Rapidity Renormalization Group and pT Resummation

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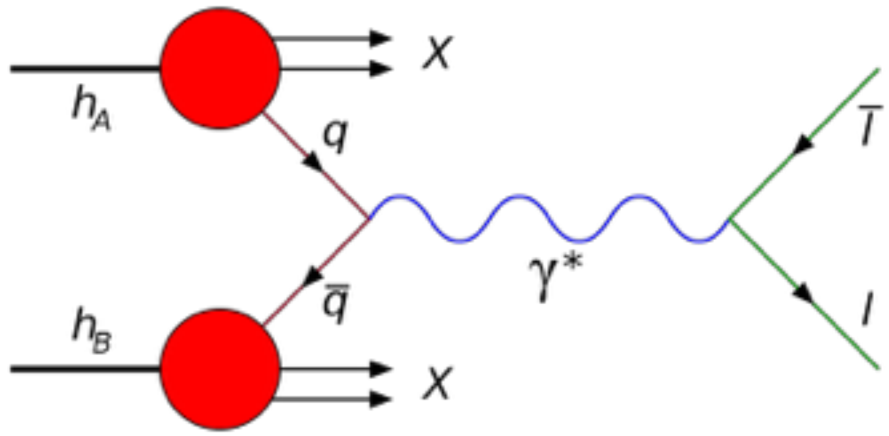
Based on work with Y. Li, D. Neill, M. Schulze, I. Stewart
Refs: 1604.00392, 1604.01404, and work in progress

Transverse momentum of color neutral system



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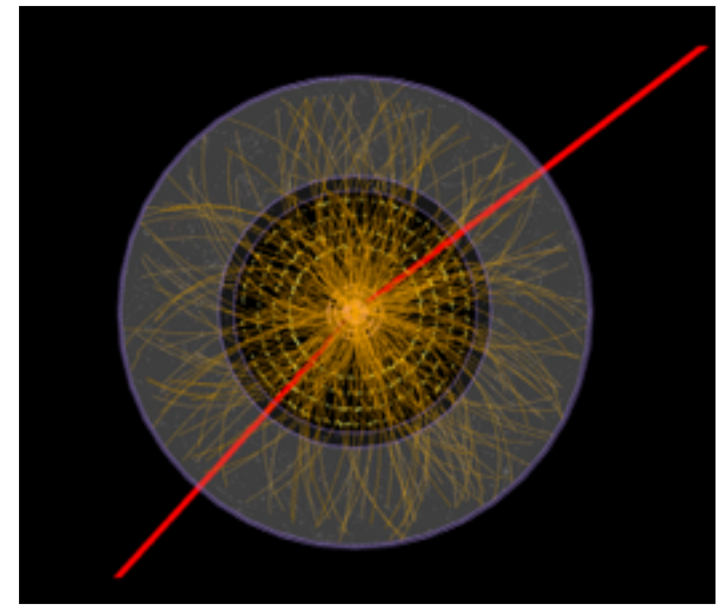


✦ Definition of the observable (Drell-Yan case):

$$\vec{q}_\perp = \vec{p}_{l^+,\perp} + \vec{p}_{l^-,\perp}$$

$$\frac{d\sigma}{d\vec{q}_\perp^{\,2}\,dY} = \sum_{i,j}\int_0^1 dx_a\,dx_b\,f_{i/h_1}(x_a,\mu_f)f_{j/h_2}(x_b,\mu_f)\,\frac{d\hat{\sigma}}{d\vec{q}_\perp^{\,2}\,dY}(\hat{s},\hat{t},\hat{u},Q^2)$$

