_{30} = -[h'' - nh^3 + 2\beta h^3]\left(\frac{P_0}{e}\right)$$

$$A'_{10} = h'\left(\frac{P_0}{e}\right)$$

$$A'_{11} = -[2nhh' + n'h^2]\left(\frac{P_0}{e}\right) \quad (20)$$

To second order the expansions for the magnetic field components may now be expressed in the form:

$$B_x(x, y, t) = (P_0/e)[-nh^2y + 2\beta h^3xy + \cdots]$$

$$B_y(x, y, t) = (P_0/e)[h - nh^2x + \beta h^3x^2 - \tfrac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2 + \cdots]$$

$$B_t(x, y, t) = (P_0/e)[h'y - (n'h^2 + 2nhh' + hh')xy + \cdots] \quad (21)$$

where P_0 is the momentum of the central trajectory.

5. Identification of n and β with Pure Quadrupole and Sextupole Fields

The scalar potential of a pure quadrupole field in cylindrical and in rectangular coordinates is given by:

$$\varphi = (B_0 r^2/2a)\sin 2\alpha = B_0 xy/a \tag{22a}$$

where B_0 is the field at the pole, a is the radius of the quadrupole aperture and r and α are the cylindrical coordinates, such that $x = r\cos\alpha$ and $y = r\sin\alpha$. From $\mathbf{B} = \nabla\varphi$, it follows that

$$B_x = B_0 y/a \quad \text{and} \quad B_y = B_0 x/a \tag{22b}$$

Using the second of Eqs. (20) and Eqs. (22a) and (22b),

$$\left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=0\\y=0}} = \frac{B_0}{a} = -nh^2\left(\frac{P_0}{e}\right)$$

where now we define the quantity k_q^2 as follows:

$$k_q^2 = -nh^2 = (B_0/a)(e/P_0) = (B_0/a)(1/B\rho) \tag{23}$$

Similarly for a pure sextupole field,

$$\varphi = (B_0 r^3/3a^2)\sin 3\alpha = (B_0/3a^2)[3x^2y - y^3]$$

$$B_x = \frac{\partial\varphi}{\partial x} = \frac{2B_0xy}{a^2} \quad \text{and} \quad B_y = \frac{B_0}{a^2}(x^2 - y^2) \tag{24}$$

where B_0 is the field at the pole and a is the radius of the sextupole aperture.

Using the third-part of Eqs. (20) and Eqs. (24)

$$\frac{1}{2!}\left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=0\\y=0}} = \frac{B_0}{a^2} = \beta h^3\left(\frac{P_0}{e}\right)$$

where we now define the quantity k_s^2 as follows:

$$k_s^2 = \beta h^3 = (B_0/a^2)(e/P_0) = (B_0/a^2)(1/B\rho) \tag{25}$$

These identities, Eqs. (23) and (25), are useful in the derivation of the equations of motion and the matrix elements for pure quadrupole and sextupole fields.

6. The Equations of Motion in Their Final Form to Second Order

Having derived Eq. (21), we are now in a position to substitute into the general second-order equations of motion, Eq. (6). Combining

Eq. (6) (the equation of motion) with the expanded field components of Eq. (21), we find for x

$$x'' - h(1 + hx) - x'(hx' + h'x)$$
$$= (P_0/P)T'\{(1 + hx)[-h + nh^2x - \beta h^3x^2 + \tfrac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2] + h'yy' + \cdots\}$$

and for y

$$y'' - y'(hx' + h'x)$$
$$= (P_0/P)T'\{-x'h'y - (1 + hx)[nh^2y - 2\beta h^3xy] + \cdots\}$$

Note that we have eliminated the charge of the particle e in the equations of motion. This has resulted from the use of the equation of motion of the central trajectory.

Inserting a second-order expansion for $T' = (x'^2 + y'^2 + (1 + hx)^2)^{1/2}$ and letting

$$P_0/P = P_0/P_0(1 + \delta) \cong 1 - \delta + \delta^2 + \cdots \tag{26}$$

we finally express the differential equations for x and y to second order as follows:

$$x'' + (1 - n)h^2x = h\delta + (2n - 1 - \beta)h^3x^2 + h'xx' + \tfrac{1}{2}hx'^2 + (2 - n)h^2x\delta + \tfrac{1}{2}(h'' - nh^3 + 2\beta h^3)y^2 + h'yy' - \tfrac{1}{2}hy'^2 - h\delta^2 + \text{higher-order terms} \tag{27}$$

$$y'' + nh^2y = 2(\beta - n)h^3xy + h'xy' - h'x'y + hx'y' + nh^2y\delta + \text{higher-order terms} \tag{28}$$

From Eqs. (27) and (28) the familiar equations of motion for the first-order terms may be extracted:

$$x'' + (1 - n)h^2x = h\delta \quad \text{and} \quad y'' + nh^2y = 0 \tag{29}$$

Substituting $k_q^2 = -nh^2$ from Eq. (23) into Eqs. (27) and (28), the second-order equations of motion for a pure quadrupole field result by taking the limit $h \to 0$, $h' \to 0$ and $h'' \to 0$. We find that

$$x'' + k_q^2x = k_q^2x\delta$$
$$y'' - k_q^2y = -k_q^2y\delta$$

where

$$k_q^2 = (B_0/a)(e/P_0) = (B_0/a)(1/B\rho_0) \tag{30}$$

Similarly, to find the second-order equations of motion for a pure sextupole field, we make use of Eq. (25) $\beta h^3 = k_s^2$ and, again, take the limit $h \to 0$, $h' \to 0$, and $h'' \to 0$. The results are:

$$x'' + k_s^2(x^2 - y^2) = 0$$
$$y'' - 2k_s^2 xy = 0$$

where

$$k_s^2 = \beta h^3 = (B_0/a^2)(e/P_0) = (B_0/a^2)(1/B\rho_0) \tag{31}$$

7. The Description of the Trajectories and the Coefficients of the Taylor's Expansion

The deviation of an arbitrary trajectory from the central trajectory is described by expressing x and y as functions of t. The expressions will also contain x_0, y_0, x'_0, y'_0 and δ, where the subscript 0 indicates that the quantity is evaluated at $t = 0$; these five boundary values will have the value zero for the central trajectory itself. The procedure for expressing x and y as a fivefold Taylor expansion will be considered in a general way using these boundary values, and detailed formulas will be developed for the calculations of the coefficients through the quadratic terms. The expansions are written:

$$x = \sum (x \mid x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi$$
$$y = \sum (y \mid x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi) x_0^\kappa y_0^\lambda x_0'^\mu y_0'^\nu \delta^\chi \tag{32}$$

Here, the parentheses are symbols for the Taylor coefficients; the first part of the symbol identifies the coordinate represented by the expansion, and the second indicates the term in question. These coefficients are functions of t to be determined. The $\sum$ indicates summation over zero and all positive integer values of the exponents κ, λ, μ, ν, χ: however, the detailed calculations will involve only the terms up to the second power. The constant term is zero, and the terms that would indicate a coupling between the coordinates x and y are also zero; this results from the midplane symmetry. Thus we have

$$(x \mid 1) = (y \mid 1) = 0$$
$$(x \mid y_0) = (y \mid x_0) = 0$$
$$(x \mid y'_0) = (y \mid x'_0) = 0 \tag{33}$$

Here, the first line is a consequence of choosing $x_0 = y_0 = 0$, while the second and third lines follow directly from considerations of symmetry, or, more formally, from the formulas at the end of this section.

As mentioned in the introduction, it is convenient to introduce the following abbreviations for the first-order Taylor coefficients:

$$\begin{aligned} &(x \mid x_0) = c_x(t) \qquad (x \mid x_0') = s_x(t) \qquad (x \mid \delta) = d(t) \\ &(y \mid y_0) = c_y(t) \qquad (y \mid y_0') = s_y(t) \end{aligned} \tag{34}$$

Retaining terms to second order and using Eqs. (33) and (34), the Taylor's expansions of Eq. (32) reduce to the following terms:

$$\begin{aligned} x = \; & \overbrace{(x \mid x_0)}^{c_x} x_0 + \overbrace{(x \mid x_0')}^{s_x} x_0' + \overbrace{(x \mid \delta)}^{d_x} \delta \\ & + (x \mid x_0^2) x_0^2 + (x \mid x_0 x_0') x_0 x_0' + (x \mid x_0 \delta) x_0 \delta \\ & + (x \mid x_0'^2) x_0'^2 + (x \mid x_0' \delta) x_0' \delta + (x \mid \delta^2) \delta^2 \\ & + (x \mid y_0^2) y_0^2 + (x \mid y_0 y_0') y_0 y_0' + (x \mid y_0'^2) y_0'^2 \end{aligned}$$

and

$$\begin{aligned} y = \; & \overbrace{(y \mid y_0)}^{c_y} y_0 + \overbrace{(y \mid y_0')}^{s_y} y_0' \\ & + (y \mid x_0 y_0) x_0 y_0 + (y \mid x_0 y_0') x_0 y_0' + (y \mid x_0' y_0) x_0' y_0 \\ & + (y \mid x_0' y_0') x_0' y_0' + (y \mid y_0 \delta) y_0 \delta + (y \mid y_0' \delta) y_0' \delta \end{aligned} \tag{35}$$

Substituting these expansions into Eqs. (27) and (28), we derive a differential equation for each of the first- and second-order coefficients contained in the Taylor's expansions for x and y. When this is done, a systematic pattern evolves, namely,

$$\begin{aligned} c_x'' + k_x^2 c_x &= 0 & & c_y'' + k_y^2 c_y = 0 \\ s_x'' + k_x^2 s_x &= 0 & \text{or} \quad & s_y'' + k_y^2 s_y = 0 \\ q_x'' + k_x^2 q_x &= f_x & & q_y'' + k_y^2 q_y = f_y \end{aligned} \tag{36}$$

where $k_x^2 = (1 - n)h^2$ and $k_y^2 = nh^2$ for the x and y motions, respectively. The first two of these equations represent the equations of motion for the first-order monoenergetic terms s_x, c_x, s_y, and c_y. That there are two solutions, one for c and one for s, is a manifestation of the fact that the differential equation is second order; hence, the two solutions differ only by the initial conditions of the characteristic s and c functions.

The third differential equation for q is a type form which represents the solution for the first-order dispersion d_x and for any one of the coefficients of the second-order aberrations in the system where the driving term f has a characteristic form for each of these coefficients. The driving function f for each aberration is obtained from the substitution of the Taylor's expansions of Eq. (35) into the general differential Eqs. (27) and (28).

The coefficients satisfy the boundary conditions:

$$\begin{aligned} c(0) &= 1 & c'(0) &= 0 \\ s(0) &= 0 & s'(0) &= 1 \\ d(0) &= 0 & d'(0) &= 0 \\ q(0) &= 0 & q'(0) &= 0 \end{aligned} \tag{37}$$

The driving term f is a polynomial, peculiar to the particular q, whose terms are the coefficients of order less than that of q, and their derivatives. The coefficients in these polynomials are themselves polynomials in h, h', ..., with coefficients that are linear functions of n, β, ... For example, for $q = (x \mid x_0^2)$, we have

$$f = (2n - 1 - \beta)h^3 c_x^2 + h' c_x c_x' + \tfrac{1}{2} h c_x'^2 \tag{38}$$

In Table I are listed the f functions for the remaining linear coefficient, the momentum dispersion $d(t)$ and all of the nonzero quadratic coefficients, shown in Eq. (35), which represent the second-order aberrations of a system.

The coefficients c and s (with identical subscripts) satisfy the same differential equation which has the form of the homogeneous equation of a harmonic oscillator. Here, the stiffness k^2 is a function of t and may be of either sign. In view of their boundary conditions, it is natural to consider c and s as the analogs of the two fundamental solutions of a simple harmonic oscillator, namely $\cos \omega t$ and $(\sin \omega t)/\omega$. The function q is the response of the hypothetical oscillator when, starting at equilibrium and at rest, it is subjected to a driving force f.

