Reference Parkes Radio School *Fundamentals of Radio Astronomy II*

https://www.atnf.csiro.au/research/radio-school/2011/talks/Parkes-school-Fundamental-II.pdf

Parkes School - Fundamentals of Radio Astronomy II

Any antenna can be thought of as a transmitter as well as a receiver. The *directivity gain* (or *directivity function*) gives the angular distribution of power radiated and is defined as

$$D[\nu, \hat{\mathbf{n}}] \equiv 4\pi \frac{p_\nu[\hat{\mathbf{n}}]}{\int d^2\hat{\mathbf{n}}\, p_\nu[\hat{\mathbf{n}}]}$$

where $p_\nu[\hat{\mathbf{n}}]$ is the power radiated per solid angle. The *directivity* is the maximum of this function

$$D[\nu] \equiv \max_{\hat{\mathbf{n}}} D[\nu, \hat{\mathbf{n}}].$$

The *beam solid angle* of an antenna (in steradians) is

$$\Omega_{\text{eff}}[\nu] \equiv \frac{4\pi}{D[\nu]}.$$

Note that by construction the angle averaged directivity gain is unity:

$$\int \frac{d\hat{\mathbf{n}}}{4\pi} D[\nu, \hat{\mathbf{n}}] = 1$$

The previous formulae had to do with the radiated power which is related to the power density supplied to the transmitter by an efficiency factor $\eta_R[\nu] \in [0, 1]$. $\eta_R[\nu]$ describes losses in the system. some of which may be intentional *e.g.* as a filter. The *power gain function* is given by

$$G[\nu, \hat{\mathbf{n}}] \equiv \eta_R[\nu]\, D[\nu, \hat{\mathbf{n}}].$$

Thinking of the antenna as a receiver, the power frequency density (per unit frequency) received from an unpolarized sky is

$$P_\nu = \frac{1}{2} \eta_R[\nu] \int d^2\hat{\mathbf{n}}\, A_{\text{eff}}[\hat{\mathbf{n}}]\, I_\nu[\hat{\mathbf{n}}]$$

where A_{eff} is the *effective area* (*reception pattern*) of the antenna. Using the *Rayleigh-Jeans brightness temperature* pattern

$$T_{RJ}[\nu, \hat{\mathbf{n}}] \equiv \frac{\lambda^2}{2 k_B} I_\nu[\hat{\mathbf{n}}]$$

where k_B is Boltzmann's constant and $\lambda \equiv c/\nu$ is the wavelength. Thus

$$P_\nu = \eta_R[\nu] \frac{k_B}{\lambda^2} \int d^2\hat{\mathbf{n}}\, A_{\text{eff}}[\nu, \hat{\mathbf{n}}]\, T_{RJ}[\nu, \hat{\mathbf{n}}].$$

The effective area is related to the directivity gain by

$$A_{\text{eff}}[\nu, \hat{\mathbf{n}}] = \frac{\lambda^2}{4\pi} D[\nu, \hat{\mathbf{n}}].$$

Thus one finds

$$P_\nu = \eta_R[\nu] \int \frac{d^2\hat{\mathbf{n}}}{4\pi} D[\nu, \hat{\mathbf{n}}]\, k_B\, T_{RJ}[\nu, \hat{\mathbf{n}}].$$

Auto-correlation Visibility

An auto-correlation *visibility* is a number $V[\nu] = c[\nu]\, P_\nu$ where the proportionality constant, $c[\nu]$, defines a calibration.

Temperature Calibration

An auto-correlation *visibility* is a number $V[\nu] = k[\nu]\, P_\nu$ where the proportionality constant, $k[\nu]$, defines a calibration. A *calibrated visibility in temperature units* , which we denote by V_T, is defined by $k[\nu] = k_T[\nu] \equiv \frac{1}{k_B\, \eta_R[\nu]}$ so that if illuminated by a uniform brightness pattern $T_{RJ}[\nu, \hat{\mathbf{n}}] = T$ will take the value $V_T = T$. With this calibration

$$V_T[\nu] = \int \frac{d^2\hat{\mathbf{n}}}{4\pi} D[\nu, \hat{\mathbf{n}}]\, T_{RJ}[\nu, \hat{\mathbf{n}}] = \frac{1}{2 k_B} \left(\frac{c}{\nu}\right)^2 \int \frac{d^2\hat{\mathbf{n}}}{4\pi} D[\nu, \hat{\mathbf{n}}]\, I_\nu[\hat{\mathbf{n}}]$$

for an arbitrary unpolarized illumination pattern. In some cases it is impractical to empirically calibrate an antenna by illuminating with a uniform brightness pattern.