Analytical results at NLO available for long time

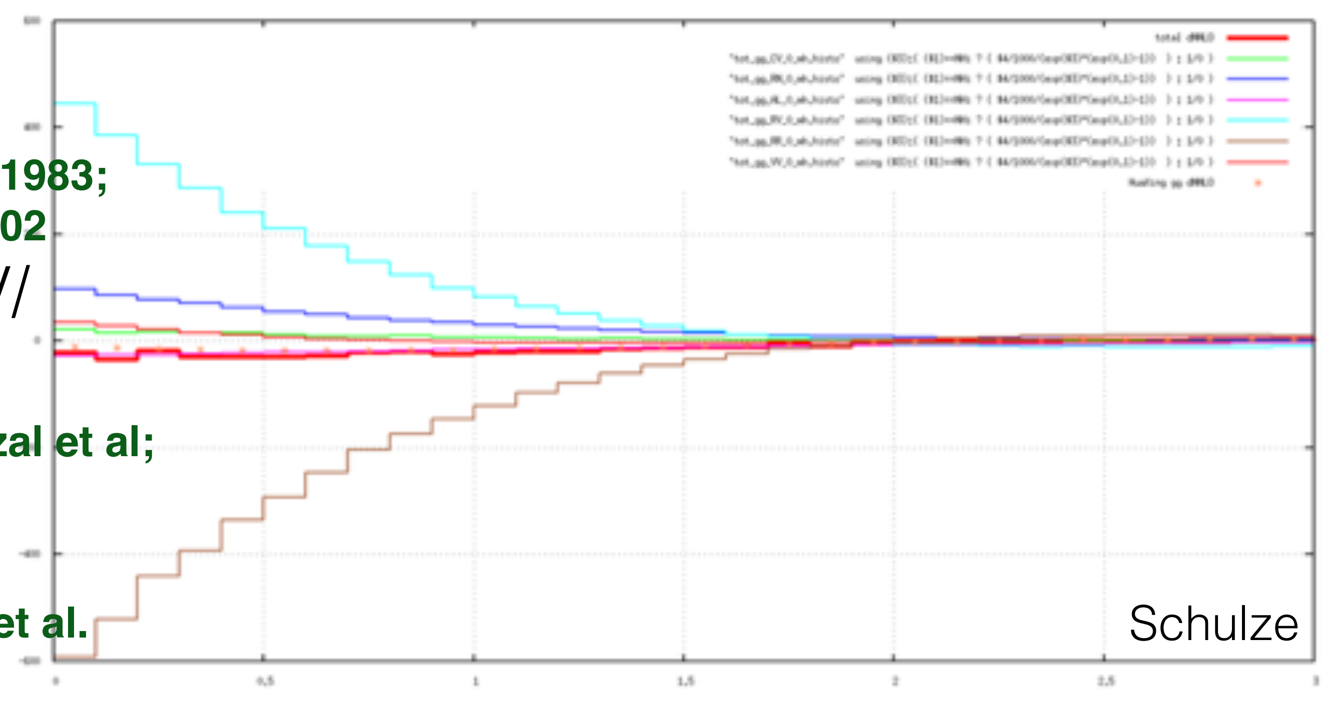
DY: Ellis, Martinelli, Petronzio, 1983;
Higgs: Glosser, Schmidt, 2002

Recent development: numerical calculation of V/H + jet at NNLO at large pT

Antenna subtraction: Gehrmann et al; N-jettiness, Boughezal et al;
STRIPPER subtraction: Boughezal et al;

Progress towards analytical NNLO V/H+jet

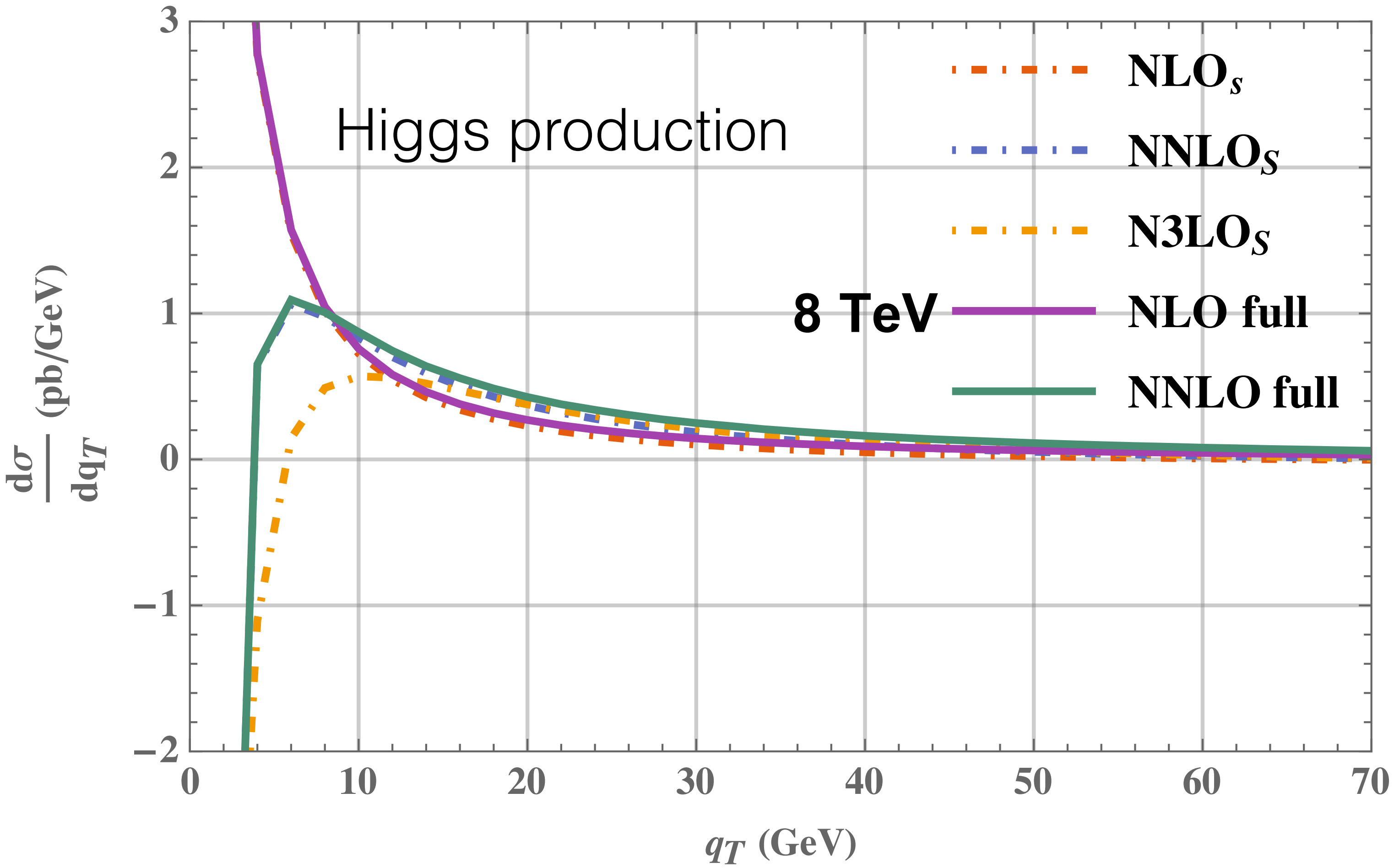
Anastasiou et al.