The stiffness parameters k_x^2 and k_y^2 represent the converging powers of the field for the two respective coordinates. It is possible for either to be negative, in which case it actually represents a diverging effect. Addition of k_x^2 and k_y^2 yields

$$k_x^2 + k_y^2 = h^2 \tag{39}$$

TABLE I
The Driving Terms for the Coefficients

Listed in the first column are the coefficients in the expressions for the coordinates x and y; they are indicated by means of the notation introduced in Eq. (32); in addition, the abbreviations given in Eq. (34) are used. For general considerations, q has been used to represent any one of these coefficients. Listed in the second column are the corresponding driving functions f, which are related to the coefficients as shown by Eq. (36). This list includes all those functions f for the linear and quadratic coefficients which do not vanish identically.

q	f			
$d = (x \mid \delta)$	h			
$(x \mid x_0^2)$		$+ (2n - 1 - \beta)h^3c_x^2$	$+ h'c_xc_x'$	$+ \frac{1}{2}hc_x'^2$
$(x \mid x_0x_0')$		$+ 2(2n - 1 - \beta)h^3c_xs_x$	$+ h'(c_xs_x' + c_x's_x)$	$+ hc_x's_x'$
$(x \mid \delta x_0)$	$(2 - n)h^2c_x$	$+ 2(2n - 1 - \beta)h^3c_xd$	$+ h'(c_xd' + c_x'd)$	$+ hc_x'd'$
$(x \mid x_0'^2)$		$+ (2n - 1 - \beta)h^3s_x^2$	$+ h's_xs_x'$	$+ \frac{1}{2}hs_x'^2$
$(x \mid \delta x_0')$	$(2 - n)h^2s_x$	$+ 2(2n - 1 - \beta)h^3s_xd$	$+ h'(s_xd' + s_x'd)$	$+ hs_x'd'$
$(x \mid \delta^2)$	$- h + (2 - n)h^2d$	$+ (2n - 1 - \beta)h^3d^2$	$+ h'dd'$	$+ \frac{1}{2}hd'^2$
$(x \mid y_0^2)$		$+ \frac{1}{2}(h'' - nh^3 + 2\beta h^3)c_y^2$	$+ h'c_yc_y'$	$- \frac{1}{2}hc_y'^2$
$(x \mid y_0y_0')$		$+ (h'' - nh^3 + 2\beta h^3)c_ys_y$	$+ h'(c_ys_y' + c_y's_y)$	$- hc_y's_y'$
$(x \mid y_0'^2)$		$+ \frac{1}{2}(h'' - nh^3 + 2\beta h^3)s_y^2$	$+ h's_ys_y'$	$- \frac{1}{2}hs_y'^2$
$(y \mid x_0y_0)$		$2(\beta - n)h^3c_xc_y$	$+ h'(c_xc_y' - c_x'c_y)$	$+ hc_x'c_y'$
$(y \mid x_0y_0')$		$2(\beta - n)h^3c_xs_y$	$+ h'(c_xs_y' - c_x's_y)$	$+ hc_x's_y'$
$(y \mid x_0'y_0)$		$2(\beta - n)h^3s_xc_y$	$+ h'(s_xc_y' - s_x'c_y)$	$+ hs_x'c_y'$
$(y \mid x_0'y_0')$		$2(\beta - n)h^3s_xs_y$	$+ h'(s_xs_y' - s_x's_y)$	$+ hs_x's_y'$
$(y \mid \delta y_0)$	nh^2c_y	$+ 2(\beta - n)h^3c_yd$	$- h'(c_yd' - c_y'd)$	$+ hc_y'd'$
$(y \mid \delta y_0')$	nh^2s_y	$+ 2(\beta - n)h^3s_yd$	$- h'(s_yd' - s_y'd)$	$+ hs_y'd'$

For a specific magnitude of h, k_x^2 and k_y^2 may be varied by adjusting n, but the total converging power is unchanged; any increase in one converging power is at the expense of the other. The total converging power is positive; this fact admits the possibility of double focusing.

A special case of interest is provided by the uniform field; here $h = \text{const.}$ and $n = 0$; then $k_x^2 = h^2$ and $k_y^2 = 0$. Thus, there is a converging effect for x resulting in the familiar semicircular focusing, which is accompanied by no convergence or divergence of y.

Another important special case is given by $n = \frac{1}{2}$; here, $k_x^2 = k_y^2 = h^2/2$. Thus, both coordinates experience an identical positive convergence, and $c_x = c_y$ and $s_x = s_y$; that is, in the linear approximation, the two coordinates behave identically, and if the trajectory continues through a sufficiently extended field, a double focus is produced.

The method of solution of the equations for c and s will not be discussed here, since they are standard differential equations. The most suitable approach to the problem must be determined in each case. In many cases it will be a satisfactory approximation to consider h and n, and therefore k^2 also, as uniform piecewise. Then, c and s are represented in each interval of uniformity by a sinusoidal function, a hyperbolic function, or a linear function of t, or simply a constant. Using Eq. (36), it follows for either the x or y motions that:

$$\frac{d}{dt}(cs' - c's) = 0$$

Upon integrating and using the initial conditions on c and s in Eq. (37), we find

$$cs' - c's = 1 \tag{40}$$

This expression is just the determinant of the first-order transport matrix representing either the x or y equations of motion. It can be demonstrated that the fact that the determinant is equal to one is equivalent to Liouville's theorem, which states that phase areas are conserved throughout the system in either the x or y plane motions.

The coefficients q are evaluated using a Green's function integral

$$q = \int_0^t f(\tau)G(t, \tau)\, d\tau \tag{41}$$

where

$$G(t, \tau) = s(t)c(\tau) - s(\tau)c(t) \tag{42}$$

and

$$q = s(t)\int_0^t f(\tau)c(\tau)\, d\tau - c(t)\int_0^t f(\tau)s(\tau)\, d\tau \tag{43}$$

To verify this result, it should be noted that this equation, in conjunction with Eq. (40), reduces the last of Eq. (36) to an identity, and that the last pair of Eq. (37) follows readily from this proposed solution. In particular, if $f = 0$, then $q = 0$. Then it will be seen from Table I that several coefficients are absent, including the linear terms that would represent a coupling between x and y. Frequently, the absence of a particular coefficient is obvious from considerations of symmetry.

Differentiation of Eq. (43) yields

$$q' = s'(t) \int_0^t f(\tau)c(\tau)\, d\tau - c'(t) \int_0^t f(\tau)s(\tau)\, d\tau \tag{44}$$

and

$$q'' = f + s''(t) \int_0^t f(\tau)c(\tau)\, d\tau - c''(t) \int_0^t f(\tau)s(\tau)\, d\tau$$

The driving terms tabulated in Table I, combined with Eqs. (43) and (44), complete the solution of the general second-order theory. It now remains to find explicit solutions for specific systems or elements of systems.

8. *Transformation from Curvilinear Coordinates to a Rectangular Coordinate System and TRANSPORT Notation*

All results so far have been expressed in terms of the general curvilinear coordinate system (x, y, t). It is useful to transform these results to the rectangular coordinate system (x, y, z), shown in Figure 4, to facilitate matching boundary conditions between the various components comprising a beam transport system. This is accomplished by introducing the angular coordinates θ and φ defined as follows (again, using the paraxial ray approximation $\tan\theta = \theta$ and $\tan\varphi = \varphi$):

$$\theta = \frac{dx}{dz} = \frac{x'}{z'} = \frac{x'}{1 + hx}$$

$$\varphi = \frac{dy}{dz} = \frac{y'}{z'} = \frac{y'}{1 + hx} \tag{45}$$

where, as before, prime means the derivative with respect to t.

Using these definitions and those of Eqs. (34) and (35), it is now possible to express the Taylor's expansions for x, θ, y, and φ in terms of the rectangular coordinate system. For the sake of completeness and to clearly define the notation used, the complete Taylor's expansions for

x, θ, y, and φ at the end of a system as a function of the initial variables are given below:

$$\begin{aligned} x = {} & \overbrace{(x \mid x_0)}^{c_x} x_0 + \overbrace{(x \mid \theta_0)}^{s_x}\theta_0 + \overbrace{(x \mid \delta)}^{d_x}\delta \\ & + (x \mid x_0^2)x_0^2 + (x \mid x_0\theta_0)x_0\theta_0 + (x \mid x_0\delta)x_0\delta \\ & + (x \mid \theta_0^2)\theta_0^2 + (x \mid \theta_0\delta)\theta_0\delta + (x \mid \delta^2)\delta^2 \\ & + (x \mid y_0^2)y_0^2 + (x \mid y_0\varphi_0)y_0\varphi_0 + (x \mid \varphi_0^2)\varphi_0^2 \end{aligned}$$

$$\begin{aligned} \theta = {} & \overbrace{(\theta \mid x_0)}^{c'_x} x_0 + \overbrace{(\theta \mid \theta_0)}^{s'_x}\theta_0 + \overbrace{(\theta \mid \delta)}^{d'_x}\delta \\ & + (\theta \mid x_0^2)x_0^2 + (\theta \mid x_0\theta_0)x_0\theta_0 + (\theta \mid x_0\delta)x_0\delta \\ & + (\theta \mid \theta_0^2)\theta_0^2 + (\theta \mid \theta_0\delta)\theta_0\delta + (\theta \mid \delta^2)\delta^2 \\ & + (\theta \mid y_0^2)y_0^2 + (\theta \mid y_0\varphi_0)y_0\varphi_0 + (\theta \mid \varphi_0^2)\varphi_0^2 \end{aligned}$$

$$\begin{aligned} y = {} & \overbrace{(y \mid y_0)}^{c_y} y_0 + \overbrace{(y \mid \varphi_0)}^{s_y}\varphi_0 \\ & + (y \mid x_0y_0)x_0y_0 + (y \mid x_0\varphi_0)x_0\varphi_0 + (y \mid \theta_0y_0)\theta_0y_0 \\ & + (y \mid \theta_0\varphi_0)\theta_0\varphi_0 + (y \mid y_0\delta)y_0\delta + (y \mid \varphi_0\delta)\varphi_0\delta \end{aligned}$$

$$\begin{aligned} \varphi = {} & \overbrace{(\varphi \mid y_0)}^{c'_y} y_0 + \overbrace{(\varphi \mid \varphi_0)}^{s'_y}\varphi_0 \\ & + (\varphi \mid x_0y_0)x_0y_0 + (\varphi \mid x_0\varphi_0)x_0\varphi_0 + (\varphi \mid \theta_0y_0)\theta_0y_0 \\ & + (\varphi \mid \theta_0\varphi_0)\theta_0\varphi_0 + (\varphi \mid y_0\delta)y_0\delta + (\varphi \mid \varphi_0\delta)\varphi_0\delta \end{aligned} \tag{46}$$

Using the definitions of Eq (45), the coefficients appearing in Eq. (46) may be easily related to those appearing in Eq. (35). At the same time, we will introduce the abbreviated notation used in the Stanford TRANSPORT Program[(3)] where the subscript 1 means x; 2 means θ, 3 means y; 4 means Φ, and 6 means δ. The subscript 5 is the path length difference l between an arbitrary ray and the central trajectory. R_{ij} will be used to signify a first-order matrix element and T_{ijk} will signify a second-order matrix element. Thus, we may write Eq. (46) in the general form

$$x_i = \sum_{j=1}^{6} R_{ij}x_j(0) + \sum_{j=1}^{6}\sum_{k=j}^{6} T_{ijk}x_j(0)x_k(0) \tag{47}$$

where

$$x_1 = x,\ x_2 = \theta,\ x_3 = y,\ x_4 = \varphi,\ x_5 = l, \text{ and } x_6 = \delta$$

denotes the subscript notation.

