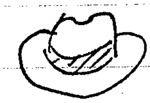
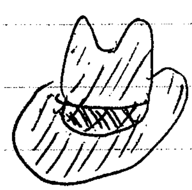
C.F. Berger

What you've always wanted to know
about on-shell methods &
but were afraid to ask



Outline

- Decomposition of 1-loop amplitudes
  - Blackhat vs. Rocket vs. CutTools
- Generalized Unitarity
  - Box example
  - Triangles & Bubbles
- Recursion
  - Tree Level Example
  - Loops
- D-dim unitarity
- Blackhat ⇒ Sherpa, ...



CFB, Zvi Bern, Lance Dixon
Fernando Febres Cordero, Darren Forde
Harald Ita, David Kosower
Daniel Maître
& Tanju Gleisberg (Sherpa)

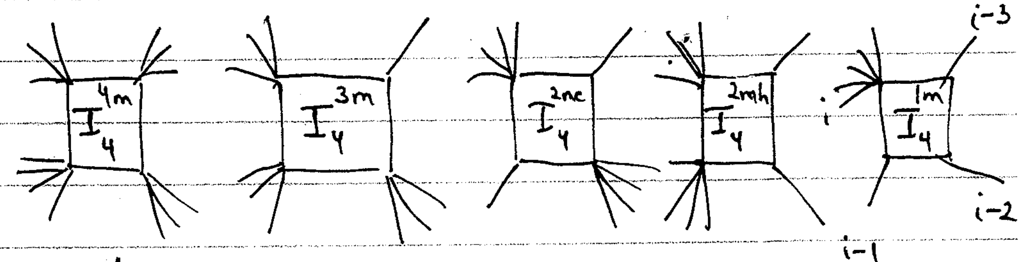
Decomposition of 1-loop Amplitudes

Any n-leg massless 1-loop amplitude can be decompose as (Bern, Dixon, Dunbar, Kosower)

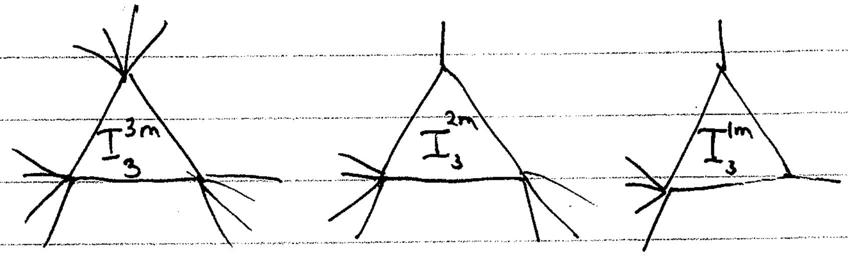
$$(1)\quad A_n = \sum_i c_i I_4^i + \sum_i d_i I_3^i + \sum_i b_i I_2^i + R$$

$\llcorner D\to 4$ rational   "$\frac{\varepsilon}{\varepsilon}$"

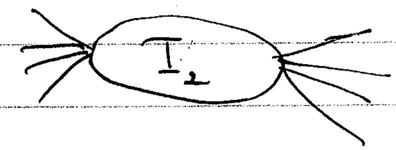
I_4^i :



I_3^i :



I_2^i :



for massive partons $\Rightarrow$ also tadpoles I_1^i

e.g. $$I_3^{2m}(s,t) = \frac{1}{(-s)-(-t)}\left(\frac{(-s)^{-\varepsilon}}{\varepsilon^2} - \frac{(-t)^{-\varepsilon}}{\varepsilon^2}\right)\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

in dim reg $D = 4-2\varepsilon$

$$(2)\quad A_n = \sum_i c_i^D I_4^{iD} + \sum_i d_i^D I_3^{Di} + \sum_i b_i^D I_2^{Di}$$