Break down of fixed order P.T. at small pT

✦ **Fixed order perturbation theory exhibits large logs at small pT**

$$\frac{d\sigma}{dp_T} = \alpha_s \left(c_{1,1} \frac{\ln p_T}{p_T} + c_{1,0} \frac{1}{p_T} + \ldots \right)$$
$$\alpha_s^2 \left(c_{2,3} \frac{\ln^3 p_T}{p_T} + c_{2,2} \frac{\ln^2 p_T}{p_T} + \ldots \right)$$
$$\alpha_s^3 \left(c_{3,5} \frac{\ln^5 p_T}{p_T} + c_{3,4} \frac{\ln^4 p_T}{p_T} + \ldots \right)$$



✦ **Impact parameter space**

$$\sigma(\vec{b}_\perp) = \int d^2\vec{q}_\perp \frac{d\sigma}{d^2\vec{q}_\perp} \exp(-i\vec{b}_\perp \cdot \vec{q}_\perp) \qquad \frac{\ln^{k-1} \vec{q}_\perp^{\,2}}{\vec{q}_\perp^{\,2}} \sim \ln^k \vec{b}_\perp^{\,2} \qquad \text{small } \vec{q}_\perp \Leftrightarrow \text{ large } \vec{b}_\perp$$

✦ **Cross section in b-space exponentiated to all orders**

$$\sigma(\vec{b}_\perp) \sim \exp\left(A(\alpha_s) \ln^2 \vec{b}_\perp^{\,2} + B(\alpha_s) \ln \vec{b}_\perp^{\,2} \right) + \text{non-singular terms}$$

