# Nucleon Distribution Amplitudes 

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In collaboration with:
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- Lightcone quantization : $z^{0} \rightarrow z^{+}=z^{0}+z^{3}$
- Lightcone-QCD allows decomposition of hadrons in Fock states:

$$
\begin{gathered}
|P, \pi\rangle \propto \sum_{\beta} \Psi_{\beta}^{q \bar{q}}|q \bar{q}\rangle+\sum_{\beta} \Psi_{\beta}^{q \bar{q}, q \bar{q}}|q \bar{q}, q \bar{q}\rangle+\ldots \\
|P, N\rangle \propto \sum_{\beta} \Psi_{\beta}^{q q q}|q q q\rangle+\sum_{\beta} \Psi_{\beta}^{q q q, q \bar{q}}|q q q, q \bar{q}\rangle+\ldots
\end{gathered}
$$

- Often restricted to the first term, i.e. $\Psi_{\beta}^{q \bar{q}}$ and $\Psi_{\beta}^{q q q}$.
- Schematically (disregarding twist decomposition), the DA $\varphi$ :

$$
\varphi(x) \propto \int \frac{\mathrm{d}^{2} k_{\perp}}{(2 \pi)^{2}} \Psi\left(x, k_{\perp}\right)
$$

## Evolution

- DA are scale dependent objects
- They obey evolution equations and can be written as:

$$
\varphi_{\pi}\left(x, \mu^{2}\right)=\varphi_{\pi}^{A s}(x)\left(1+\sum_{j=2,4 \ldots}^{\infty} a_{j}^{\left(\frac{3}{2}\right)}\left(\mu^{2}\right) C_{j}^{\left(\frac{3}{2}\right)}(x)\right)
$$

Efremov and Radyushkin (1980)
Lepage and Brodsky (1980)

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## Caveat

What does large enough mean?

Fock space at high $Q^{2}$


- At large $Q^{2}$,

$$
F\left(Q^{2}\right) \simeq \int[\mathrm{d} x][\mathrm{d} y] \varphi^{*}(y) T(x, y) \varphi(x)
$$

- Higher Fock states suppressed by $\left(\frac{\alpha_{S}\left(Q^{2}\right)}{Q^{2}}\right)$ per additional constituent.
- $T$ can be computed through perturbation theory.


## From DA to Form factors

- Pion case:

$$
Q^{2} F_{\pi}\left(Q^{2}\right)=16 \pi \alpha_{S}\left(Q^{2}\right) f_{\pi} \omega_{\varphi}^{2} \quad \text { for large enough } Q^{2}
$$

with

$$
\omega_{\varphi}=\frac{1}{3} \int \mathrm{~d} x \frac{\varphi\left(x, Q^{2}\right)}{x}, \quad \omega_{A s}=1
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Farrar and Jackson (1979),
Efremov and Radyushkin (1980),
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Farrar and Jackson (1979), Efremov and Radyushkin (1980),

Lepage and Brodsky (1980).

- Proton case:
- same reasoning but absolute normalisation unknown,
- when assuming isospin symmetry, the ratio between the magnetic form factors of the proton and neutron can be predicted.


## Pion distribution amplitude

$$
\phi_{A s}(x)=6 x(1-x)
$$



Chang et al. (2013)

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- Broad DSE pion DA is much more consistent with the form factor than the asymptotic one.
- The scale when the asymptotic DA become relevant is huge.


## Proton distribution amplitude



## Proton distribution amplitude



## Proton distribution amplitude



## Proton distribution amplitude


$\varphi_{A s}\left(x_{1}, x_{2}, x_{3}\right)=120 x_{1} x_{2} x_{3}$
Lepage and Brodsky (1980)
What happens when computing the Proton DA within DSEs framework?

- Modern diquark: strong correlations between two quarks inside a nucleon.

Cahill et al., (1987)

- Two types of diquark correlations inside the nucleon:
- Scalar diquarks.
- Axial-Vector diquarks.
- This allows to solve a simplified Faddeev equation...
- .. and to compute in the DSE framework of different baryon observables, including the nucleon form factors.

We would like to apply this approximation to compute nucleon DA.

## Leading Twist Nucleon DA

- Parameterisation of non-local matrix element in 24 invariant functions:

$$
\begin{aligned}
\langle 0| \epsilon^{i j k} u_{\alpha}^{i} & \left(z_{1}\right) u_{\beta}^{j}\left(z_{2}\right) d_{\gamma}^{k}\left(z_{3}\right)|P\rangle \\
= & \frac{1}{4}\left[(\not p C)_{\alpha \beta}\left(\gamma_{5} N^{+}\right)_{\gamma} V\left(z_{i}^{-}\right)+\left(\not p \gamma_{5} C\right)_{\alpha \beta}\left(N^{+}\right)_{\gamma} A\left(z_{i}^{-}\right)\right. \\
& \left.-\left(i p^{\mu} \sigma_{\mu \nu} C\right)_{\alpha \beta}\left(\gamma^{\nu} \gamma_{5} N^{+}\right)_{\gamma} T\left(z_{i}^{-}\right)\right]+ \text {higher twist. }
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Chernyak and Zhitnitsky (1983) Braun et al. (2000)