(BH3)

kind of (-2ε)-prod as "small mass"

$$\left(l^{D}\right)^2 = \left(l^{[4]}\right)^2 + \left(l^{[-2\varepsilon]}\right)^2 = \left(l^{[4]}\right)^2 + \mu^2$$

$$(3)\quad A_n = \sum_i c_i^{[4]} I_4^{D\,i}[1] + \sum_i c_i^{[\mu^2]} I_4^{D\,i}[\mu^2] + \sum_i c_i^{[\mu^4]} I_4^{D\,i}[\mu^4]$$
$$+ \sum_i d_i^{[4]} I_3^{D\,i}[1] + \sum_i d_i^{[\mu^2]} I_3^{D\,i}[\mu^2]$$
$$+ \sum_i b_i^{[4]} I_2^{D\,i}[1] + \sum_i b_i^{[\mu^2]} I_2^{D\,i}[\mu^2]$$

$$(1)\quad \overset{D\to 4}{=} \sum_i c_i^{[4]} I_4^{D\to 4\,i}[1] + \sum_i d_i^{[4]} I_3^{D\to 4\,i}[1] + \sum_i b_i^{[4]} I_2^{D\to 4\,i}[1]$$
$$+ R$$

$$\Rightarrow (*)\quad \boxed{R = \lim_{D\to 4}\left\{ \sum_i c_i^{[\mu^2]} I_4^{D\,i}[\mu^2] + \sum_i c_i^{[\mu^4]} I_4^{D\,i}[\mu^4] + \sum_i d_i^{[\mu^2]} I_3^{D\,i}[\mu^2] + \sum_i b_i^{[\mu^2]} I_2^{D\,i}[\mu^2]\right\}}$$

Use dimension-shifting relations (Bern, Morgan; Bern, Dixon, Dunbar, Kosower)

$$I_n^{D}[\mu^{2r}] = \frac{1}{2^r} I_n^{D+2r}[1] \prod_{l=0}^{r-1} (D-4+l)$$

$$R = \lim_{D\to 4}\left\{ \sum_i c_i^{[\mu^2]} \frac{D-4}{2} I_4^{D+2\,i}[1] + \sum_i c_i^{[\mu^4]} \frac{(D-4)(D-2)}{4} I_4^{D+ \dots} \right.$$
$$\left. + \sum_i d_i^{[\mu^2]} \frac{D-4}{2} I_3^{D+2\,i}[1] + \sum_i b_i^{[\mu^2]} \frac{D-4}{2} I_2^{D+2\,i}[1]\right\}$$

$$\lim_{D\to 4} I_4^D[\mu^4] = -\frac{1}{6} \qquad \lim_{D\to 4} I_4^D[\mu^2] = O(\varepsilon)$$

$$\lim_{D\to 4} I_3^D[\mu^2] = -\frac{1}{2}$$

$$\lim_{D\to 4} I_2^D[\mu^2] = -\frac{1}{6}\left(s - 3(m_1^2 + m_2^2)\right)$$

(Badger)
(Forde)

$\Rightarrow$ $\boxed{\text{BH bootstrap version}}$

$$A_n = \underbrace{\sum_i c_i^{[4]} I_4^{D\to 4\,i} + \sum_i d_i^{[4]} I_3^{D\to 4\,i} + \sum_i b_i I_2^{D\to 4\,i}}_{D=4 \text{ unitarity } \equiv C_n} + R_n^{\text{recursive}}$$

$\boxed{\text{BH rational extraction version}}$

$$A_n = C_n^{[4]} + \underbrace{\sum_i c_i^{[\mu^4]}\left(-\frac{1}{6}\right) + \sum_i d_i^{[\mu^2]}\left(-\frac{1}{2}\right) + \sum_i b_i^{[\mu^2]}\left(\frac{3(m_1^2+m_2^2)-s}{6}\right)}_{D=4 \text{ unitarity w. small "mass" } (\mu^2)}$$

$\boxed{\text{Rocket}}$ (Ellis, Giele, Kunszt, Melnikov, Zanderighi)

$$A_n = C_n^{[4]} + (D=6, D=8) \text{ unitarity}$$

$\boxed{\text{CutTools}}$ (van Hameren, Ossola, Papadopoulos, Pittau)

Analogous to Rocket, except at <u>integrand</u> level

(BH 5)