Collins, Soper, Sterman, 1985 …

✦ Hard, collinear, and soft modes dominate singular Xsec

$$p^\mu = (p^+, p^-, p_\perp) = (p_0 + p_z, p_0 - p_z, p_\perp)$$

$$n^\mu = (2,0,0) \quad \bar{n}^\mu = (0,2,0)$$

power counting parameter $\lambda \sim \dfrac{q_\perp}{Q}$

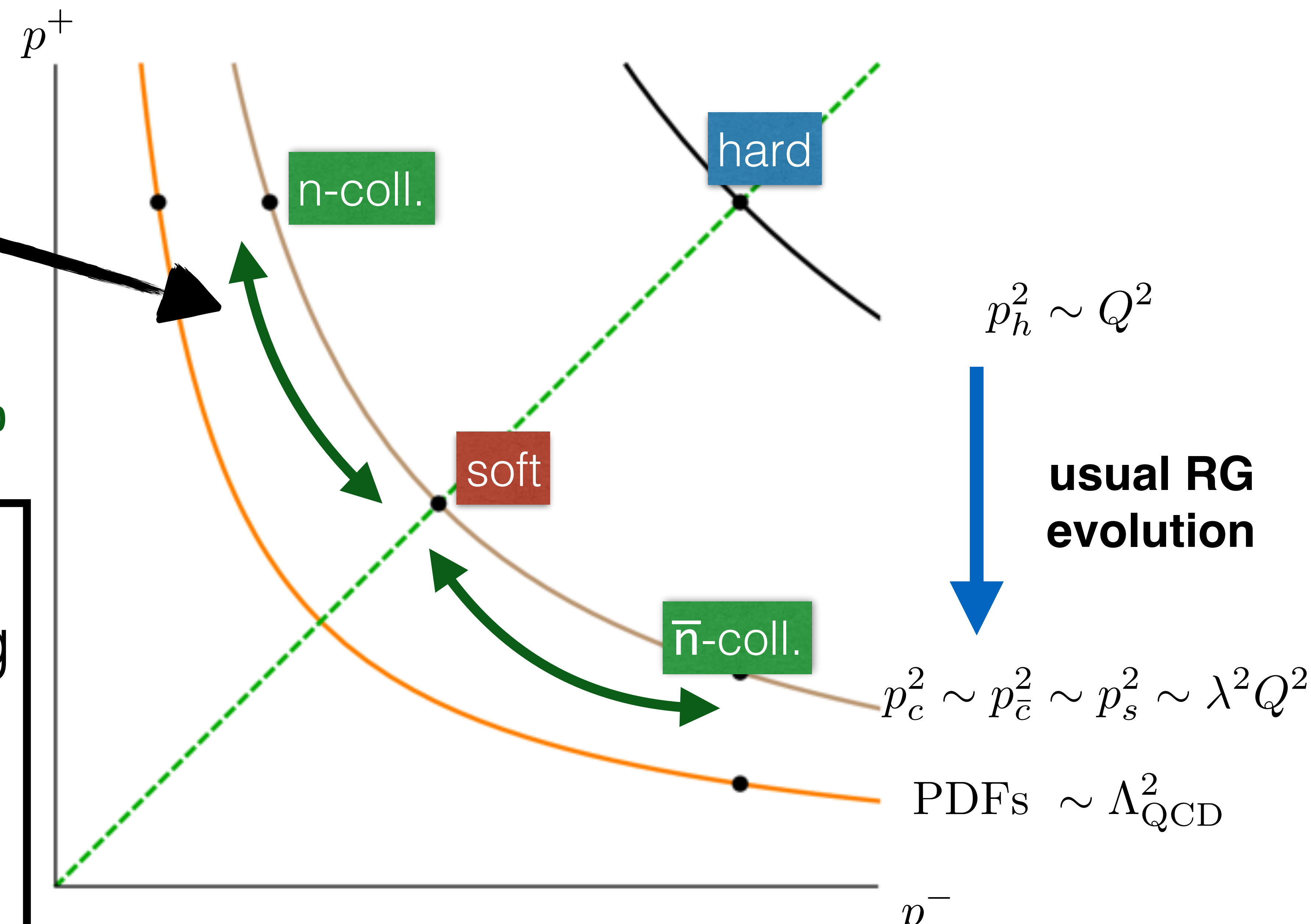
hard: $Q(1,1,1)$

collinear: $Q(1,\lambda^2,\lambda), \quad Q(\lambda^2,1,\lambda)$

soft: $Q(\lambda,\lambda,\lambda)$

Rapidity evolution between soft and collinear/anti-collinear modes

Collins-Soper equation, 82
Rapidity RG, Chiu et al, 2011,12
Collinear anomaly, Becher et al, 2010
…



Large logs are resumed by solving usual RG equation and rapidity RG equation

pT resummation in Effective theory

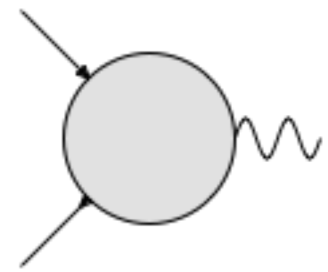
- pT resummation in the SCET rapidity RG formalism

Chiu, Jain, Neill, Rothstein, 2012

$$\frac{d\sigma_{\rm DY}}{dQ^2 dY d^2\vec{q}_\perp} = \sigma_0 \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{q}_\perp} H(Q,\mu_h) B_q(x_A, Q, \vec{b}_\perp, \mu_b, \nu_b) B_{\bar{q}}(x_B, Q, \vec{b}_\perp, \mu_b, \nu_b) S_\perp(\vec{b}_\perp, \mu_s, \nu_s)$$

$$\cdot \exp\left\{ -\int_{b_0^2/\vec{b}_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\left(\Gamma_{\rm cusp}\left[\alpha_s(\bar{\mu})\right] + \frac{d\gamma^r\left[\alpha_s(\bar{\mu})\right]}{d\ln\bar{\mu}^2} \right) \ln\frac{Q^2}{\bar{\mu}^2} + \left(\gamma^V\left[\alpha_s(\bar{\mu})\right] - \gamma^r\left[\alpha_s(\bar{\mu})\right] \right) \right] \right\}$$

- **Hard function H: quark/gluon form factor**



- **Beam function B: quark/gluon correlator (unrenormlized)**

$$W_n(x) = \mathrm{P}\exp\left(ig\int_{-\infty}^{0} ds\, \bar{n}\cdot A(x+s\bar{n})\right)$$

$$B_{q/N}(z, Q, \vec{b}_\perp) = \int dx^+ e^{izP^- x^+/2} \left\langle P \middle| (\bar{\psi}_n W_n)(x^+, 0, \vec{b}_\perp) \frac{\bar{n}_\mu \gamma^\mu}{2} (W_n^\dagger \psi_n)(0) \middle| P \right\rangle$$