Using Eq. (45) defining θ and φ, the following identities among the various matrix element definitions result:

For the Taylor's expansions for x we have:

$$\begin{aligned}
R_{11} &= (x \mid x_0) = c_x \\
R_{12} &= (x \mid \theta_0) = (x \mid x'_0) = s_x \\
R_{16} &= (x \mid \delta) = d_x \\
T_{111} &= (x \mid x_0^2) \\
T_{112} &= (x \mid x_0\theta_0) = (x \mid x_0x'_0) + h(0)s_x \\
T_{116} &= (x \mid x_0\delta) \\
T_{122} &= (x \mid \theta_0^2) = (x \mid x_0'^2) \\
T_{126} &= (x \mid \theta_0\delta) = (x \mid x'_0\delta) \\
T_{166} &= (x \mid \delta^2) \\
T_{133} &= (x \mid y_0^2) \\
T_{134} &= (x \mid y_0\varphi_0) = (x \mid y_0y'_0) \\
T_{144} &= (x \mid \varphi_0^2) = (x \mid y_0'^2)
\end{aligned} \tag{48}$$

For the θ terms we have.

$$\begin{aligned}
R_{21} &= (\theta \mid x_0) = (x' \mid x_0) = \frac{d}{dt}(x \mid x_0) = c'_x \\
R_{22} &= (\theta \mid \theta_0) = (x' \mid x'_0) = s'_x \\
R_{26} &= (\theta \mid \delta) = (x' \mid \delta) = d'_x \\
T_{211} &= (\theta \mid x_0^2) = (x' \mid x_0^2) - h(t)c_xc'_x \\
T_{212} &= (\theta \mid x_0\theta_0) = (x' \mid x_0x'_0) + h(0)s'_x - h(t)[c_xs'_x + c'_xs_x] \\
T_{216} &= (\theta \mid x_0\delta) = (x' \mid x_0\delta) - h(t)[c_x d'_x + c'_x d_x] \\
T_{222} &= (\theta \mid \theta_0^2) = (x' \mid x_0'^2) - h(t)s_xs'_x \\
T_{226} &= (\theta \mid \theta_0\delta) = (x' \mid x'_0\delta) - h(t)[s_x d'_x + s'_x d_x]
\end{aligned}$$

$$T_{266} = (\theta \mid \delta^2) = (x' \mid \delta^2) - h(t)\, d_x\, d'_x$$
$$T_{233} = (\theta \mid y_0^2) = (x' \mid y_0^2)$$
$$T_{234} = (\theta \mid y_0\varphi_0) = (x' \mid y_0 y'_0)$$
$$T_{244} = (\theta \mid \varphi_0^2) = (x' \mid y_0'^2) \qquad (49)$$

For the y terms in the Taylor's expansion:

$$R_{33} = (y \mid y_0) = c_y$$
$$R_{34} = (y \mid \varphi_0) = (y \mid y'_0) = s_y$$
$$T_{313} = (y \mid x_0 y_0)$$
$$T_{314} = (y \mid x_0\varphi_0) = (y \mid x_0 y'_0) + h(0)s_y$$
$$T_{323} = (y \mid \theta_0 y_0) = (y \mid x'_0 y_0)$$
$$T_{324} = (y \mid \theta_0\varphi_0) = (y \mid x'_0 y'_0)$$
$$T_{336} = (y \mid y_0\delta)$$
$$T_{346} = (y \mid \varphi_0\delta) = (y \mid y'_0\delta) \qquad (50)$$

and finally for the φ terms we have:

$$R_{43} = (\varphi \mid y_0) = (y' \mid y_0) = \frac{d}{dt}(y \mid y_0) = c'_x$$
$$R_{44} = (\varphi \mid \varphi_0) = (y' \mid y'_0) = s'_y$$
$$T_{413} = (\varphi \mid x_0 y_0) = (y' \mid x_0 y_0) - h(t)c_x c'_y$$
$$T_{414} = (\varphi \mid x_0\varphi_0) = (y' \mid x_0 y'_0) + h(0)s'_y - h(t)c_x s'_y$$
$$T_{423} = (\varphi \mid \theta_0 y_0) = (y' \mid x'_0 y_0) - h(t)s_x c'_y$$
$$T_{424} = (\varphi \mid \theta_0\varphi_0) = (y' \mid x'_0 y'_0) - h(t)s_x s'_y$$
$$T_{436} = (\varphi \mid y_0\delta) = (y' \mid y_0\delta) - h(t)c'_y\, d_x$$
$$T_{446} = (\varphi \mid \varphi_0\delta) = (y' \mid y'_0\delta) - h(t)s'_y\, d_x \qquad (51)$$

All of the above terms are understood to be evaluated at the terminal point of the system except for the quantity $h(0)$ which is to be evaluated at the beginning of the system. In practice, $h(0)$ will usually be equal to $h(t)$; but to retain the formalism, we show them as being different here.

All nonlisted matrix elements are equal to zero.

9. *First- and Second-Order Matrix Formalism of Beam Transport Optics*

The solution of first-order beam transport problems using matrix algebra has been extensively documented.[(4-6)] However, it does not seem to be generally known that matrix methods may be used to solve second- and higher-order beam transport problems. A general proof of the validity of extending matrix algebra to include second-order terms has been given by Brown, Belbeoch, and Bounin[(7)] the results of which are summarized below in the notation of this report and in TRANSPORT notation.

Consider again Eq. (47). From ref. 3, the matrix formalism may be logically extended to include second-order terms by extending the definition of the column matrices x_i and x_j in the first-order matrix algebra to include the second-order terms as shown in Tables II–V. In addition, it is necessary to calculate and include the coefficients shown in the lower right-hand portion of the square matrix such that the set of simultaneous equations represented by Tables II–V are valid. Note that the second-order equations, represented by the lower right-hand portion of the matrix, are derived in a straightforward manner from the first-order equations, represented by the upper left-hand portion of the matrix. For example, consider the matrix in Table II; we see from row 1 that

$$x = c_x x_0 + s_x \theta_0 + d_x \delta + \text{second-order terms}$$

Hence, row 4 is derived directly by squaring the above equation as follows:

$$\begin{aligned} x^2 &= (c_x x_0 + s_x \theta_0 + d_x \delta)^2 \\ &= c_x^2 x_0^2 + 2 c_x s_x x_0 \theta_0 + 2 c_x\, d_x x_0 \delta + s_x^2 \theta_0^2 + 2 s_x\, d_x \theta_0 \delta + d_x^2 \delta^2 \end{aligned}$$

The remaining rows are derived in a similar manner.

If now $x_1 = M_1 x_0$ represents the complete first- and second-order transformation from 0 to 1 in a beam transport system and $x_2 = M_2 x_1$ is the transformation from 1 to 2, then the first- and second-order transformation from 0 to 2 is simply $x_2 = M_2 x_1 = M_2 M_1 x_0$; where M_1 and M_2 are matrices fabricated as shown in Tables II and III in our notation or as shown in Tables IV and V in TRANSPORT notation.

TABLE II

Formulation of the Second-Order Matrix for the Bend (x)-Plane

$$
\begin{bmatrix} x \\ \theta \\ \delta \\ \hline x^2 \\ x\theta \\ x\delta \\ \theta^2 \\ \theta\delta \\ \delta^2 \\ y^2 \\ y\varphi \\ \varphi^2 \end{bmatrix}
=
\left[\begin{array}{ccc|ccccccccc}
c_x & s_x & d_x & (x \mid x_0^2) & (x \mid x_0\theta_0) & (x \mid x_0\delta) & (x \mid \theta_0^2) & (x \mid \theta_0\delta) & (x \mid \delta^2) & (x \mid y_0^2) & (x \mid y_0\varphi_0) & (x \mid \theta_0^2) \\
c'_x & s'_x & d'_x & (\theta \mid x_2^0) & (\theta \mid x_0\theta_0) & \text{etc.} & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & c_x^2 & 2s_x c_x & 2c_x d_x & s_x^2 & 2s_x d_x & d_x^2 & 0 & 0 & 0 \\
 & & & c_x c'_x & c_x s'_x + c'_x s_x & \text{etc.} & & & & & & \\
 & 0 & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}\right]
\begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ \hline x_0^2 \\ x_0\theta_0 \\ x_0\delta \\ \theta_0^2 \\ \theta_0\delta \\ \delta^2 \\ y_2^0 \\ y_0\varphi_0 \\ \varphi_0^2 \end{bmatrix}
$$

TABLE III

Formulation of the Second-Order Matrix for the Nonbend (y) Plane

$$
\begin{bmatrix} y \\ \varphi \\ \hline xy \\ x\varphi \\ \theta y \\ \theta\varphi \\ y\delta \\ \varphi\delta \end{bmatrix}
=
\left[\begin{array}{cc|cccccc}
c_y & s_y & (y \mid x_0 y_0) & (y \mid x_0 y_0) & (y \mid \theta_0 y_0) & (y \mid \theta_0 \varphi_0) & (y \mid y_0 \delta) & (y \mid \varphi_0 \delta) \\
c_y' & s_y' & (\varphi \mid x_0 y_0) & (\varphi \mid x_0 \varphi_0) & (\varphi \mid \theta_0 y_0) & (\varphi \mid \theta_0 \varphi_0) & (\varphi \mid y_0 \delta) & (\varphi \mid \varphi_0 \delta) \\
\hline
 & & c_x c_y & c_x s_y & s_x c_y & s_x s_y & d_x c_y & d_x s_y \\
 & & c_x c_y' & c_x s_y' & s_x c_y' & \text{etc.} & & \\
 & 0 & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right]
\begin{bmatrix} y_0 \\ \varphi_0 \\ \hline x_0 y_0 \\ x_0 \varphi_0 \\ \theta_0 y_0 \\ \theta_0 \varphi_0 \\ y_0 \delta \\ \varphi_0 \delta \end{bmatrix}
$$

TABLE IV

Formulation of Second-Order Matrix in the Bend (x) Plane Using TRANSPORT Notation

$$
\begin{bmatrix} x \\ \theta \\ \delta \\ \hline x^2 \\ x\theta \\ x\delta \\ \theta^2 \\ \theta\delta \\ \delta^2 \\ \nu^2 \\ y\varphi \\ \varphi^2 \end{bmatrix}
=
\left[\begin{array}{ccc|ccccccccc}
R_{11} & R_{12} & R_{16} & T_{111} & T_{112} & T_{116} & T_{122} & T_{126} & T_{166} & T_{133} & T_{134} & T_{144} \\
R_{21} & R_{22} & R_{26} & T_{211} & T_{212} & T_{216} & T_{222} & T_{226} & T_{266} & T_{233} & T_{234} & T_{244} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & R_{11}^2 & 2R_{11}R_{12} & 2R_{11}R_{16} & R_{12}^2 & 2R_{12}R_{16} & R_{16}^2 & 0 & 0 & 0 \\
 & & & R_{11}R_{21} & R_{11}R_{22} + R_{12}R_{21} & R_{11}R_{26} + R_{16}R_{21} & \text{etc.} & & & 0 & 0 & 0 \\
 & 0 & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}\right]
\begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ \hline x_0^2 \\ x_0\theta_0 \\ x_0\delta \\ \theta_0^2 \\ \theta_0\delta \\ \delta^2 \\ y_0^2 \\ y_0\varphi_0 \\ \varphi_0^2 \end{bmatrix}
$$

TABLE V

Formulation of Second-Order Matrix in Nonbend (y) Plane Using TRANSPORT Notation

$$
\begin{bmatrix} y \\ \varphi \\ \hline xy \\ x\varphi \\ \theta y \\ \theta\varphi \\ y\delta \\ \varphi\delta \end{bmatrix}
=
\left[\begin{array}{cc|cccccc}
R_{33} & R_{34} & T_{313} & T_{314} & T_{323} & T_{324} & T_{336} & T_{346} \\
R_{43} & R_{44} & T_{413} & T_{414} & T_{423} & T_{424} & T_{436} & T_{446} \\
\hline
 & & R_{11}R_{33} & R_{11}R_{34} & R_{12}R_{33} & R_{12}R_{34} & R_{16}R_{33} & R_{16}R_{34} \\
 & & R_{11}R_{43} & R_{11}R_{44} & R_{12}R_{43} & \text{etc.} & & \\
 & 0 & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array}\right]
\begin{bmatrix} y_0 \\ \varphi_0 \\ \hline x_0y_0 \\ x_0\varphi_0 \\ \theta_0y_0 \\ \theta_0\varphi_0 \\ y_0\delta \\ \varphi_0\delta \end{bmatrix}
$$

III. Reduction of the General First- and Second-Order Theory to the Case of the Ideal Magnet

Section II of this report was devoted to the derivation of the general second-order differential equations of motion of charged particles in a static magnetic field. In Section II no restrictions were placed on the variation of the field along the central orbit, i.e., h, n, and β were assumed to be functions of t. As such, the final results were left in either a differential equation form or expressed in terms of an integral containing the driving function $f(t)$, and a Green's function $G(t, \tau)$ derived from the first-order solutions of the homogeneous equations. We now limit the generality of the problem by assuming h, n, and β to be constants over the interval of integration. With this restriction, the solutions to the homogeneous differential equation [Eq. (36) of Sec. II] are the following simple trigonometric functions:

$$\begin{aligned} c_x(t) &= \cos k_x t \qquad & s_x(t) &= (1/k_x) \sin k_x t \\ c_y(t) &= \cos k_y t \qquad & s_y(t) &= (1/k_y) \sin k_y t \end{aligned} \tag{52}$$

where now

$$k_x^2 = (1 - n)h^2, \quad k_y^2 = nh^2, \quad \text{and} \quad h = 1/\rho_0$$

become constants of the motion. ρ_0 is the radius of curvature of the central trajectory.

The solution of the inhomogeneous differential equations [the third of Eqs. (36)] for the remaining matrix elements is solved as indicated in Section II, using the Green's functions integral Eq. (41) and the driving functions listed in Table I. With the restrictions that k_x and k_y are constants, the Green's functions reduce to the following simple trigonometric forms:

$$G_x(t, \tau) = (1/k_x) \sin k_x(t - \tau)$$

and

$$G_y(t, \tau) = (1/k_y) \sin k_y(t - \tau) \tag{53}$$

The resulting matrix elements are tabulated below in terms of the key integrals listed in Table VI, the five characteristic first-order matrix elements s_x, c_x, d_x, c_y, and s_y and the constants h, n, and β.