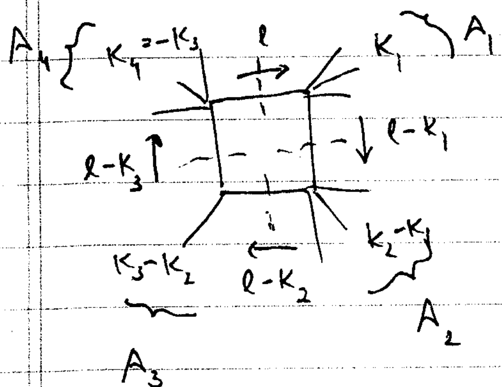
Generalized Unitarity

(Britto, Cachazo, Feng)

Boxes $\quad c_4 I_4 = c_4 \int d^4\ell \, \frac{1}{\ell^2 (\ell-K_1)^2 (\ell-K_2)^2 (\ell-K_3)^2}$

$$\frac{1}{\ell^2 + i\epsilon} = \frac{1}{\ell^2} + i\delta^+(\ell^2)$$

Suffices to replace full propagators with δ-fns to detect which box one has $\Rightarrow$ loop momenta set <u>on-shell</u>



$$= \int d^4\ell \, A_1(\ell) A_2(\ell) A_3(\ell) A_4(\ell) \, \delta^+(\ell^2) \, \delta^+((\ell-K_1)^2) \, \delta^+((\ell-K_2)^2) \, \delta^+((\ell-K_3)^2)$$

4-D integral, 4-δ fns $\Rightarrow \int$ collapses

$$c_4 = A_1(\ell_{sol}) A_2(\ell_{sol}) A_3(\ell_{sol}) A_4(\ell_{sol})$$

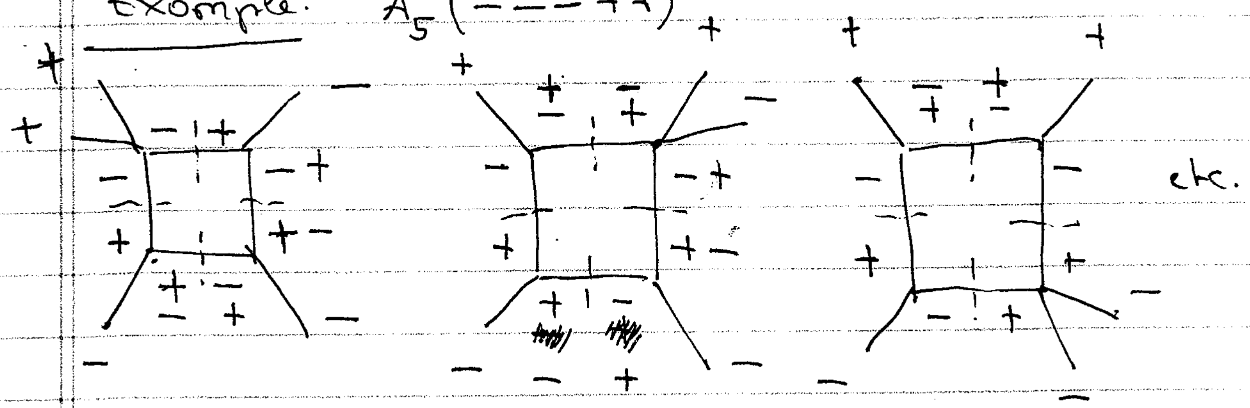
where ℓ^μ_{sol} solves the 4 δ-fn constraints
(2 solutions in general, quadratic eq)