- **Soft function S: VEV. of light-like Wilson loop (unrenormalized)**

$$S_\perp = \frac{\rm Tr}{C} \langle 0 | T\{S_{\bar{n}}^\dagger S_n(0,0,0)\} \overline{T}\{S_n^\dagger S_{\bar{n}}(0,0,\vec{b}_\perp)|0\rangle$$

$$S_n(x) = \mathrm{P}\exp\left(ig\int_{-\infty}^{0} ds\, n\cdot A(x+sn)\right)$$

Anomalous dimension for resummation

✦ Resummation formulae in the SCET formalism at canonical scale

$$\frac{d\sigma_{\rm DY}}{dQ^2\,dY\,d^2\vec{q}_\perp} = \sigma_0 \int \frac{d^2\vec{b}_\perp}{(2\pi)^2}\, e^{i\vec{b}_\perp\cdot\vec{q}_\perp}\, H(Q,\mu_h)\, B_q(x_A,Q,\vec{b}_\perp,\mu_b,\nu_b)\, B_{\bar{q}}(x_B,Q,\vec{b}_\perp,\mu_b,\nu_b)\, S_\perp(\vec{b}_\perp,\mu_s,\nu_s)$$

$$\cdot \exp\left\{ -\int_{b_0^2/\vec{b}_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\left(\Gamma_{\rm cusp}\left[\alpha_s(\bar{\mu})\right] + \frac{d\gamma^r\left[\alpha_s(\bar{\mu})\right]}{d\ln\bar{\mu}^2} \right) \ln\frac{Q^2}{\bar{\mu}^2} + \left(\gamma^V\left[\alpha_s(\bar{\mu})\right] - \gamma^r\left[\alpha_s(\bar{\mu})\right] \right) \right] \right\}$$

$A\ln^2\vec{b}_\perp^2$ — double log.

$B\ln\vec{b}_\perp^2$ — single log.

cusp anomalous dim.
three-loop

form factor
normal anomalous dim.
three-loop

rapidity anomalous dim.

Largest uncertainty source
at N3LL

pT distribution as a precision probe of N.P. QCD

$$\frac{d\sigma_{\rm DY}}{dQ^2 dY d^2\vec{q}_\perp} = \sigma_0 \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{q}_\perp} H(Q,\mu_h) B_q(x_A, Q, \vec{b}_\perp, \mu_b, \nu_b) B_{\bar{q}}(x_B, Q, \vec{b}_\perp, \mu_b, \nu_b) S_\perp(\vec{b}_\perp, \mu_s, \nu_s)$$

$$\cdot \exp\left\{ -\int_{b_0^2/\vec{b}_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\left(\Gamma_{\rm cusp}\left[\alpha_s(\bar{\mu})\right] + \frac{d\gamma^r\left[\alpha_s(\bar{\mu})\right]}{d\ln\bar{\mu}^2} \right) \ln\frac{Q^2}{\bar{\mu}^2} + \left(\gamma^V\left[\alpha_s(\bar{\mu})\right] - \gamma^r\left[\alpha_s(\bar{\mu})\right] \right) \right] \right\}$$

$\cdot e^{-S_{\rm np}}$ (non-perturbative modification at large impact parameter)

- **b* prescription:** $b^* = \dfrac{b_\perp}{\sqrt{1 + b_\perp^2/b_{\rm max}^2}}$
- **Commonly used N.P. model:** $S_{\rm N.P.} = \exp\left[-\left(g_1 + g_2 \ln\frac{Q}{2Q_0} + g_1 g_3 \ln(100 x_A x_B) \right) b_\perp^2 \right]$

Different functional form for global fit

Landry, Brock, Nadolsky, Yuan, 2002;
Konychev, Nadolsky, 2005;
Qiu, Zhang, 2001;
Echevarria, Idilbi, Schafer, Scimemi, 2011;
Sun, Isaacson, Yuan, Yuan, 2014;
…

- **Quadratic form at small b** — Korchemsky, Sterman, 94; Scimemi, Vladimirov, 16
- **No first principle prediction at large b**
  - **quadratic: original CSS parameterization**
  - **linear: Tafat, 2002**
  - **constant: Collins, Rogers, 2014**
  - **Logarithmic: Collins, Soper, 82; SIYY, 2014**
- **Need truly non-perturbative prediction. Lattice? integrability?**

$$\frac{d\sigma_{\rm DY}}{dQ^2dY\,d^2\vec{q}_\perp} = \sigma_0 \int \frac{d^2\vec{b}_\perp}{(2\pi)^2} e^{i\vec{b}_\perp\cdot\vec{q}_\perp} H(Q,\mu_h) B_q(x_A,Q,\vec{b}_\perp,\mu_b,\nu_b) B_{\bar{q}}(x_B,Q,\vec{b}_\perp,\mu_b,\nu_b) S_\perp(\vec{b}_\perp,\mu_s,\nu_s)$$

$$\cdot \exp\left\{ -\int_{b_0^2/\vec{b}_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\left(\Gamma_{\rm cusp}[\alpha_s(\bar{\mu})] + \frac{d\gamma^r[\alpha_s(\bar{\mu})]}{d\ln\bar{\mu}^2} \right) \ln\frac{Q^2}{\bar{\mu}^2} + \left(\gamma^V[\alpha_s(\bar{\mu})] - \gamma^r[\alpha_s(\bar{\mu})] \right) \right] \right\}$$

$B \ln \vec{b}_\perp^2$

$$B = \alpha_s B_1 + \alpha_s^2 B_2 + \cdots$$

$$\gamma_r = \gamma_0^r B_1 + \alpha_s^2 \gamma_1^r + \cdots$$

- **Process dependent. Two loops known:**
  - DY: Davies, Stirling, 1984
  - Higgs: de Florian, Grazzini, 2000