Unpolarized Point Source

A unpolarized point source at position $\hat{\mathbf{n}}_\star$ has brightness pattern

$$I_\nu[\hat{\mathbf{n}}] = f_\nu\, \delta^{(2)}[\hat{\mathbf{n}}, \hat{\mathbf{n}}_\star]$$

where f_ν is it's flux density. It's contribution to the visibility is

$$V_T^\star[\nu] = \frac{1}{8\pi} \frac{f_\nu}{k_B} \left(\frac{c}{\nu}\right)^2 D[\nu, \hat{\mathbf{n}}_\star]$$

Antenna with a Beam Center

If the directive gain is maximized at all frequencies in the same direction, $\hat{\mathbf{n}}_{bc}$, which we call the *beam center*. Even without a detailed knowledge of the beam pattern one can often determine the beam center from the symmetry of the antenna. If there is a beam center then $D[\nu] \equiv D[\nu, \hat{\mathbf{n}}_{bc}]$. For an antenna with a beam center, $\hat{\mathbf{n}}_{bc}$, which is the direction in which the antenna is *pointed*, especially if the antenna is steerable.

If one points an antenna toward a point source, $\hat{\mathbf{n}}_\star = \hat{\mathbf{n}}_{bc}$, then

$$V_T^\star[\nu] = \frac{1}{8\pi} \frac{f_\nu}{k_B} \left(\frac{c}{\nu}\right)^2 D[\nu] = \frac{1}{2} \frac{f_\nu}{k_B} \left(\frac{c}{\nu}\right)^2 \frac{1}{\Omega_{\text{eff}}[\nu]}.$$

Flux Calibration

One often calibrates a receiver with a beam center by pointing it at a source with a known flux density $f_\nu^\star$ that is bright enough to dominate all other illumination and then measuring the value of P_ν which we denote by $P_\nu^\star$. This is given by

$$P_\nu^\star = \frac{1}{8\pi} \eta_R[\nu] \left(\frac{c}{\nu}\right)^2 D[\nu]\, f_\nu^\star.$$

In this situation one knows $f_\nu^\star$ and measures $P_\nu^\star$. If for other observations one defines the *calibrated visibility in flux units* by

$$V_F[\nu] = \frac{P_\nu}{P_\nu^\star} f_\nu^\star$$

one is using

$$k[\nu] = k_F[\nu] \equiv \frac{1}{\eta_R[\nu]} \frac{8\pi}{D[\nu]} \left(\frac{\nu}{c}\right)^2 = \frac{2}{\eta_R[\nu]} \frac{1}{\Omega_{\text{eff}}[\nu]} \left(\frac{\nu}{c}\right)^2$$

and

$$V_F[\nu] = 2 \left(\frac{\nu}{c}\right)^2 \int d^2\hat{\mathbf{n}} \frac{D[\nu, \hat{\mathbf{n}}]}{D[\nu]} k_B\, T_{RJ}[\nu, \hat{\mathbf{n}}] = \int d^2\hat{\mathbf{n}} \frac{D[\nu, \hat{\mathbf{n}}]}{D[\nu]} I_\nu[\hat{\mathbf{n}}]$$

so the contribution of a point source is

$$V_T^\star[\nu] = \frac{D[\nu, \hat{\mathbf{n}}_\star]}{D[\nu, \hat{\mathbf{n}}_{bc}]} f_\nu.$$

Note if one doesn't know the $D[\nu, \hat{\mathbf{n}}]$ one cannot determine f_ν from $V_T^\star[\nu]$ unless $\hat{\mathbf{n}}_\star = \hat{\mathbf{n}}_{bc}$. One can map out the beam pattern by pointing at multiple point sources if one has a steerable telescope and the beam is not much effected by pointing in different directions, which is not always the case.

Cas A calibration

We start with raw visibilities (output of the correlator) $V_{i,i}^{\text{raw}}[\nu]$. When we point the beam centers at Cas A we measure $V_{i,i}^{\text{raw,CasA}}[\nu]$ while when we point away from Cas A we obtain a much smaller number $V_{i,i}^{\text{raw},0}[\nu]$ which is mostly thermal noise from the LNA. Since we know the flux density of Cas A, which is f_ν^{CasA}, we define the visibilities in flux density units as

$$V_{i,i}^F[\nu] \equiv f_\nu^{\text{CasA}} \frac{V_{i,i}^{\text{raw}}[\nu]}{V_{i,i}^{\text{raw,CasA}}[\nu] - V_{i,i}^{\text{raw},0}[\nu]}$$

so the contribution to the visibility of Cas A when pointed at Cas A is

$$V_{i,i}^{F,\text{CasA}}[\nu] = f_\nu^{\text{CasA}}.$$

John has provided us the visibilities calibrated in flux units.