ℓ_{sol} general for 4-mass box see BCF

ℓ_{sol} general for 3-mass box see BlackHat '08

$\Rightarrow$ 2-, 1 mass box solns can be read off from these

Example: $A_5(---++)$

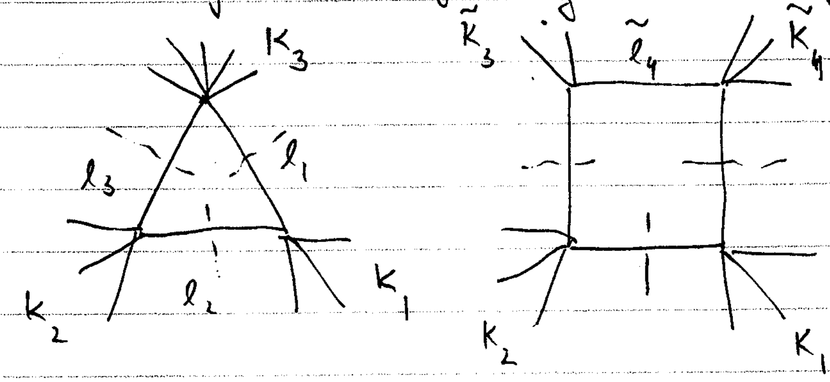


$\Rightarrow$ computation of "integrals" reduced to combinatorics!

Triangles

$$c_3 \int d^4 l \, \delta^+(l^2) \, \delta^+((l-K_1)^2) \, \delta^+((l-K_2)^2)$$

$\Rightarrow \int$ not completely localized anymore! $\Rightarrow \int dt$



contribute to <u>same</u> cut

but, we observe that the box has an extra propagator $l_i^2(t) = f_i(t-t_i)$ (similar to OPP) if we parametrize (exact form see Forde)

$$l_i^\mu = \alpha K_1^\mu + \beta K_2^\mu + \gamma t \, f^\mu(K_1 K_2) + \frac{\delta}{t} \, g^\mu(K_1, K_2)$$

$\Rightarrow$ triple cut

box coefficient!

$$C_3(t) = \sum_{j=-3}^{3} d_j t^j + \sum_i \frac{c_i}{\xi_i (t - t_i)}$$

poles in t!

We know the boxes $\to$ subtract them off!

$$T_3(t) = \sum_{j=-3}^{3} d_j t^j$$

d_0 is the triangle coefficient we want $\Rightarrow$ simple polynomial. Can extract d_0 via discrete Fourier transform

$$\boxed{d_0 = \frac{1}{7} \sum_{j=0}^{6} T_3\left(t_0\, e^{2\pi i j/7}\right)}$$ (Mackeod 08)

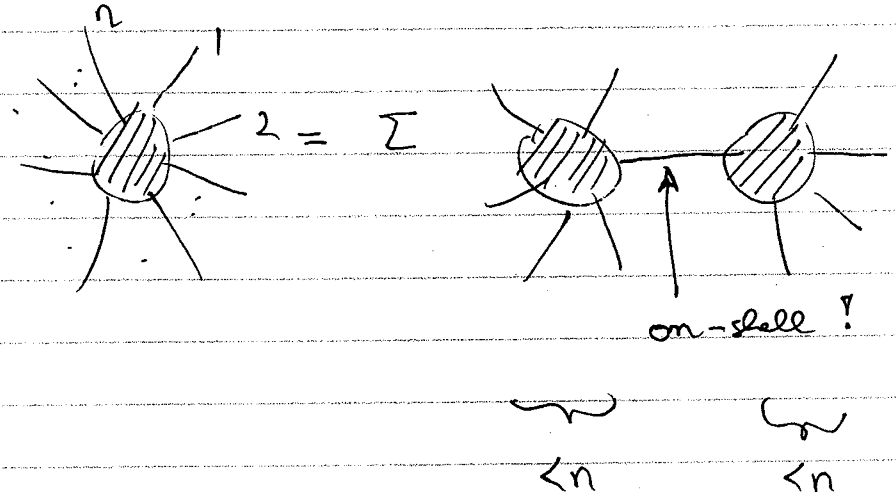
(see also OPP)

Similarly for bubbles, except 2 free parameters $\Rightarrow$ extraction somewhat more involved but similar subtract off triangles and do a double discrete FT For details see Forde

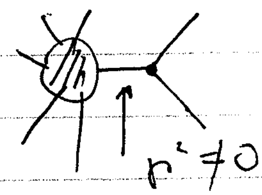
Ingredients in Rocket and CutTools somewhat similar in spirit. Details of implementation differ

Recursion

Tree Level (Britto, Cachazo, Feng, Witten)



Doesn't work with real momenta:



momentum conservation

$\Rightarrow$ complex continue momenta (spinors)

$$p_i \to p_i + zq \qquad p_j \to p_j - zq$$

$$p_i + p_j \to p_i + p_j$$ works for complex momenta!