- **Obey Casimir scaling to the known perturbative order. Two loops:**
  - Gehrmann, Lubbert, L.L.Yang (2012,2014)
  - Echevarria, Scimemi, Vladimirov (2015)
  - Luebbert, Oredsson, Stahlhofen (2016)

Three-loop knowledge of rapidity anomalous dimension important for reduce perturbative uncertainty, and may shed light on non-perturbative large b behavior

- Hard function (form factor) free from rapidity evolution

- Consistency relation between Beam and soft function

$$\nu \frac{d}{d\nu} \left[B B S_{\perp} \right] = 0 \qquad \nu \text{ rapidity evolution scale}$$

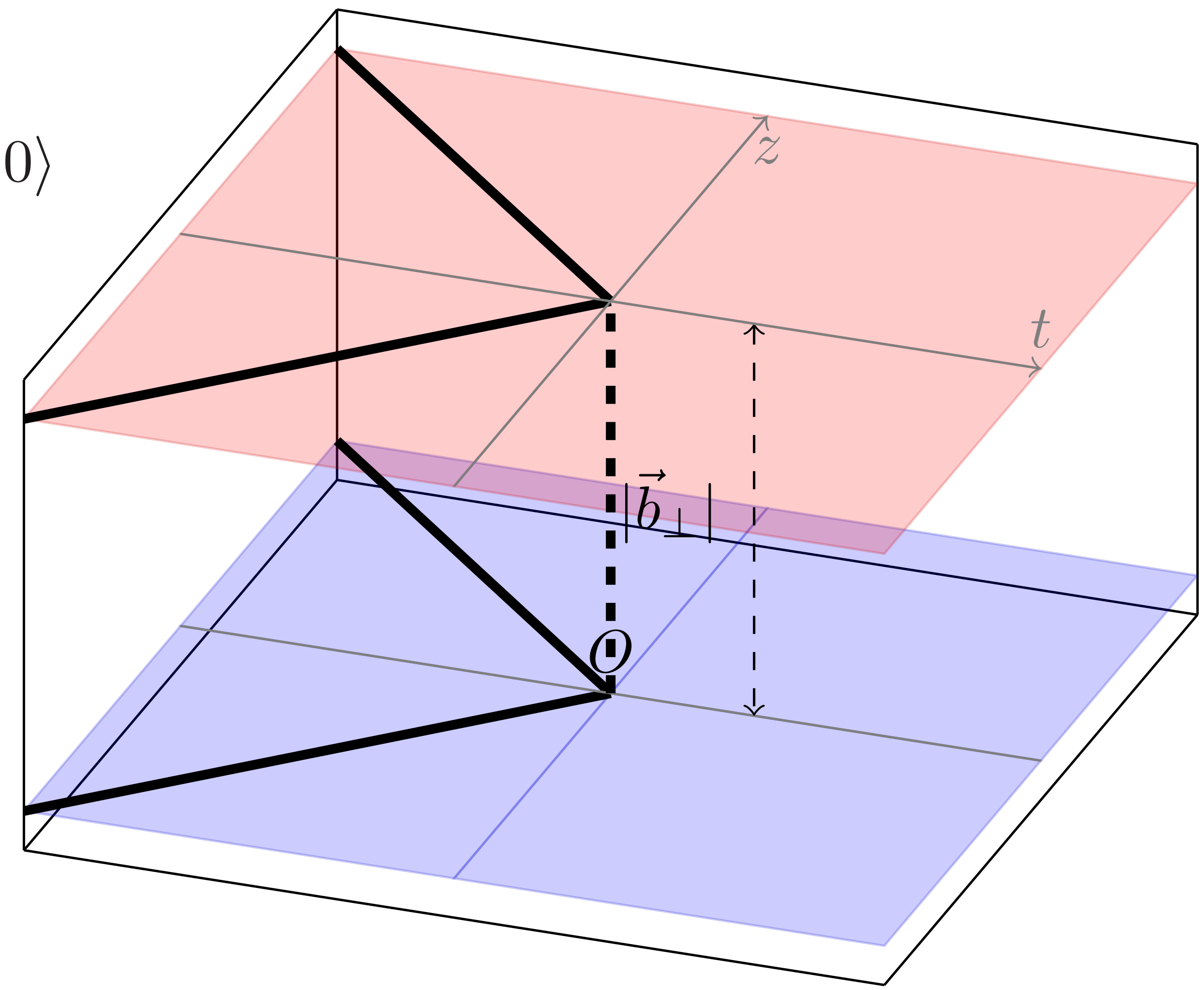
- Can compute either Beam function or soft function to obtain rapidity anomalous dimension

- The calculation would be simplest using soft function - vev. of light-like Wilson loop.