TABLE VIa

Tabulation of the First- and Second-Order Matrix Elements for an Ideal Magnet in Terms of the Key Integrals Listed in Table VIb

Definitions:

$$k_x^2 = (1 - n)h^2 \qquad h = 1/\rho_0$$
$$k_y^2 = nh^2$$

$$R_{11} = (x \mid x_0) = c_x(t) = \cos k_x t$$
$$R_{12} = (x \mid \theta_0) = s_x(t) = (1/k_x)\sin k_x t$$
$$R_{16} = (x \mid \delta) = d_x(t) = (h/k_x^2)[1 - c_x(t)]$$

$$T_{111} = (x \mid x_0^2) = (2n - 1 - \beta)h^3 I_{111} + \tfrac{1}{2}k_x^4 h I_{122}$$
$$T_{112} = (x \mid x_0\theta_0) = hs_x(t) + 2(2n - 1 - \beta)h^3 I_{112} - k_x^2 h I_{112}$$
$$T_{116} = (x \mid x_0\delta) = (2 - n)h^2 I_{11} + 2(2n - 1 - \beta)h^3 I_{116} - k_x^2 h^2 I_{122}$$
$$T_{122} = (x \mid \theta_0^2) = (2n - 1 - \beta)h^3 I_{122} + \tfrac{1}{2}h I_{111}$$
$$T_{126} = (x \mid \theta_0\delta) = (2 - n)h^2 I_{12} + 2(2n - 1 - \beta)h^3 I_{126} + h^2 I_{112}$$
$$T_{166} = (x \mid \delta^2) = -hI_{10} + (2 - n)h^2 I_{16} + (2n - 1 - \beta)h^3 I_{166} + \tfrac{1}{2}h^3 I_{122}$$
$$T_{133} = (x \mid y_0^2) = \beta h^3 I_{133} - \tfrac{1}{2}k_y^2 h I_{10}$$
$$T_{134} = (x \mid y_0\varphi_0) = 2\beta h^3 I_{134}$$
$$T_{144} = (x \mid \varphi_0^2) = \beta h^3 I_{144} - \tfrac{1}{2}h I_{10}$$

$$R_{21} = (\theta \mid x_0) = c_x'(t) = -k_x^2 s_x(t)$$
$$R_{22} = (\theta \mid \theta_0) = s_x'(t) = c_x(t)$$
$$R_{26} = (\theta \mid \delta) = d_x'(t) = hs_x(t)$$

$$T_{211} = (\theta \mid x_0^2) = (2n - 1 - \beta)h^3 I_{211} + \tfrac{1}{2}k_x^4 h I_{222} - hc_x(t)c_x'(t)$$
$$T_{212} = (\theta \mid x_0\theta_0) = hs_x'(t) \quad 2(2n - 1 - \beta)h^3 I_{212} - k_x^2 h I_{212} - h[c_x(t)s_x'(t) + c_x'(t)s_x(t)]$$
$$T_{216} = (\theta \mid x_0\delta) = (2 - n)h^2 I_{21} + 2(2n - 1 - \beta)h^3 I_{216} - k_x^2 h^2 I_{222} - h[c_x(t)\, d_x'(t) + c_x'(t)\, d_x(t)]$$
$$T_{222} = (\theta \mid \theta_0^2) = (2n - 1 - \beta)h^3 I_{222} + \tfrac{1}{2}h I_{211} - hs_x(t)s_x'(t)$$
$$T_{226} = (\theta \mid \theta_0\delta) = (2 - n)h^2 I_{22} + 2(2n - 1 - \beta)h^3 I_{226} + h^2 I_{212} - h[s_x(t)\, d_x'(t) + s_x'(t)\, d_x(t)]$$

$$
\begin{aligned}
T_{266} &= (\theta \mid \delta^2) &&= -hI_{20} + (2-n)h^2 I_{26} + (2n-1-\beta)h^3 I_{266} + \tfrac{1}{2}h^3 I_{222} - h\, d_x(t)\, d_x'(t)\\
T_{233} &= (\theta \mid y_0^2) &&= \beta h^3 I_{233} - \tfrac{1}{2}k_y^2 h I_{20}\\
T_{234} &= (\theta \mid y_0\varphi_0) &&= 2\beta h^3 I_{234}\\
T_{244} &= (\theta \mid \varphi_0^2) &&= \beta h^3 I_{244} - \tfrac{1}{2}hI_{20}\\
R_{33} &= (y \mid y_0) &&= c_y(t) = \cos k_y t\\
R_{34} &= (y \mid \varphi_0) &&= s_y(t) = (1/k_y)\sin k_y t\\
T_{313} &= (y \mid x_0 y_0) &&= + 2(\beta - n)h^3 I_{313} + k_x^2 k_y^2 h I_{324}\\
T_{314} &= (y \mid x_0\varphi_0) &&= hs_y(t) + 2(\beta - n)h^3 I_{314} - k_x^2 h I_{323}\\
T_{323} &= (y \mid \theta_0 y_0) &&= + 2(\beta - n)h^3 I_{323} - k_y^2 h I_{314}\\
T_{324} &= (y \mid \theta_0\varphi_0) &&= + 2(\beta - n)h^3 I_{324} + hI_{313}\\
T_{336} &= (y \mid y_0\delta) &&= k_y^2 I_{33} + 2(\beta - n)h^3 I_{336} - k_y^2 h^2 I_{324}\\
T_{346} &= (y \mid \varphi_0\delta) &&= k_y^2 I_{34} + 2(\beta - n)h^3 I_{346} + h^2 I_{323}\\
R_{43} &= (\varphi \mid y_0) &&= c_y'(t) = -k_y^2 s_y(t)\\
R_{44} &= (\varphi \mid \varphi_0) &&= s_y'(t) = c_y(t)\\
T_{413} &= (\varphi \mid x_0 y_0) &&= 2(\beta - n)h^3 I_{413} + k_x^2 k_y^2 h I_{424} - hc_x(t)c_y'(t)\\
T_{414} &= (\varphi \mid x_0\varphi_0) &&= hs_y'(t) + 2(\beta - n)h^3 I_{414} - k_x^2\, hI_{423} - hc_x(t)s_y'(t)\\
T_{423} &= (\varphi \mid \theta_0 y_0) &&= 2(\beta - n)h^3 I_{423} - k_y^2 h I_{414} - hs_x(t)c_y'(t)\\
T_{424} &= (\varphi \mid \theta_0\varphi_0) &&= 2(\beta - n)h^3 I_{424} + hI_{413} - hs_x(t)s_y'(t)\\
T_{436} &= (\varphi \mid y_0\delta) &&= k_y^2 I_{43} + 2(\beta - n)h^3 I_{436} - k_y^2 h^2 I_{424} - h\, d_x(t)c_y'(t)\\
T_{446} &= (\varphi \mid \varphi_0\delta) &&= k_y^2 I_{44} + 2(\beta - n)h^3 I_{446} + h^2 I_{423} - h\, d_x(t)s_y'(t)
\end{aligned}
\qquad (54)
$$

TABLE VIb

Tabulation of Key Integrals Required for the Numerical Evaluation of the Second-Order Aberrations of Ideal Magnets

The results are expressed in terms of the five characteristic first-order matrix elements $s_x(t)$, $c_x(t)$, $d_x(t)$, $c_y(t)$, and $s_y(t)$ and the quantities h and n (assumed to be constant for the ideal magnet over the interval of integration $\tau = 0$ to $\tau = t$). The path length of the central trajectory is t. From the solutions of the differential equations [Eq. (29) of Sec. II], the first-order matrix elements for the ideal magnet are:

$c_x(t) = \cos k_x t \qquad s_x(t) = (1/k_x) \sin k_x t \qquad d_x(t) = (h/k_x^2)[1 - c_x(t)] \qquad c_y(t) = \cos k_y t \qquad s_y(t) = (1/k_y) \sin k_y t$

where

$$k_x^2 = (1 - n)h^2, \quad k_y^2 = nh^2, \quad \text{and} \quad h = 1/\rho_0$$

ρ_0 is the radius of curvature of the central trajectory.

$$I_{10} = \int_0^t G_x(t, \tau)\, d\tau = \left[\frac{d_x(t)}{h}\right]$$

$$I_{11} = \int_0^t c_x(\tau) G_x(t, \tau)\, d\tau = \frac{1}{2} t s_x(t)$$

$$I_{12} = \int_0^t s_x(\tau) G_x(t, \tau)\, d\tau = \frac{1}{2k_x^2} [s_x(t) - t c_x(t)]$$

$$I_{16} = \int_0^t d_x(\tau) G_x(t, \tau)\, d\tau = \frac{h}{k_x^2} (I_{10} - I_{11}) = \frac{h}{k_x^2} \left[\frac{d_x(t)}{h} - \frac{t}{2} s_x(t)\right]$$

$$I_{111} = \int_0^t c_x^2(\tau) G_x(t, \tau)\, d\tau = \frac{1}{3} \left[s_x^2(t) + \frac{d_x(t)}{h}\right]$$

$$I_{112} = \int_0^t c_x(\tau) s_x(\tau) G_x(t, \tau)\, d\tau = \frac{1}{3} s_x(t) \left[\frac{d_x(t)}{h}\right]$$

$$I_{116} = \int_0^t c_x(\tau)\, d_x(\tau) G_x(t, \tau)\, d\tau = \frac{h}{k_x^2}(I_{11} - I_{111}) = \frac{h}{k_x^2}\left\{\frac{t}{2} s_x(t) - \frac{1}{3}\left[s_x^2(t) + \frac{d_x(t)}{h}\right]\right\}$$

$$I_{122} = \int_0^t s_x^2(\tau) G_x(t, \tau)\, d\tau = \frac{1}{k_x^2}(I_{10} - I_{111}) = \frac{1}{3k_x^2}\left[2\frac{d_x(t)}{h} - s_x^2(t)\right]$$

$$I_{126} = \int_0^t s_x(\tau)\, d_x(\tau) G_x(t, \tau)\, d\tau = \frac{h}{k_x^2}(I_{12} - I_{112}) = \frac{h}{k_x^2}\left\{\frac{1}{2k_x^2}[s_x(t) - tc_x(t)] - \frac{1}{3} s_x(t)\left[\frac{d_x(t)}{h}\right]\right\}$$

$$= \frac{h}{6k_x^4}[s_x(t) + 2s_x(t)c_x(t) - 3tc_x(t)]$$

$$I_{166} = \int_0^t d_x^2(\tau) G_x(t, \tau)\, d\tau = \frac{h^2}{k_x^4}(I_{10} - 2I_{11} + I_{111}) = \frac{h^2}{k_x^4}\left\{\frac{4}{3}\left[\frac{d_x(t)}{h}\right] + \frac{1}{3} s_x^2(t) - ts_x(t)\right\}$$

$$I_{133} = \int_0^t c_y^2(\tau) G_x(t, \tau)\, d\tau = \left[\frac{d_x(t)}{h}\right] - \left[\frac{k_y^2}{k_x^2 - 4k_y^2}\right]\left[s_y^2(t) - 2\frac{d_x(t)}{h}\right]$$

$$I_{134} = \int_0^t c_y(\tau) s_y(\tau) G_x(t, \tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}[s_y(t)c_y(t) - s_x(t)]$$

$$I_{144} = \int_0^t s_y^2(\tau) G_x(t, \tau)\, d\tau = \frac{1}{k_x^2 - 4k_y^2}\left[s_y^2(t) - 2\frac{d_x(t)}{h}\right]$$

$$I_{20} = I'_{10} = \frac{d}{dt}\int_0^t G_x(t, \tau)\, d\tau = s_x(t)$$

$$I_{21} = I'_{11} = \frac{d}{dt}\int_0^t c_x(\tau) G_x(t, \tau)\, d\tau = \frac{1}{2}[s_x(t) + tc_x(t)]$$

$$I_{22} = I'_{12} = \frac{d}{dt}\int_0^t s_x(\tau) G_x(t, \tau)\, d\tau = \frac{1}{2} ts_x(t) = I_{11}$$

$$I_{26} = I'_{16} = \frac{d}{dt}\int_0^t d_x(\tau) G_x(t, \tau)\, d\tau = \frac{h}{2k_x^2}[s_x(t) - tc_x(t)]$$

$$I_{211} = I'_{111} = \frac{d}{dt}\int_0^t c_x^2(\tau) G_x(t, \tau)\, d\tau = \frac{s_x(t)}{3}[1 + 2c_x(t)]$$

(continued)