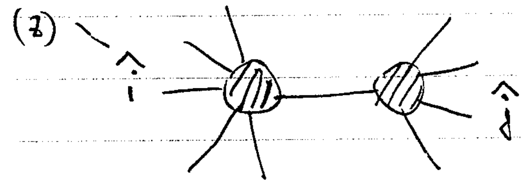
$$p_{ij}^2 = 0 \qquad p_{ij}^2(z) = 0 \qquad (q^2 = 0,\ q \cdot p_{ij} = 0)$$

$\Rightarrow$ amplitude becomes complex because propagators become complex

$$(p_l + \dots + p_i + \dots p_m)^2 \equiv P^2_{l..i...m} \to P^2_{l...i...m}(z) = (p_l + \dots + p_i + zq + \dots + p_m)^2$$

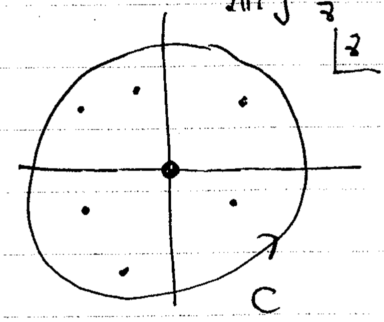
(BH9)

$$A_n(z) = \sum_{\text{partitions}} \sum_{\substack{\text{helicities}\\ \lambda}} A_L^{\lambda}(z, p_i(z)) \frac{1}{P^2_{\ell\ldots i\ldots m}(z)} A_R^{-\lambda}(z, p_j)$$



We want $A_n(z=0)$

Cauchy's theorem: If $A_n(z) \xrightarrow{z\to\infty} 0$ then $\frac{1}{2\pi i}\oint \frac{dz}{z} A(z) = 0$



$$\sum_{\substack{\text{poles}\\ \alpha}} \operatorname{Res}_{z=z_\alpha} \frac{A(z)}{z} = 0$$

one pole at $z=0$

$$A(0) = -\sum_{\substack{\text{poles}\\ \alpha}} \operatorname{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

Poles where $P^2_{\ell\ldots i\ldots m}(z) = 0 = (p_\ell + \ldots + p_i + zq + \ldots + p_m)^2$

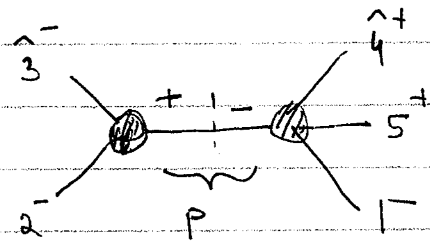
$$0 = (P_{\ell\ldots i\ldots m})^2 + 2z\, q\cdot(P_{\ell\ldots i\ldots m}$$

$$\Rightarrow \boxed{A(0) = -\sum_{\substack{\text{poles}\\ \alpha}} \sum_{\text{partitions}} \sum_{\lambda} A_L^{\lambda}(z_\alpha) \frac{1}{P^2_{\ell\ldots i\ldots m}} A_R^{-\lambda}(z_\alpha)}$$

Relies only on:

- $A(z) \xrightarrow{z\to\infty} 0$
- simple poles $\Leftarrow$ follows from factorization properties