- Problem: light-cone singularity not regularized by dimensional regularization (problem also presented in the beam function)

(un-regulated) Rapidity singularity

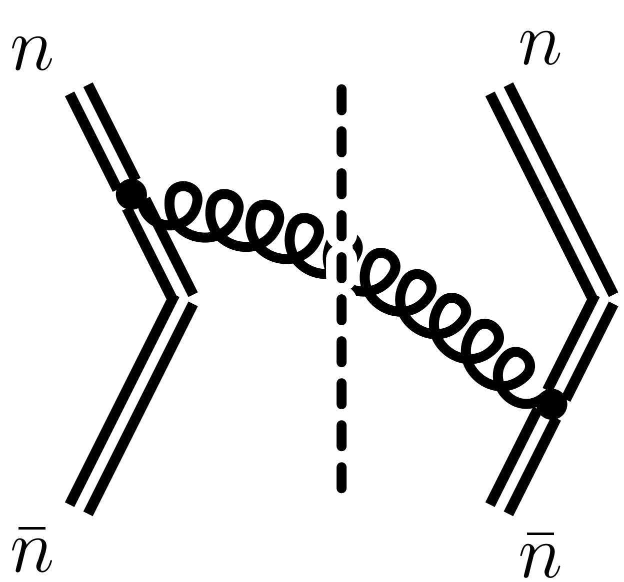
$$S_\perp = \frac{\text{Tr}}{C}\langle 0|T\{S_{\bar n}^\dagger S_n(0,0,0)\}\overline{T}\{S_n^\dagger S_{\bar n}(0,0,\vec b_\perp)|0\rangle$$



$S_{n,\bar n}$ light-like Wilson line to -∞

invariant under arbitrary z boost

one-loop example:



$$\sim \int dx_a\, dx_b D_+(x_{ab}^2)$$

$$\sim \int_0^\infty dt_1 \int_0^\infty dt_2 \frac{1}{(t_1 t_2 + \vec b_\perp^2)^{1-\epsilon}}$$

rapidity divergence In momentum space:

$$\int_0^\infty \frac{dk^+}{k^+}$$

- **Several rapidity regulators have been proposed**
  - **Tilting the Wilson line off light cone:** Ji, Ma, Yuan (2004); Collins (2011)
  - **analytic regulator:** Becher, Neubert (2009); Becher, Bell (2011); **two-loop calculation:** Gehrmann, Lubbert, Yang (2012,2014)

$$\int d^d k \to \int d^d k \left(\frac{\nu}{k^+}\right)^\alpha$$

  - **delta regulator (mass regulator):** Echevarria, Idilbi and Scimemi (2011); **two-loop calculation:** Echevarria, Scimemi, Vladimirov (2015)

$$\frac{1}{k^+ + i\varepsilon} \to \frac{1}{k^+ + \delta}$$

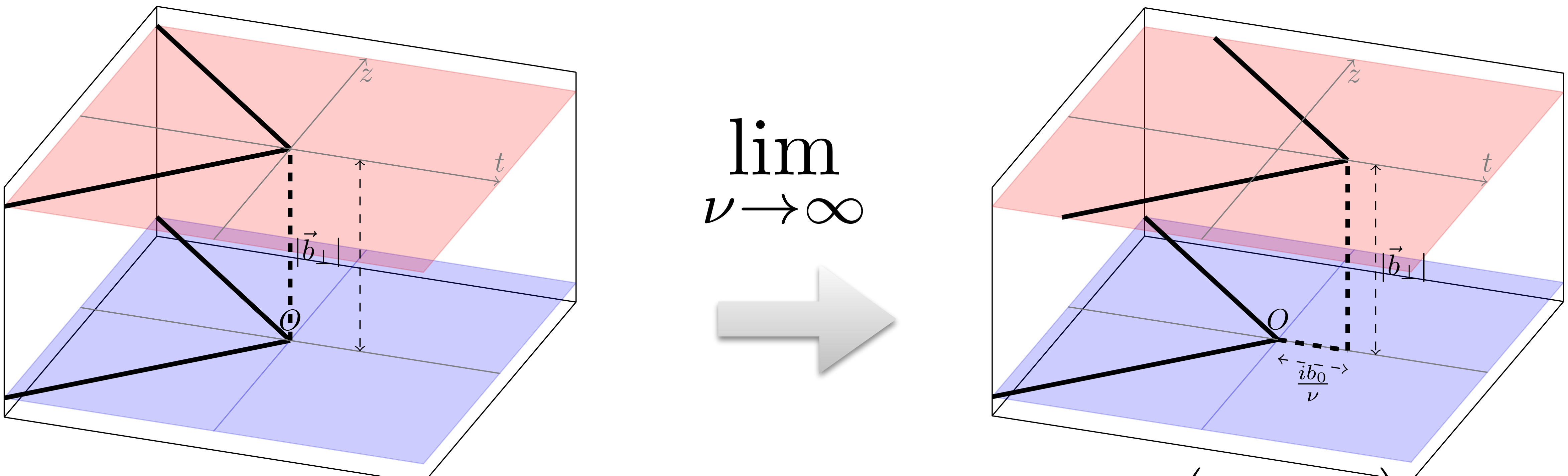
  - **rapidity renormalization group:** Chiu, Jain, Neill, Rothstein (2011,2012); **two-loop calculation:** Luebbert, Oredsson, Stahlhofen (2016)

$$\int d^d k \to \int d^d k \left(\frac{\nu}{|k_z|}\right)^\eta$$

A new regulator for rapidity divergence

1604.00392, Y. Li, Neill, HXZ

✦ The regulator: an infinitesimal shift to in Euclidean time



✦ In momentum space: $\prod_i \int \frac{d^{d-1}k_i}{2k_i^0(2\pi)^3} \Rightarrow \left(\prod_i \int \frac{d^{d-1}k_i}{2k_i^0(2\pi)^3}\right) \exp\left(-\sum_j \frac{b^0 k_j^0}{\nu}\right)$