TABLE VIb (*continued*)

$$I_{212} = I'_{112} = \frac{d}{dt}\int_0^t c_x(\tau)s_x(\tau)G_x(t,\tau)\,d\tau = \frac{1}{3}\left[2s_x^2(t) - \frac{d_x(t)}{h}\right]$$

$$I_{216} = I'_{116} = \frac{d}{dt}\int_0^t c_x(\tau)\,d_x(\tau)G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left[\frac{tc_x(t)}{2} + \frac{s_x(t)}{6} - \frac{2s_x(t)c_x(t)}{3}\right]$$

$$I_{222} = I'_{122} = \frac{d}{dt}\int_0^t s_x^2(\tau)G_x(t,\tau)\,d\tau = \frac{2}{3}\,s_x(t)\left[\frac{d_x(t)}{h}\right]$$

$$I_{226} = I'_{126} = \frac{d}{dt}\int_0^t s_x(\tau)\,d_x(\tau)G_x(t,\tau)\,d\tau = \frac{h}{k_x^2}\left\{\frac{1}{2}\,ts_x(t) - \frac{2}{3}\,s_x^2(t) + \frac{1}{3}\left[\frac{d_x(t)}{h}\right]\right\}$$

$$I_{266} = I'_{166} = \frac{d}{dt}\int_0^t d_x^2(\tau)G_x(t,\tau)\,d\tau = \frac{h^2}{k_x^4}\left[\frac{1}{3}\,s_x(t) + \frac{2}{3}\,s_x(t)c_x(t) - tc_x(t)\right]$$

$$I_{233} = I'_{133} = \frac{d}{dt}\int_0^t c_y^2(\tau)G_x(t,\tau)\,d\tau = s_x(t) - \frac{2k_y^2}{k_x^2 - 4k_y^2}\,[s_y(t)c_y(t) - s_x(t)]$$

$$I_{234} = I'_{134} = \frac{d}{dt}\int_0^t c_y(\tau)s_y(\tau)G_x(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\,[2c_y^2(t) - 1 - c_x(t)] = \frac{1}{k_x^2 - 4k_y^2}\left\{k_x^2\left[\frac{d_x(t)}{h}\right] - 2k_y^2 s_y^2(t)\right\}$$

$$I_{244} = I'_{144} = \int_0^t s_y^2(\tau)G_x(t,\tau)\,d\tau = \frac{2}{k_x^2 - 4k_y^2}\,[s_y(t)c_y(t) - s_x(t)]$$

$$I_{30} = \int_0^t G_y(t,\tau)\,d\tau = \frac{1 - c_y(t)}{k_y^2}$$

$$I_{33} = \int_0^t c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{2}\,ts_y(t)$$

$$I_{34} = \int_0^t s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{2k_y^2}\,[s_y(t) - tc_y(t)]$$

$$I_{313} = \int_0^t c_x(\tau)c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\,\{c_y(t)[1 - c_x(t)] - 2k_y^2 s_x(t)s_y(t)\}$$

$$= \frac{1}{k_x^2 - 4k_y^2}\left\{k_x^2 c_y(t)\left[\frac{d_x(t)}{h}\right] - 2k_y^2 s_x(t)s_y(t)\right\}$$

$$I_{314} = \int_0^t c_x(\tau)s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\{2s_x(t)c_y(t) - s_y(t)[1 + c_x(t)]\}$$

$$I_{323} = \int_0^t s_x(\tau)c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\left\{2\left[\frac{k_y^2}{k_x^2}\right]s_y(t)[1 + c_x(t)] - s_x(t)c_y(t)\right\} + \frac{s_y(t)}{k_x^2}$$

$$I_{324} = \int_0^t s_x(\tau)s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\left\{\frac{2c_y(t)[1 - c_x(t)]}{k_x^2} - s_x(t)s_y(t)\right\} = \frac{1}{k_x^2 - 4k_y^2}\left\{2c_y(t)\left[\frac{d_x(t)}{h}\right] - s_x(t)s_y(t)\right\}$$

$$I_{336} = \int_0^t c_y(\tau)d_x(\tau)G_y(t,\tau)\,d\tau = \frac{h}{k_x^2}(I_{33} - I_{313}) = \frac{h}{k_x^2}\left[\frac{t}{2}s_y(t) - \frac{1}{k_x^2 - 4k_y^2}\{c_y(t)[1 - c_x(t)] - 2k_y^2 s_x(t)s_y(t)\}\right]$$

$$I_{346} = \int_0^t s_y(\tau)\,d_x(\tau)G_y(t,\tau)\,d\tau = \frac{h}{k_x^2}(I_{34} - I_{314}) = \frac{h}{k_x^2}\left[\frac{1}{2k_y^2}[s_y(t) - tc_y(t)] - \frac{1}{k_x^2 - 4k_y^2}\{2s_x(t)c_y(t) - s_y(t)[1 + c_x(t)]\}\right]$$

$$I_{40} = I'_{30} = \frac{d}{dt}\int_0^t G_y(t,\tau)\,d\tau = s_y(t)$$

$$I_{43} = I'_{33} = \frac{d}{dt}\int_0^t c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{2}[s_y(t) + tc_y(t)]$$

$$I_{44} = I'_{34} = \frac{d}{dt}\int_0^t s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{2}ts_y(t) = I_{33}$$

$$I_{413} = I'_{313} = \frac{d}{dt}\int_0^t c_x(\tau)s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\{(k_x^2 - 2k_y^2)s_x(t)c_y(t) - k_y^2 s_y(t)[1 + c_x(t)]\}$$

$$I_{414} = I'_{314} = \frac{d}{dt}\int_0^t c_x(\tau)c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\{(k_x^2 - 2k_y^2)s_x(t)s_y(t) - c_y(t)[1 - c_x(t)]\}$$

$$I_{423} = I'_{323} = \frac{d}{dt}\int_0^t s_x(\tau)c_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\left\{2\left(\frac{k_y^2}{k_x^2}\right)c_y(t)[1 + c_x(t)] - c_x(t)c_y(t) - k_y^2 s_x(t)s_y(t)\right\} + \frac{c_y(t)}{k_x^2}$$

$$I_{424} = I'_{324} = \frac{d}{dt}\int_0^t s_x(\tau)s_y(\tau)G_y(t,\tau)\,d\tau = \frac{1}{k_x^2 - 4k_y^2}\left\{c_y(t)s_x(t) - c_x(t)s_y(t) - 2k_y^2 s_y(t)\left[\frac{d_x(t)}{h}\right]\right\}$$

$$I_{436} = I'_{336} = \frac{d}{dt}\int_0^t c_y(\tau)\,d_x(\tau)G_y(t,\tau)\,d\tau = \frac{h}{k_x^2}\left[\frac{t}{2}c_y(t) + \frac{s_y(t)}{2} + \frac{1}{k_x^2 - 4k_y^2}\{k_y^2 s_y(t)[1 + c_x(t)] - (k_x^2 - 2k_y^2)s_x(t)c_y(t)\}\right]$$

$$I_{446} = I'_{346} = \frac{d}{dt}\int_0^t s_y(\tau)\,d_x(\tau)G_y(t,\tau)\,d\tau = \frac{h}{k_x^2}\left[\frac{ts_y(t)}{2} - \frac{1}{k_x^2 - 4k_y^2}\{(k_x^2 - 2k_y^2)s_x(t)s_y(t) - c_y(t)[1 - c_x(t)]\}\right]$$

The constants n and β are defined by the midplane field expansion [Eq. (18) of Sec. II]:

$$B_y(x, 0, t) = (B_y(0, 0, t)[1 - nhx + \beta h^2 x^2 + \gamma h^3 x^3 + \cdots] \quad (18)$$

or, from Eq. (19) of Section II:

$$n = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=0\\y=0}} \quad \text{and} \quad \beta = \left[\frac{1}{2!h^2 B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=0\\y=0}} \quad (19)$$

1. Matrix Elements for a Pure Quadrupole Field

For a pure quadrupole, the matrix elements are derived from those of the general case by letting $\beta = 0$, $k_x^2 = k_q^2$ and $k_y^2 = -k_q^2$, where

$$k_q^2 = -nh^2 = (B_0/a)(1/B\rho)$$

and then taking the limit $h \rightarrow 0$. The results are:

$$\begin{aligned}
R_{11} &= \cos k_q t \\
R_{12} &= (1/k_q) \sin k_q t \\
T_{116} &= \tfrac{1}{2} k_q t \sin k_q t \\
T_{126} &= (1/2k_q) \sin k_q t - (t/2) \cos k_q t \\
R_{21} &= -k_q \sin k_q t \\
R_{22} &= \cos k_q t \\
T_{216} &= (k_q/2)[k_q t \cos k_q t + \sin k_q t] \\
T_{226} &= \tfrac{1}{2} k_q t \sin k_q t \\
R_{33} &= \cosh k_q t \\
R_{34} &= (1/k_q) \sinh k_q t \\
T_{336} &= -\tfrac{1}{2} k_q t \sinh k_q t \\
T_{346} &= \tfrac{1}{2}\left[\frac{1}{k_q} \sinh k_q t - t \cosh k_q t\right] \\
R_{43} &= k_q \sinh k_q t \\
R_{44} &= \cosh k_q t \\
T_{436} &= -(k_q/2)[k_q t \cosh k_q t + \sinh k_q t] \\
T_{446} &= -\tfrac{1}{2} k_q t \sinh k_q t
\end{aligned} \quad (55)$$

all nonlisted matrix elements are identically zero.

2. Matrix Elements for a Pure Sextupole Field

For a pure sextupole, the matrix elements are derived from those of the general case by letting

$$\beta h^3 = k_s^2 = (B_0/a^2)(1/B\rho)$$

and then taking the limit $h \to 0$. The results are:

$$
\begin{aligned}
R_{11} &= 1\\
R_{12} &= t\\
T_{111} &= -\tfrac{1}{2}k_s^2 t^2\\
T_{112} &= -\tfrac{1}{3}k_s^2 t^3\\
T_{122} &= -\tfrac{1}{12}k_s^2 t^4\\
T_{133} &= \tfrac{1}{2}k_s^2 t^2\\
T_{134} &= \tfrac{1}{3}k_s^2 t^3\\
T_{144} &= \tfrac{1}{12}k_s^2 t^4\\
R_{21} &= 0\\
R_{22} &= 1\\
T_{211} &= -k_s^2 t\\
T_{212} &= -k_s^2 t^2\\
T_{222} &= -\tfrac{1}{3}k_s^2 t^3\\
T_{233} &= k_s^2 t\\
T_{234} &= k_s^2 t^2\\
T_{244} &= \tfrac{1}{3}k_s^2 t^3\\
R_{33} &= 1\\
R_{34} &= t\\
T_{313} &= k_s^2 t^2\\
T_{314} &= \tfrac{1}{3}k_s^2 t^3\\
T_{323} &= \tfrac{1}{3}k_s^2 t^3\\
T_{324} &= \tfrac{1}{6}k_s^2 t^4\\
R_{43} &= 0\\
R_{44} &= 1\\
T_{413} &= 2k_s^2 t\\
T_{414} &= k_s^2 t^2\\
T_{423} &= k_s^2 t^2\\
T_{424} &= \tfrac{2}{3}k_s^2 t^3
\end{aligned}
\tag{56}
$$

All nonlisted matrix elements are identically zero.

3. First- and Second-Order Matrix Elements for a Curved, Inclined Magnetic Field Boundary

Matrix elements for the fringing fields of bending magnets have been derived using an impulse approximation.(7,8) These computations, combined with a correction term(9) to the R_{43} elements (to correct for the finite extent of actual fringing fields), have produced results which are in substantial agreement with precise ray-tracing calculations and with experimental measurements made on actual magnets.

We introduce four new variables (illustrated in Fig. 11); the angle of inclination β_1 of the entrance face of a bending magnet, the radius of curvature R_1 of the entrance face, the angle of inclination β_2 of the exit face, and the radius of curvature R_2 of the exit face. The sign convention of β_1 and β_2 is considered positive for positive focusing in the transverse (y) direction. The sign convention for R_1 and R_2 is positive if the field boundary is convex outward; (a positive R represents a negative sextupole component of strength $k_s^2 L = -(h/2R)\sec^3\beta$). The sign conventions adopted here are in agreement with Penner,(4) and Brown, Belbeoch, and Bounin.(7)

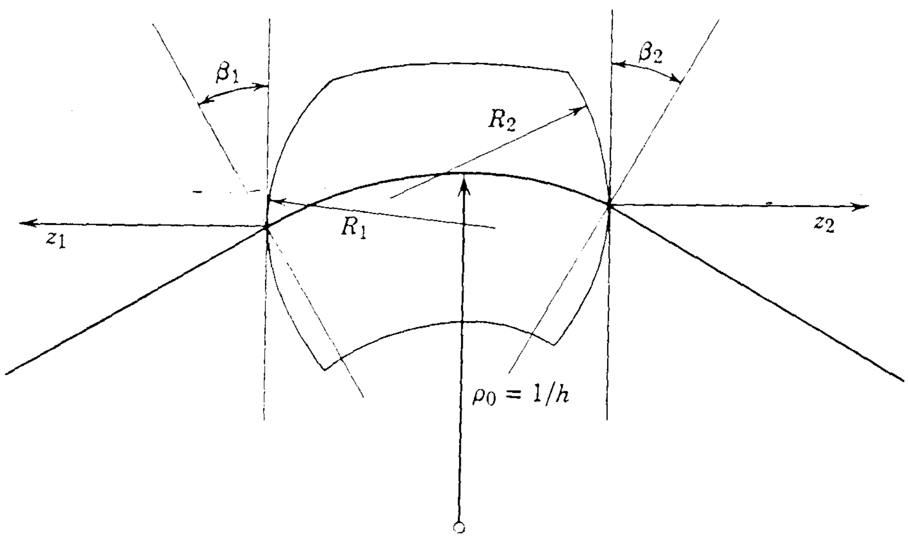

FIG. 11. Field boundaries for bending magnets. Definition of the quantities β_1, β_2, R_1, and R_2 used in the matrix elements for field boundaries of bending magnets. The quantities have a positive sign convention as illustrated in the figure.