Example $(- - - + +)$



$$3 \to 3 - zq = \hat{3} \qquad (q\cdot 3 = 0)$$
$$4 \to 4 + zq = \hat{4}$$

pole at $(p_2 + p_3 - zq)^2 = 0$

$$(p_2+p_3)^2 - 2z\, q\cdot(p_2+p_3) = 0$$

$$z_\alpha = \frac{(p_2+p_3)^2}{2\, q\cdot p_2}$$

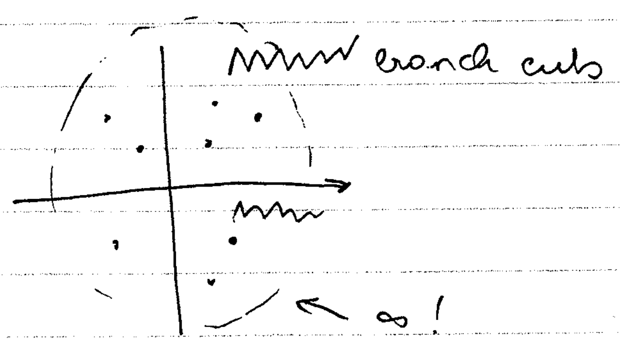
$$\Rightarrow A_5(1^- 2^- 3^- 4^+ 5^+) = A_3(2^-, \hat{3}^-, P^+) \frac{1}{(p_2+p_3)^2} A_4(1^-, P^-, \hat{4}^+, 5^+)$$

where $\hat{3}^\mu = 3^\mu - z_\alpha q^\mu$

$$\hat{4}^\mu = 4^\mu + z_\alpha q^\mu$$

$\boxed{\text{Loop level bootstrap}}$ (CFB, Bern, Dixon, Forde, Kosower)

Pole structure:



- $A(z) \not\to 0$ as $z \to \infty$
- branch cuts
- nonfactorizing contributions

BH 11

- branch cuts from I^i's $\Rightarrow$ know them from generalized unitarity $\Rightarrow$ subtract them off

$$A(z=0) = \overbrace{C(z=0)}^{\text{unitarity}} + \overbrace{\sum_{\text{poles}} \frac{\mathrm{Res}\, R_n(z)}{z}}^{(*)} + \mathrm{Inf}\, A$$

$$+ \sum_{\text{poles}} \text{nonfactorizing channels}$$

"pole" at ∞
(has contribution from

$R(z \to \infty)$ and $C(z \to \infty)$

?   know this!

$$(*) = \sum_{\substack{\text{poles}\\ \alpha}} R_\ell(z_\alpha) \frac{1}{p^2} A^{\text{tree}}(z_\alpha) + \sum_{\substack{\text{poles}\\ \alpha'}} A^{\text{tree}}(z_{\alpha'}) \frac{1}{p^2} R(z$$

empirical study:

- can find complex shift where $\cancel{\mathrm{Inf}\, A}$
  but nonfactorizing channels

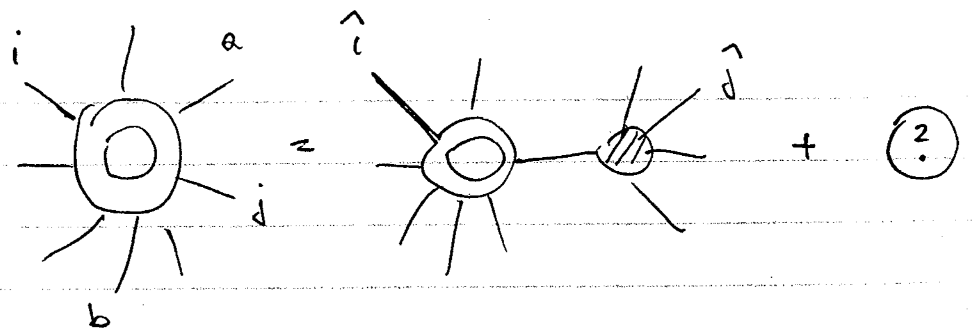
$$R_n = R^{\text{recursive}} + \sum_{\text{poles}} \text{nonfactorizing channels}$$