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- Nucleon leading twist DA defined as:

$$
\varphi\left(x_{i}\right)=V\left(x_{i}\right)-A\left(x_{i}\right)
$$

## Matrix element of the leading twist DA

- Definition of the leading twist DA in terms of matrix element:

$$
\begin{aligned}
\langle 0| \epsilon^{i j k} & \left(u_{\uparrow}^{i}\left(z_{1}\right) C \not \ddagger u_{\downarrow}^{j}\left(z_{2}\right)\right) \not \ddagger d_{\uparrow}^{k}\left(z_{3}\right)|P\rangle \\
& =-\frac{1}{2}(p \cdot z) \neq N^{\uparrow} \int \mathcal{D} x_{i} \varphi\left(x_{1}, x_{2}, x_{3}\right) e^{-i \sum_{i} x_{i} P \cdot z_{i}} .
\end{aligned}
$$

- quark of given chirality: $q^{\uparrow(\downarrow)}=\frac{1 \pm \gamma_{5}}{2} q$
- momentum conservation: $\mathcal{D} x_{i}=\mathrm{d} x_{i} \delta\left(1-x_{1}-x_{2}-x_{3}\right)$
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- Different possible contractions for $|u(\uparrow) u(\downarrow) d(\uparrow)\rangle$ :



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- Different possible contractions for $|u(\uparrow) u(\downarrow) d(\uparrow)\rangle$ :
- one spin aligned quark $u$ with a scalar diquark $|u(\uparrow),[u d]\rangle$
- one spin aligned quark $u$ with a longitudinally polarised AV diquark: $\left|u(\uparrow),\{u d\}_{L}\right\rangle$
- one spin aligned quark d with a longitudinally polarised AV diquark: $\left|d(\uparrow),\{u u\}_{\llcorner }\right\rangle$
- one spin anti-aligned quark $u$ with a transversaly polarised AV diquark: $\left|u(\downarrow),\{u d\}_{T}\right\rangle$


## Quark-Diquark DA



- Need of specific ingredients:
- quark propagator $S_{u}\left(S_{d}\right)$,
- AV diquark propagator $S_{u u}$,
- diquark Bethe-Salpeter amplitude $\Gamma_{u u}$,
- nucleon Bethe-Salpeter amplitude $\Gamma_{d ; u u}$.


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All these objects can be computed non-pertubatively using DSEs-BSEs.

## Nakanishi representation: Quark-Diquark Amplitudergonne

- quark propagator:

$$
S_{q}(q)=\frac{-i \gamma \cdot q+M}{q^{2}+M^{2}}
$$

- diquark propagator:

$$
S_{q q}(K)=\frac{1}{K^{2}+\widetilde{M}^{2}}\left(\delta_{\mu \nu}+\frac{K^{\mu} K^{\nu}}{K^{2}}\right)
$$

- Nakanishi representation for the quark-diquark Bethe-Salpeter Amplitude:

$$
\mathcal{A}_{\mu}(K, P)=i \gamma_{5} P_{\mu} \frac{\bar{M}}{f_{N}} \bar{M}^{2 \sigma} \int_{-1}^{+1} \mathrm{~d} z \rho_{\sigma}(z)\left[\frac{1}{\left(\left(K-\frac{1-z}{2} P\right)^{2}+\Lambda_{N}^{2}\right)}\right]^{\sigma}
$$

## Nakanishi representation: Diquark Amplitude

- Quark propagator:

$$
S_{q}(q)=\frac{-i \gamma \cdot q+M}{q^{2}+M^{2}}
$$

- Extended Bethe-Salpeter Amplitude:

$$
\begin{gathered}
\Gamma_{\mu}(q, K)=i \tilde{\gamma}_{\mu} C \frac{M}{f} M^{2 \nu} \int_{-1}^{+1} \mathrm{~d} z \rho_{\nu}(z)\left[\frac{1}{\left(\left(q-\frac{1-z}{2} K\right)^{2}+\Lambda_{q}^{2}\right)}\right]^{\nu} \\
K \cdot \tilde{\gamma}=0
\end{gathered}
$$

- Computation of the Mellin moment of the nucleon DA:

$$
\widetilde{\varphi}\left(n_{1}, n_{2}, n_{3}\right)=\int \mathcal{D} x_{i} x_{1}^{n_{1}} x_{2}^{n_{2}} x_{3}^{n_{3}} \varphi\left(x_{1}, x_{2}, x_{3}\right)
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## Analytical results for very simple Ansätze.

## Preliminary results

Caveat: transversely polarised diquark is missing


Asymptotic


70\%Scalar 30\% AV



100\% Scalar


100\% AV

- At high enough $Q^{2}$, it is possible to compute the form factor through the DA.
- Results on the pion show that at available energy, the asymptotic DA is not relevant for such a computation.
- To compute the nucleon DA and see how it differs from the asymptotic one.
- We developed algebraic models as a first step.
- Results are encouraging.
- Short term outlooks:
- Finish the algebraic computations.
- Numerical computation using solution of the DSEs.
- Comparison with lattice data
- Computation of the ratio of the proton and neutron magnetic form factors.
- Longer term outlooks $\rightarrow$ computations of other matrix elements:
- Valence nucleon PDF.
- Valence nucleon GPD,
$\rightarrow$ following the methods highlighted in arXiv:1602.07722 for the pion.
$\rightarrow$ using the PARTON Software developed at Saclay (1512.06174).


## Thank you for your attention!