The results of these calculations yield the following matrix elements for the fringing fields of the entrance face of a bending magnet:

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= 0 \\
T_{111} &= -(h/2)\tan^2\beta_1 \\
T_{133} &= (h/2)\sec^2\beta_1 \\
R_{21} &= -(1/f_x) = h\tan\beta_1 \\
R_{22} &= 1 \\
T_{211} &= (h/2R_1)\sec^3\beta_1 - nh^2\tan\beta_1 \\
T_{212} &= h\tan^2\beta_1 \\
T_{216} &= -h\tan\beta_1 \\
T_{233} &= h^2(n + \tfrac{1}{2} + \tan^2\beta_1)\tan\beta_1 - (h/2R_1)\sec^3\beta_1 \\
T_{234} &= -h\tan^2\beta_1 \\
R_{33} &= 1 \\
R_{34} &= 0 \\
T_{313} &= h\tan^2\beta_1 \\
R_{43} &= -(1/f_y) = -h\tan(\beta_1 - \psi_1) \\
R_{44} &= 1 \\
T_{413} &= -(h/R_1)\sec^3\beta_1 + 2h^2 n\tan\beta_1 \\
T_{414} &= -h\tan^2\beta_1 \\
T_{423} &= -h\sec^2\beta_1 \\
T_{436} &= h\tan\beta_1 - h\psi_1\sec^2(\beta_1 - \psi_1)
\end{aligned}
\qquad (57)
$$

All nonlisted matrix elements are equal to zero. The quantity ψ_1 is the correction to the transverse focal length when the finite extent of the fringing field is included.[9]

$$\psi_1 = Khg\sec\beta_1(1 + \sin^2\beta_1) + \text{higher order terms in } (hg)$$

where g = the distance between the poles of the magnet at the central orbit (i.e., the magnet gap) and

$$K = \int_{-\infty}^{+\infty} \frac{B_y(z)[B_0 - B_y(z)]}{gB_0^2}\,dz$$

$B_y(z)$ is the magnitude of the fringing field on the magnetic mid-plane at a position z. z is the perpendicular distance measured from the entrance face of the magnet to the point in question. B_0 is the asymptotic value of $B_y(z)$ well inside the magnet entrance. Typical values of K for actual magnets may range from 0.3 to 1.0 depending upon the detailed shape of the magnet profile and the location of the energizing coils.

The matrix elements for the fringing fields of the exit face of a bending magnet are:

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= 0 \\
T_{111} &= (h/2)\tan^2\beta_2 \\
T_{133} &= -(h/2)\sec^2\beta_2 \\
R_{21} &= -1/f_x = h\tan\beta_2 \\
R_{22} &= 1 \\
T_{211} &= (h/2R_2)\sec^3\beta_2 - h^2(n + \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2 \\
T_{212} &= -h\tan^2\beta_2 \\
T_{216} &= -h\tan\beta_2 \\
T_{233} &= h^2(n - \tfrac{1}{2}\tan^2\beta_2)\tan\beta_2 - (h/2R_2\sec^3\beta_2 \\
T_{234} &= h\tan^2\beta_2 \\
R_{33} &= 1 \\
R_{34} &= 0 \\
T_{313} &= -h\tan^2\beta_2 \\
R_{43} &= -1/f_y = -h\tan(\beta_2 - \psi_2) \\
R_{44} &= 1 \\
T_{413} &= -(h/R_2)\sec^3\beta_2 + h^2(2n + \sec^2\beta_2)\tan\beta_2 \\
T_{414} &= h\tan^2\beta_2 \\
T_{423} &= h\sec^2\beta_2 \\
T_{436} &= h\tan\beta_2 - h\psi_2\sec^2(\beta_2 - \psi_2)
\end{aligned}
\tag{58}
$$

All nonlisted matrix elements are zero.

$$\psi_2 = Khg\sec\beta_2(1 + \sin^2\beta_2) + \text{higher order terms in } (hg)$$

and K is evaluated for the exit fringing field.

4. *Matrix Elements for a Drift Distance*

For a drift distance of length L, the matrix elements are simply as follows:

$$R_{11} = R_{22} = R_{33} = R_{44} = R_{55} = R_{66} = 1$$

$$R_{12} = R_{34} = L$$

All remaining first- and second-order matrix elements are zero.

IV. Some Useful First-Order Optical Results Derived from the General Theory of Section II [10,11]

We have shown in Section II, Eq. (47), that beam transport optics may be reduced to a process of matrix multiplication. To first order, this is represented by the matrix equation

$$x_i(t) = \sum_{j=1}^{6} R_{ij}x_j(0) \tag{59}$$

where

$$x_1 = x,\ x_2 = \theta,\ x_3 = y,\ x_4 = \varphi,\ x_5 = l, \text{ and } x_6 = \delta$$

We have also proved that the determinant $|R| = 1$ results from the basic equation of motion and is a manifestation of Liouville's theorem of conservation of phase space volume.

The six simultaneous linear equations represented by Eq. (59) may be expanded in matrix form as follows:

$$\begin{bmatrix} x(t) \\ \theta(t) \\ y(t) \\ \varphi(t) \\ l(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \theta_0 \\ y_0 \\ \varphi_0 \\ l_0 \\ \delta_0 \end{bmatrix} \tag{60}$$

where the transformation is from an initial position $\tau = 0$ to a final position $\tau = t$.

The zero elements $R_{13} = R_{14} = R_{23} = R_{24} = R_{31} = R_{32} = R_{41} = R_{42} = R_{36} = R_{46} = 0$ in the R matrix are a direct consequence of midplane symmetry. If midplane symmetry is destroyed, these elements will in general become nonzero. The zero elements in column five occur because the variables x, θ, y, φ, and δ are independent of the path length difference l. The zeros in row six result from the fact that we have restricted the problem to static magnetic fields, i.e., the scalar momentum is a constant of the motion.

We have already attached a physical significance to the nonzero matrix elements in the first four rows in terms of their identification with characteristic first-order trajectories. We now wish to relate the elements appearing in column six with those in row five and calculate both sets in terms of simple integrals of the characteristic first-order elements $c_x(t) = R_{11}$ and $s_x(t) = R_{12}$. In order to do this, we make use of the Green's integral, Eq. (43) of Section II, and of the expression for the differential path length in curvilinear coordinates

$$dT = [(dx)^2 + (dy)^2 + (1 + hx)^2(dt)^2]^{1/2} \quad (61)$$

used in the derivation of the equation of motion.

1. First-Order Dispersion

The spacial dispersion $d_x(t)$ of a system at position t is derived using the Green's function integral, Eq. (43), and the driving term $f = h(\tau)$ for the dispersion (see Table I). The result is

$$d_x(t) = R_{16} = s_x(t) \int_0^t c_x(\tau)h(\tau)\,d\tau - c_x(t) \int_0^t s_x(\tau)h(\tau)\,d\tau \quad (62)$$

where τ is the variable of integration. Note that $h(\tau)\,d\tau = d\alpha$ is the differential angle of bend of the central trajectory at any point in the system. Thus first-order dispersion is generated only in regions where the central trajectory is deflected (i.e., in dipole elements.) The angular dispersion is obtained by direct differentiation of $d_x(t)$ with respect to t;

$$d'_x(t) = R_{26} = s'_x(t) \int_0^t c_x(\tau)h(\tau)\,d\tau - c'_x(t) \int_0^t s_x(\tau)h(\tau)\,d\tau \quad (63)$$

where

$$c'_x(t) = R_{21} \quad \text{and} \quad s'_x(t) = R_{22}.$$

2. First-Order Path Length

The first-order path length difference is obtained by expanding Eq. (61) and retaining only the first-order term, i.e.,

$$l - l_0 = (T - t) = \int_0^t x(\tau)h(\tau)\,d\tau + \text{higher order terms}$$

from which

$$\begin{aligned} l &= x_0 \int_0^t c_x(\tau)h(\tau)\,d\tau + \theta_0 \int_0^t s_x(\tau)h(\tau)\,d\tau + l_0 + \delta \int_0^t d_x(\tau)h(\tau)\,d\tau \\ &= R_{51}x_0 + R_{52}\theta_0 + l_0 + R_{56}\delta \end{aligned} \quad (64)$$

Inspection of Eqs. (62)–(64) yields the following useful theorems:

A. Achromaticity

A system is defined as being achromatic if $d_x(t) = d'_x(t) = 0$. Therefore it follows from Eqs. (62) and (63) that the necessary and sufficient conditions for achromaticity are that

$$\int_0^t s_x(\tau)h(\tau)\,d\tau = \int_0^t c_x(\tau)h(\tau)\,d\tau = 0 \tag{65}$$

By comparing Eq. (64) with Eq. (65), we note that if a system is achromatic, all particles of the same momentum will have equal (first-order) path lengths through the system.

B. Isochronicity

It is somewhat unfortunate that this word has been used in the literature, since it is applicable only to highly relativistic particles. Nevertheless, from Eq. (64) the necessary and sufficient conditions that the first-order path length of all particles (independent of their initial momentum) will be the same through a system are that

$$\int_0^t c_x(\tau)h(\tau)\,d\tau = \int_0^t s_x(\tau)h(\tau)\,d\tau = \int_0^t d_x(\tau)h(\tau)\,d\tau = 0 \tag{66}$$

3. First-Order Imaging

First-order point-to-point imaging in the x plane occurs when $x(t)$ is independent of the initial angle θ_0. This can only be so when

$$s_x(t) = R_{12} = 0 \tag{67}$$

Similarly, first-order point-to-point imaging occurs in the y plane when

$$s_y(t) = R_{34} = 0 \tag{68}$$

First-order parallel-to-point imaging occurs in the x plane when $x(t)$ is independent of the initial particle position x_0. This will occur only if

$$c_x(t) = R_{11} = 0 \tag{69}$$

and correspondingly in the y plane, parallel-to-point imaging occurs when

$$c_y(t) = R_{33} = 0 \tag{70}$$

4. Magnification

For point-to-point imaging in the x plane, the magnification is given by

$$M_x = \left|\frac{x(t)}{x_0}\right| = |R_{11}| = |c_x(t)|$$

and in the y plane by

$$M_y = |R_{33}| = |c_y(t)| \tag{71}$$

5. First-Order Momentum Resolution

For point-to-point imaging the first-order momentum resolving power R_1 (not to be confused with the matrix R) is the ratio of the momentum dispersion to the image size: Thus

$$R_1 = \left|\frac{R_{16}}{R_{11}x_0}\right| = \left|\frac{d_x(t)}{c_x(t)x_0}\right|$$

For point-to-point imaging [$s_x(t) = 0$] using Eq. (62), the dispersion at an image is

$$d_x(t) = -c_x(t) \int_0^t s_x(\tau)h(\tau)\,d\tau \tag{72}$$

from which the first-order momentum resolving power R_1 becomes

$$R_1 x_0 = \left|\frac{d_x(t)}{c_x(t)}\right| = \left|\int_0^t s_x(\tau)h(\tau)\,d\tau\right| \tag{73}$$

where x_0 is the source size.

6. Zero Dispersion

For point-to-point imaging, using Eq. (72), the necessary and sufficient condition for zero dispersion at an image is

$$\int_0^t s_x(\tau)h(\tau)\,d\tau = 0 \tag{74}$$

For parallel-to-point imaging [i.e., $c_x(t) = 0$], the condition for zero dispersion at the image is

$$\int_0^t c_x(\tau)h(\tau)\,d\tau = 0 \tag{75}$$

7. Focal Length

It can be readily demonstrated from simple lens theory[4] that the physical interpretations of R_{21} and R_{43} are:

$$c'_x(t) = R_{21} = -1/f_x \quad \text{and} \quad c'_y(t) = R_{43} = -1/f_y \tag{76}$$

where f_x and f_y are the system focal lengths in the x and y planes, respectively, between $\tau = 0$ and $\tau = t$.