OR

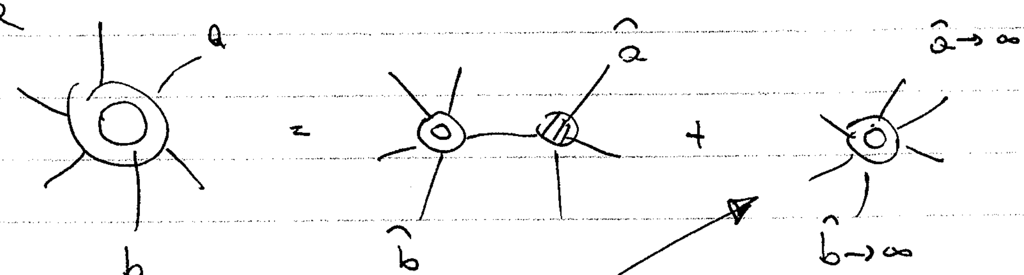
- $\mathrm{Inf}\, A_j$ but ~~nonfactorizing channels~~

$$R_n = R^{\text{recursive}'} + \mathrm{Inf}$$

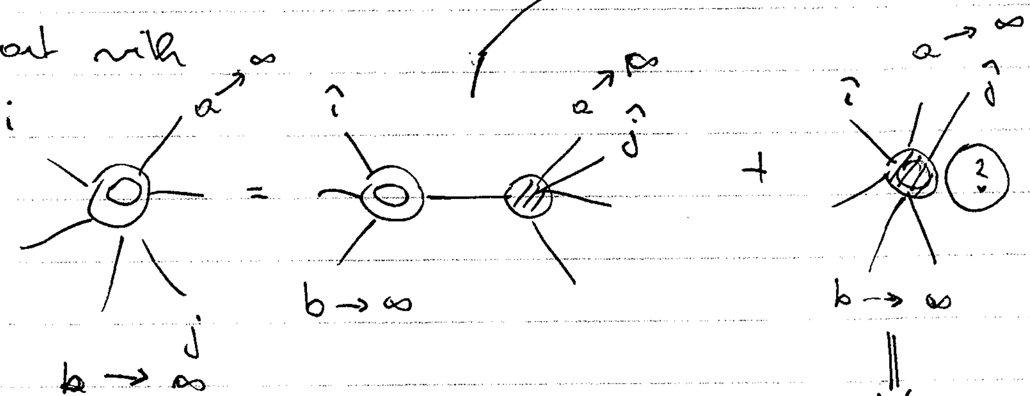
BH 12



OR



start with



vanishes!

$\Rightarrow$ $\boxed{\text{bootstrap}}$

$$\boxed{A_n = C_n(\text{unitarity}) + R_n^{[a,b\rangle} + \underset{[a,b\rangle}{\mathrm{Inf}}\left[R^{[i,j\rangle} + \cancel{2^{[i,j\rangle}} + C^{[i,j\rangle}_{\text{unitarity}} \right]}$$

BH 13

$\boxed{\text{BlackHat}}$

- computes color-ordered, color-stripped 1-loop helicity amplitudes for Ng, $NgMq$, $NgMq$ 1V
  (massless at the moment)
  => feed in momentum config → out comes complex #
- to get actual amplitude, need to dress with couplings, color

e.g.
$$A^{\text{tree}}(g_1, \dots, g) = g^{n-2} \sum_{\sigma \in S_n / \mathbb{Z}_n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}\right) A^{\text{tree}}(\sigma(1), \dots, \sigma(n))$$

currently "manually" assembled in Sherpa-specific format
↓
Will be automated soon as well

completely automated at 1 loop
⇓
$\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}, \varepsilon^0$ ($\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}$ ~ tree, known, just as a check)

- to get cross section: need to sum over helicities, interface v. real emission, and subtract divergences before $\int$ over PS

=> done with Sherpa

$$\sigma^{NLO} = \int d^D\sigma^{NLO}$$

$$= \int_{m+1} d^D\sigma^R + \int_m \int_{loop} d^D\sigma^V$$

$$= \int_{m+1} \left(d^D\sigma^R - d^D\sigma^A\right) + \int_m \Big[\underbrace{\int_1 d^D\sigma^V + \int_1 d^D\sigma^A}_{BH} \Big]_{\varepsilon=0}$$

$$d^D\sigma^A = \sum_{dipoles} d^4\sigma^B \otimes d^D V_{dipole}$$