In temperature units the contribution to the visibility of Cas A when pointed at Cas A should be

$$V_{i,i}^{T,\text{CasA}}[\nu] = \frac{1}{8\pi} \frac{f_\nu^{\text{CasA}}}{k_B} \left(\frac{c}{\nu}\right)^2 D[\nu]$$

where k_B is Boltzmann's constant and $D[\nu]$ is the directivity of the beam. The conversion factor from flux density units to temperature units is

$$\frac{V_{i,i}^{T,\text{CasA}}[\nu]}{V_{i,i}^{F,\text{CasA}}[\nu]} = \frac{1}{8\pi} \frac{1}{k_B} \left(\frac{c}{\nu}\right)^2 D[\nu]$$

so more generally

$$V_{i,i}^T[\nu] = \frac{1}{8\pi} \frac{1}{k_B} \left(\frac{c}{\nu}\right)^2 D[\nu]\, V_{i,i}^F[\nu]$$

The directivity of an azimuthally symmetric beam centered at $\hat{\mathbf{n}}_{bc}$ is

$$D[\nu] \equiv \frac{4\pi B[\nu, \hat{\mathbf{n}}_{bc}]}{\int d^2\hat{\mathbf{n}}\, B[\nu, \hat{\mathbf{n}}]}$$

For a Gaussian beam

$$B[\nu, \hat{\mathbf{n}}] \propto e^{-\frac{1}{2}\left(\frac{\angle[\hat{\mathbf{n}}, \hat{\mathbf{n}}_{bc}]}{\sigma[\nu]}\right)^2}$$

so in the small angle approximation

$$D[\nu] = \frac{4\pi}{2\pi \int_0^\infty d\theta\, \theta\, e^{-\frac{1}{2}\left(\frac{\theta}{\sigma[\nu]}\right)^2}} = \frac{2}{\sigma[\nu]^2}$$

(where $\sigma[\nu]^2$ is in radians). Thus for small angle Gaussian beams

$$V_{i,i}^T[\nu] = \frac{1}{4\pi} \frac{1}{k_B} \left(\frac{c}{\nu\, \sigma[\nu]}\right)^2 V_{i,i}^F[\nu].$$

The FWHM of a Gaussian beam is defined by

$$\frac{1}{2} = e^{-\frac{1}{2}\left(\frac{\text{FWHM}[\nu]/2}{\sigma[\nu]}\right)^2}$$

or

$$\text{FWHM}[\nu] = 2\sqrt{2 \ln[2]}\, \sigma[\nu]$$

so

$$D[\nu] = \frac{16 \ln[2]}{\sigma[\nu]^2} = \frac{16 \ln[2]^2}{\text{FWHM}[\nu]^2}$$

and

$$V_{i,i}^T[\nu] = \frac{2}{\pi} \frac{1}{k_B} \left(\frac{c\sqrt{\ln[2]}}{\nu\, \text{FWHM}[\nu]}\right)^2 V_{i,i}^F[\nu]$$

where FWHM[ν] is in radians.

For the Tianlai dishes

$$\text{FWHM}[\nu] \approx 4.36\,°\,(750\text{ MHz}/\nu)$$

so

$$D[\nu] = \frac{16 \ln[2]}{(4.36 \frac{\pi}{180})^2} \left(\frac{\nu}{750\text{ MHz}}\right)^2 = 1915.22 \left(\frac{\nu}{750\text{ MHz}}\right)^2$$

and

$$V_{i,i}^T[\nu] = \frac{2}{\pi} \frac{1}{k_B} \left(\frac{c\sqrt{\ln[2]}}{4.36 \frac{\pi}{180} 750\text{ MHz}}\right)^2 V_{i,i}^F[\nu] = 8.81888 \frac{\text{mK}}{\text{Jy}} V_{i,i}^F[\nu].$$

The $\frac{\pi}{180}$ converts from degrees to radians. There is no frequency dependence in this conversion factor.

To convert to milliKelvin units I multiply the numbers in Johns calibrated files by 8.81888. For example the mean $V_{2V,2V}^T[\nu]$ during the 9 nights is 114 442 mK = 114.442 K.

Mathematica Computation

In[•]:=

$$\frac{16\,\text{Log}[2]}{(4.36 \frac{\pi}{180})^2}$$

Out[•]= 1915.22

In[•]:=

$$\text{UnitConvert}\left[\frac{2}{\pi} \frac{1}{1\,k} \left(\frac{1\,c\sqrt{\text{Log}[2]}}{4.36 \frac{\pi}{180}\, 750\text{ MHz}}\right)^2, 1\text{ mK}/1\text{ Jy}\right]$$

Out[•]= 8.81888 mK/Jy