8. Evaluation of the First-Order Matrix for Ideal Magnets

From the results of Section III, we conclude that for an ideal magnet the matrix elements of R are simple trigonometric or hyperbolic functions. The general result for an element of length L is

$$R = \begin{bmatrix} \cos k_xL & \frac{1}{k_x}\sin k_xL & 0 & 0 & 0 & \frac{h}{k_x^2}[1 - \cos k_xL] \\ -k_x \sin k_xL & \cos k_xL & 0 & 0 & 0 & \left(\frac{h}{k_x}\right)\sin k_xL \\ 0 & 0 & \cos k_yL & \frac{1}{k_y}\sin k_yL & 0 & 0 \\ 0 & 0 & -k_y \sin k_yL & \cos k_yL & 0 & 0 \\ \frac{h}{k_x}\sin k_xL & \frac{h}{k_x^2}[1 - \cos k_xL] & 0 & 0 & 1 & \frac{h^2}{k_x^3}[k_xL - \sin k_xL] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{77}$$

where for a dipole (bending) magnet, we have defined

$$k_x^2 = (1 - n)h^2 \quad \text{and} \quad k_y^2 = nh^2$$

For a pure quadrupole, the R matrix is evaluated by letting

$$k_x^2 = k_q^2 \quad \text{and} \quad k_y^2 = -k_q^2$$

and taking the limiting case $h \to 0$, where

$$k_q^2 = -nh^2 = (B_0/a)(1/B\rho)$$

Taking these limits, the R matrix for a quadrupole is:

$$R = \begin{bmatrix} \cos k_q L & \frac{1}{k_q}\sin k_q L & 0 & 0 & 0 & 0 \\ -k_q \sin k_q L & \cos k_q L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh k_q L & \frac{1}{k_q}\sinh k_q L & 0 & 0 \\ 0 & 0 & k_q \sinh k_q L & \cosh k_q L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{78}$$

Note that the trigonometric and hyperbolic functions will interchange if the sign of B_0 is reversed.

9. The R Matrix Transformed to the Principal Planes

The positions Z of the principal planes of a magnetic element (measured from its ends) may be derived from the following matrix equation:

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ R_{21} & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{43} & 1 & 0 & 0 \\ X & X & X & X & X & X \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -Z_{2x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{2y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} R \begin{bmatrix} 1 & -Z_{1x} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -Z_{1y} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{79}$$

Solving this equation, we have

$$Z_{1x} = (R_{22}-1)/R_{21} \qquad Z_{2x} = (R_{11}-1)/R_{21}$$
$$Z_{1y} = (R_{44}-1)/R_{43} \qquad Z_{2y} = (R_{33}-1)/R_{43} \tag{80}$$

For the ideal magnet, the general result for the transformation matrix R_{pp} between the principal planes is

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k_x \sin k_x L & 1 & 0 & 0 & 0 & (h/k_x) \sin k_x L \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -k_y \sin k_y L & 1 & 0 & 0 \\ (h/k_x) \sin k_x L & 0 & 0 & 0 & 1 & (h^2/k_x^3)[k_x L - \sin k_x L] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{81}$$

and because of symmetry

$$Z_x = Z_{1x} = Z_{2x} = (1/k_x) \tan (k_x L/2)$$

and

$$Z_y = Z_{1y} = Z_{2y} = (1/k_y) \tan (k_y L/2) \tag{82}$$

Correspondingly, for the ideal quadrupole, R_{pp} is derived by letting

$$k_x^2 = k_q^2 \quad \text{and} \quad k_y^2 = -k_q^2$$

and taking the limit $h \to 0$ for each of the matrix elements. The result is:

$$R_{pp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k_q \sin k_q L & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_q \sinh k_q L & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{83}$$

where now

$$Z_x = (1/k_q) \tan (k_q L/2)$$
$$Z_y = (1/k_q) \tanh (k_q L/2) \tag{84}$$

V. Some General Second-Order Theorems Derived from the General Theory of Section II

We have established in Section II that any second-order aberration coefficient q may be evaluated via the Green's function integral, Eq. (43), i.e.,

$$q = s(t) \int_0^t f(\tau) c(\tau)\, d\tau - c(t) \int_0^t f(\tau) s(\tau)\, d\tau$$

A second-order aberration may therefore be determined as soon as a first-order solution for the system has been established, since the polynomial expressions for the driving terms $f(\tau)$ have all been expressed as functions of the characteristic first-order matrix elements (Table I). Usually one is interested in knowing the value of the aberration at an image point of which there are two cases of interest, point-to-point imaging $s(t) = 0$ and parallel-to-point imaging $c(t) = 0$.

Thus for point-to-point imaging,

$$q = -c(t)\int_0^t f(\tau)s(\tau)\, d(\tau)$$

where $\tau = t$ is the location of an image and $|c(t)| = M$ is the first-order spatial magnification at the image, and for parallel-to-point imaging,

$$q = s(t)\int_0^t f(\tau)c(\tau)\, d(\tau)$$

where $\tau = t$ is the position of the image and $s(t)$ is the angular dispersion at the image.

If a system possesses first-order optical symmetries, then it can be immediately determined if a given second-order aberration is identically zero as a consequence of the first-order symmetry. We observe that for point-to-point imaging a second-order aberration coefficient q will be identically zero if the product of the corresponding driving term $f(\tau)$ and the first-order matrix element $s(\tau)$ form an odd function about the midpoint of the system.

As an example of this, consider the transformation between principal planes for the two symmetric achromatic systems illustrated in Figures 12 and 13. We assume in both cases that the elements of the system have been chosen such as to transform an initial parallel beam of particles into a final parallel beam, i.e., $R_{21} = -1/f_x = 0$ for midplane trajectories. We further assume parallel-to-point imaging at the midpoint of the system. With these assumptions, the first-order matrix transformation for midplane trajectories between principal planes is:

$$\begin{bmatrix} x(t) \\ x'(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x'(0) \\ \delta(0) \end{bmatrix}$$

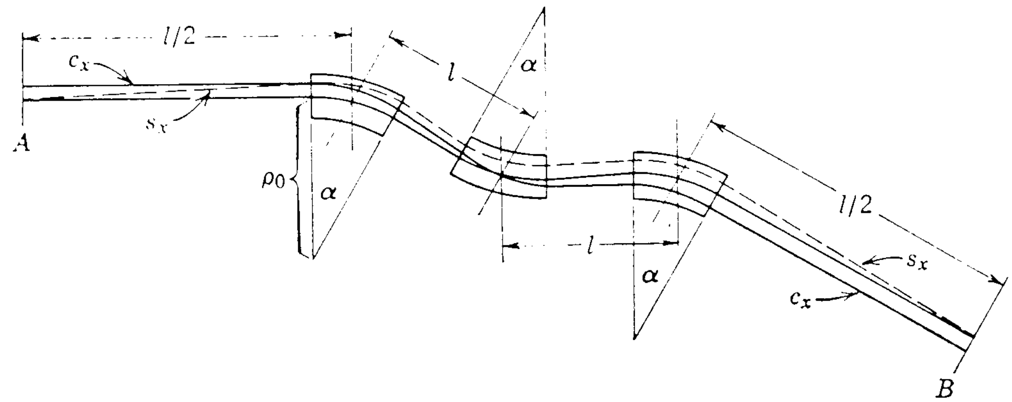

FIG. 12. Three bending magnet achromatic system. A and B are locations of principal planes.

Thus $c_x(t) = -1$, $s_x(t) = 0$, $c'_x(t) = 0$, $s'_x(t) = -1$, and of course $d_x(t) = d'_x(t) = 0$. About the midpoint of the system, the following symmetries exist for the characteristic first-order matrix elements and for the curvature $h(\tau) = 1/\rho_0$ of the central trajectory; we classify them as being either odd or even functions about the midpoint of the system. The results are:

$$c_x(\tau) = \text{odd} \qquad s_x(\tau) = \text{even} \qquad d_x(\tau) = \text{even} \qquad h(\tau) = \text{even}$$

$$c'_x(\tau) = \text{even} \qquad s'_x(\tau) = \text{odd} \qquad d'_x(\tau) = \text{odd} \qquad h'(\tau) = \text{odd}$$

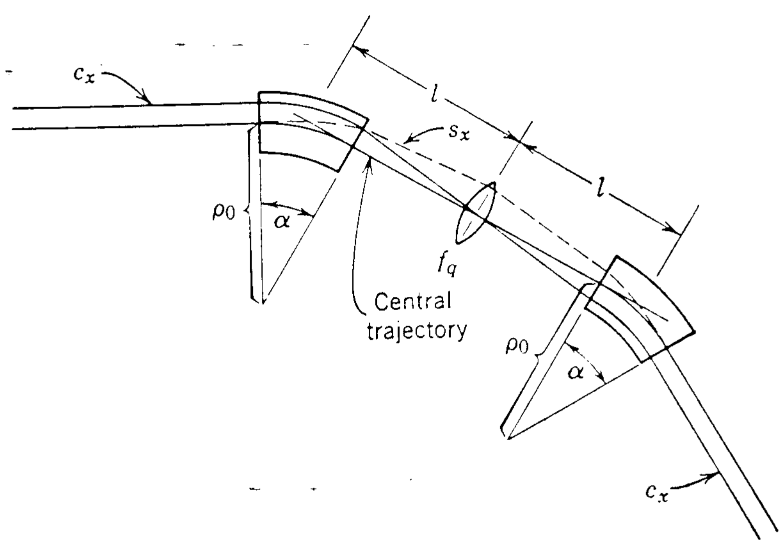

FIG. 13. Achromatic system with quadrupole at center to achieve achromatic imaging. The principal planes are located at centers of the bending magnets.

As a consequence of these symmetries, the following second-order coefficients are uniquely zero for the transformation between principal planes.

$$(x \mid x_0x_0') = (x \mid x_0\delta) = (x' \mid x_0^2) = (x' \mid x_0'^2) = (x' \mid x_0'\delta) = (x' \mid \delta^2) = 0$$

This result is valid, independent of the details of the fringing fields of the magnets, provided symmetry exists about the midpoint.

1. Optical Symmetries in $n = \frac{1}{2}$ Magnetic Systems

In magnetic optical systems composed of $n = \frac{1}{2}$ magnets having normal entry and exit of the central trajectory (i.e., nonrotated entrance and exit faces), several general mathematical relationships result from the $n = \frac{1}{2}$ symmetry. Since $k_x^2 = (1 - n)h^2$ and $k_y^2 = nh^2$, for $n = \frac{1}{2}$ it follows that $c_x(\tau) = c_y(\tau)$ and $s_x(\tau) = s_y(\tau)$ at any position τ along the system; thus as is well known, an $n = \frac{1}{2}$ system possesses first-order double focusing properties.

In addition to the above first-order results, at any point t in an $n = \frac{1}{2}$ system, the sums of the following second-order aberration coefficients are constants independent of the distribution or magnitude (βh^3) of the sextupole components throughout the system:

$$(x \mid x_0'^2) + (x \mid y_0'^2) = \text{a constant independent of } \beta h^3$$

$$2(x \mid x_0^2) + (y \mid x_0y_0) = \text{a constant independent of } \beta h^3$$

$$(x \mid x_0x_0') + (y \mid x_0y_0') = \text{a constant independent of } \beta h^3$$

$$(x \mid x_0\delta) + (y \mid y_0\delta) = \text{a constant independent of } \beta h^3$$

$$2(x \mid x_0'^2) + (y \mid x_0'y_0') = \text{a constant independent of } \beta h^3$$

$$(x \mid x_0'\delta) + (y \mid y_0'\delta) = \text{a constant independent of } \beta h^3$$

$$(x \mid x_0^2) + (x \mid y_0^2) = \text{a constant independent of } \beta h^3$$

$$(x \mid x_0x_0') + (x \mid y_0y_0') = \text{a constant independent of } \beta h^3 \quad (85)$$

Similarly,

$$(x' \mid x_0'^2) + (x' \mid y_0'^2) = \text{a constant independent of } \beta h^3$$

$$2(x' \mid x_0^2) + (y' \mid x_0y_0) = \text{a constant independent of } \beta h^3$$

$$(x' \mid x_0x_0') + (y' \mid x_0y_0') = \text{a constant independent of } \beta h^3$$

$$(x' \mid x_0\delta) + (y' \mid y_0\delta) = \text{a constant independent of } \beta h^3$$

$$2(x' \mid x_0'^2) + (y' \mid x_0' y_0') = \text{a constant independent of } \beta h^3$$

$$(x' \mid x_0'\delta) + (y' \mid y_0'\delta) = \text{a constant independent of } \beta h^3$$

$$(x' \mid x_0^2) + (x' \mid y_0^2) = \text{a constant independent of } \beta h^3$$

$$(x' \mid x_0 x_0') + (x' \mid y_0 y_0') = \text{a constant independent of } \beta h^3 \quad (86)$$

Of the above relations, the first is perhaps the most interesting in that it shows the impossibility of simultaneously eliminating both the $(x \mid x_0'^2)$ and $(x \mid y_0'^2)$ aberrations in an $n = \frac{1}{2}$ system; i.e., either $(x \mid x_0'^2)$ or $(x \mid y_0'^2)$ may be eliminated by the appropriate choice of sextupole elements, but not both.

VI. An Approximate Evaluation of the Second-Order Aberrations for High-Energy Physics

Quite often it is desirable to estimate the magnitude of various second-order aberrations in a proposed system to obtain insight into what constitutes an optimum solution to a given problem. A considerable simplification occurs in the formalism in the high-energy limit where ρ_0 is very much greater than the transverse amplitudes of the first-order trajectories and where the dipole, quadrupole, and sextupole functions are physically separated into individual elements. It is also assumed that fringing-field effects are small compared to the contributions of the various multipole elements.

Under these circumstances, the second-order chromatic aberrations are generated predominately in the quadrupole elements; the geometric aberrations are generated in the dipole elements (bending magnets); and, depending upon their location in the system, the sextupole elements couple with either the chromatic or geometric aberrations or both.

We have tabulated in Tables VII–IX the approximate formulas for the high-energy limit for three cases of interest: point-to-point imaging in the x (bend) plane, Table VII; point-to-point imaging in the y (nonbend) plane, Table VIII; and parallel-to-point imaging in the y plane, Table IX.

TABLE VII

Applying the Greens' Function Solution, Eq. (22), in the High-Energy Limit as Defined Above for Point-to-Point Imaging in the x (bend) Plane, the Second-Order Matrix Elements Reduce to the Values Shown

$$(x \mid x_0^2) \cong -\tfrac{1}{2}c_x(i)\int_0^t c_x'^2 s_x\,d\alpha + c_x(i)\sum_j S_j c_x^2 s_x$$

$$(x \mid x_0 x_0') \cong -c_x(i)\int_0^t c_x' s_x' s_x\,d\alpha + 2c_x(i)\sum_j S_j c_x s_x^2$$

$$(x \mid x_0\delta) \cong -c_x(i)\int_0^t c_x' d_x' s_x\,d\alpha + 2c_x(i)\sum_j S_j c_x s_x d_x - c_x(i)\sum_q \frac{c_x s_x}{f_q}$$

$$(x \mid x_0'^2) \cong -\tfrac{1}{2}c_x(i)\int_0^t s_x'^2 s_x\,d\alpha + c_x(i)\sum_j S_j s_x^3$$

$$(x \mid x_0'\delta) \cong -c_x(i)\int_0^t s_x' d_x' s_x\,d\alpha + 2c_x(i)\sum_j S_j s_x^2 d_x - c_x(i)\sum_q \frac{s_x^2}{f_q}$$

$$(x \mid \delta^2) \cong -\frac{c_x(i)}{2}\int_0^t (d_x')^2 s_x\,d\alpha + c_x(i)\sum_j S_j s_x d_x^2 - c_x(i)\sum_q \frac{s_x d_x}{f_q}$$

$$(x \mid y_0^2) \cong \tfrac{1}{2}c_x(i)\int_0^t c_y'^2 s_x\,d\alpha - c_x(i)\sum_j S_j c_y^2 s_x$$

$$(x \mid y_0 y_0') \cong c_x(i)\int_0^t c_y' s_y' s_x\,d\alpha - 2c_x(i)\sum_j S_j c_y s_y s_x$$

$$(x \mid y_0'^2) \cong \tfrac{1}{2}c_x(i)\int_0^t s_y'^2 s_x\,d\alpha - c_x(i)\sum_j S_j s_y^2 s_x$$

TABLE VIII

For Point-to-Point Imaging in the y (nonbend) Plane, Eq. (23), the High-Energy Limit Yields the Values Given

$$(y \mid x_0 y_0) \cong -c_y(i)\int_0^t c_x' c_y' s_y\,d\alpha - 2c_y(i)\sum_j S_j c_x c_y s_y$$

$$(y \mid x_0 y_0') \cong -c_y(i)\int_0^t c_x' s_y' s_y\,d\alpha - 2c_y(i)\sum_j S_j c_x s_y^2$$

$$(y \mid x_0' y_0) \cong -c_y(i)\int_0^t s_x' c_y' s_y\,d\alpha - 2c_y(i)\sum_j S_j s_x c_y s_y$$

$$(y \mid x_0' y_0') \cong -c_y(i)\int_0^t s_x' s_y' s_y\,d\alpha - 2c_y(i)\sum_j S_j s_x s_y^2$$

$$(y \mid y_0\delta) \cong -c_y(i)\int_0^t c_y' d_x' s_y\,d\alpha - 2c_y(i)\sum_j S_j c_y d_x s_y + c_y(i)\sum_q \frac{c_y s_y}{f_q}$$

$$(y \mid y_0'\delta) \cong -c_y(i)\int_0^t s_y' d_x' s_y\,d\alpha - 2c_y(i)\sum_j S_j d_x s_y^2 + c_y(i)\sum_q \frac{s_y^2}{f_q}$$

TABLE IX

For Parallel-(Line)-to-Point Imaging in the y (Nonbend) Plane, Eq. (24), the High-Energy Limit Yields the Values Shown

$$(y \mid x_0 y_0) \cong s_y(i) \int_0^t c_x' c_y' c_y \, d\alpha + 2s_y(i) \sum_j S_j c_x c_y^2$$

$$(y \mid x_0 y_0') \cong s_y(i) \int_0^t c_x' s_y' c_y \, d\alpha + 2s_y(i) \sum_j S_j c_x s_y c_y$$

$$(y \mid x_0' y_0) \cong s_y(i) \int_0^t s_x' c_y' c_y \, d\alpha + 2s_y(i) \sum_j S_j s_x c_y^2$$

$$(y \mid x_0' y_0') \cong s_y(i) \int_0^t s_x' s_y' c_y \, d\alpha + 2s_y(i) \sum_j S_j s_x s_y c_y$$

$$(y \mid y_0 \delta) \cong +s_y(i) \int c_y' d_x' c_y \, d\alpha + 2s_y(i) \sum_j S_j c_y^2 \, d_x - s_y(i) \sum_q \frac{c_y}{f_q}$$

$$(y \mid y_0' \delta) \cong +s_y(i) \int s_y' d_x' c_y \, d\alpha + 2s_y(i) \sum_j S_j s_y c_y \, d_x - s_y(i) \sum_q \frac{s_y c_y}{f_q}$$

For the purpose of clearly illustrating the physical principles involved, we assume that the amplitudes of the characteristic first-order matrix elements c_x, s_x, d_x, c_y, and s_y are constant within any given quadrupole or sextupole element, and we define the strengths of the quadrupole and sextupole elements as follows:

$$\int_0^L k_q^2 \, d\tau = k_q^2 L_q \cong \frac{1}{f_q}$$

where L_q is the effective length of the quadrupole, and where $1/f_q = k_q \sin k_q L$ is the reciprocal of the focal length of the qth quadrupole; and for the jth sextupole of length L_s, we define its strength as

$$\int_0^L k_s^2 \, d\tau = k_s^2 L_s = S_j$$

The results are given in the tables in terms of integrals over the bending magnets and summations over the quadrupole and sextupole elements. Note that under these circumstances the quadrupole and sextupole contributions to the aberration coefficients are proportional to the amplitudes of the characteristic first-order trajectories within these elements, whereas the dipole contributions are proportional to the derivatives of the first-order trajectories within the dipole elements.

As an example of the above concepts, we shall calculate the angle ψ between the momentum focal plane and the central trajectory for some representative cases.

For point-to-point imaging, it may be readily verified that

$$\tan\psi = -\left(\frac{d_x(i)}{c_x(i)}\right)\frac{1}{(x_i \mid x_o'\delta)} = \frac{\int_0^i s_x\, d\alpha}{(x_i \mid x_o'\delta)} = \frac{R_1 x_o}{(x_i \mid x_o'\delta)} \tag{87}$$

where the subscript o refers to the object plane and the subscript i to the image plane.

Let us now consider some representative quadrupole configurations and assume that the bending magnets are placed in a region having a large amplitude of the unit sinelike function s_x (so as to optimize the first-order momentum resolving power R_1).

1. Case I

Consider the simple quadrupole configuration shown in Figure 14 with the bending magnets located in the region between the quadrupoles and $s_x' \cong 0$ in this region. For these conditions, $f_1 = l_1$, $s_x = l_1$ at the quadrupoles, and $f_2 = l_3$. From Table VII, we have:

$$(x_i \mid x_o'\delta) \cong -c_x(i)\sum_q \frac{s_x^2}{f_q} = -c_x(i)l_1\left(1 + \frac{l_1}{l_3}\right) = l_1(1 + M_x)$$

where we make use of the fact that $(l_3/l_1) = M_x = -c_x(i)$. M_x is the first-order magnification of the system.

Hence,

$$\tan\psi = \frac{\int_0^i s_x\, d\alpha}{(x_i \mid x_o'\delta)} \cong \frac{\alpha}{(1 + M_x)} \tag{88}$$

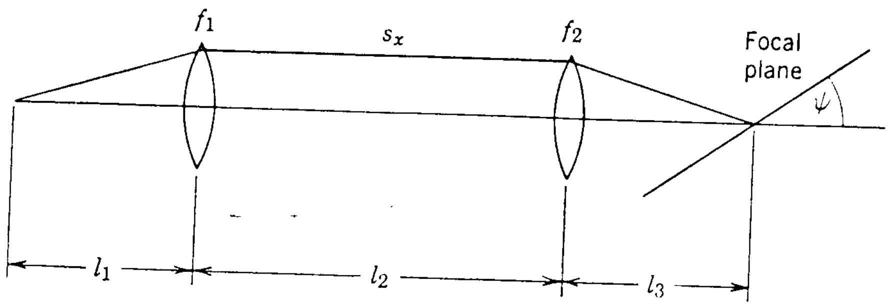

FIGURE 14

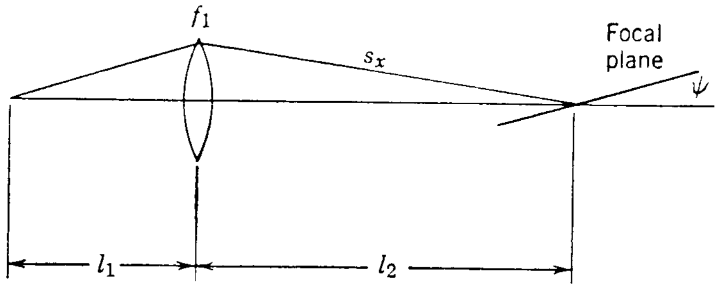

FIGURE 15

2. *Case II*

For a single quadrupole, Figure 15, the result is similar

$$\tan \psi = K\alpha/(1 + M_x) \tag{89}$$

except for the factor $K < 1$ resulting from the fact that s_x cannot have the same amplitude in the bending magnets as it does in the quadrupole. Therefore

$$\int_0^i s_x \, d\alpha = Kl_1\alpha$$

3. *Case III*

Now let us consider a symmetric four-quadrupole array, Figure 16, such that we have an intermediate image. Then

$$(x_i \mid x_0'\delta) = -2c_x(i)l_1[1 + (l_1/l_3)] = \text{twice that for Case I}$$

because of symmetry, $c_x(i) = M_x = 1$. Thus, we conclude

$$\tan \psi = -(\alpha/2)[1 + (l_1/l_3)] \tag{90}$$

In other words, the intermediate image has introduced a factor of two in the denominator and has changed the sign of ψ.

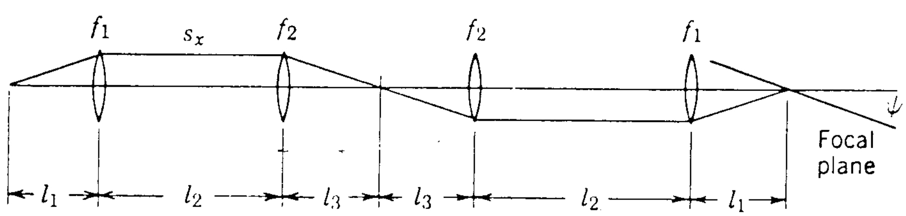

FIGURE 16

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