✦ Manifestly preserve gauge symmetry and Non-Abelian exponentiation theorem.

✦ Logarithmic like singularity log(v). Don't need O(v) terms

✦ Have operator definition. Possible to put on Lattice

Relation to other soft function: threshold

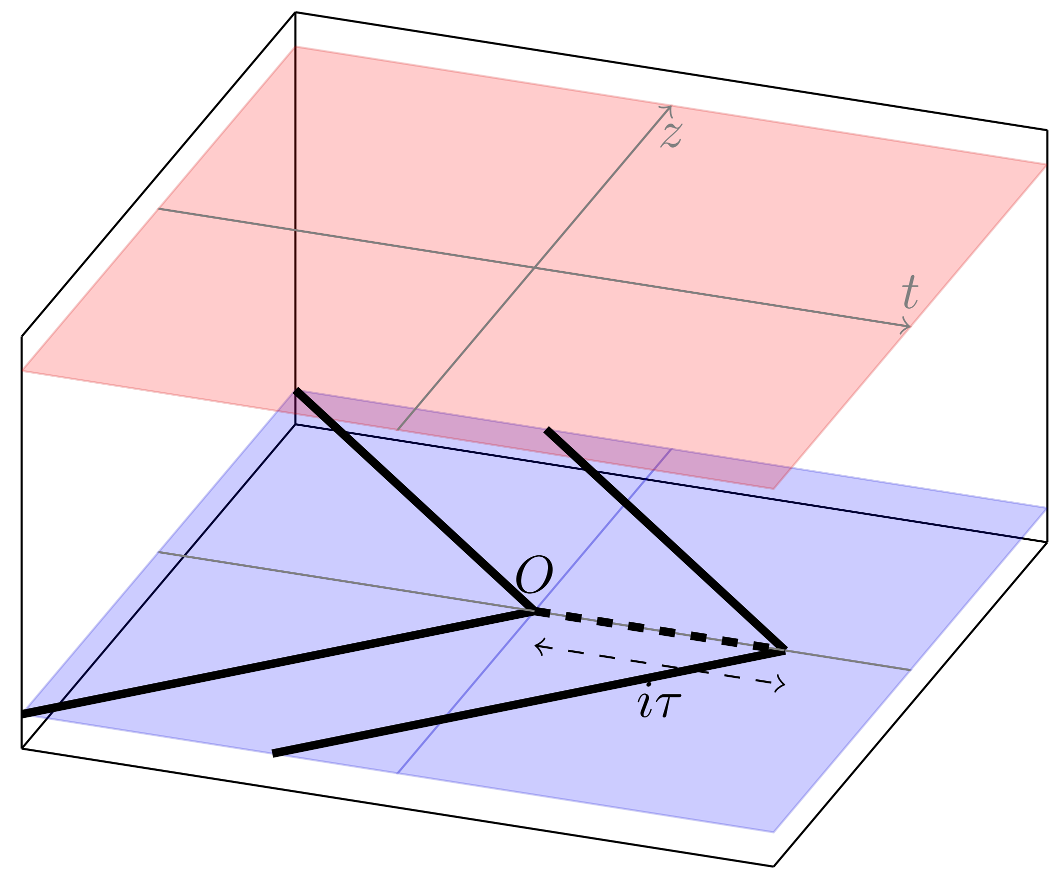
- Light-like Wilson loop separated in Euclidean time only

$$S_{\text{thr.}} = \frac{\text{Tr}}{C} \langle 0 | T\{S^\dagger_{\bar{n}} S_n(0,0,0)\} \overline{T}\{S^\dagger_n S_{\bar{n}}(i\tau, i\tau, 0)|0\rangle$$

$$\sigma = \tau \int \frac{dx}{x}\frac{dz}{z} f_1(x) f_2(\tau/x/z)\hat{\sigma}(z)$$

$$\hat{\sigma}(z) \sim \delta(1-z) + \alpha_s \left[\frac{\ln(1-z)}{1-z}\right]_+ + \cdots$$

$$1 - z = 1 - \frac{Q^2}{\hat{s}} \simeq 2\frac{k_s^0}{Q} + \ldots$$



- Useful for resummation of large logarithms of (1-z) in partonic cross section of Drell-Yan and Higgs production

Korchemsky, Marchesini, 1993
Becher, Neubert, Xu, 2007

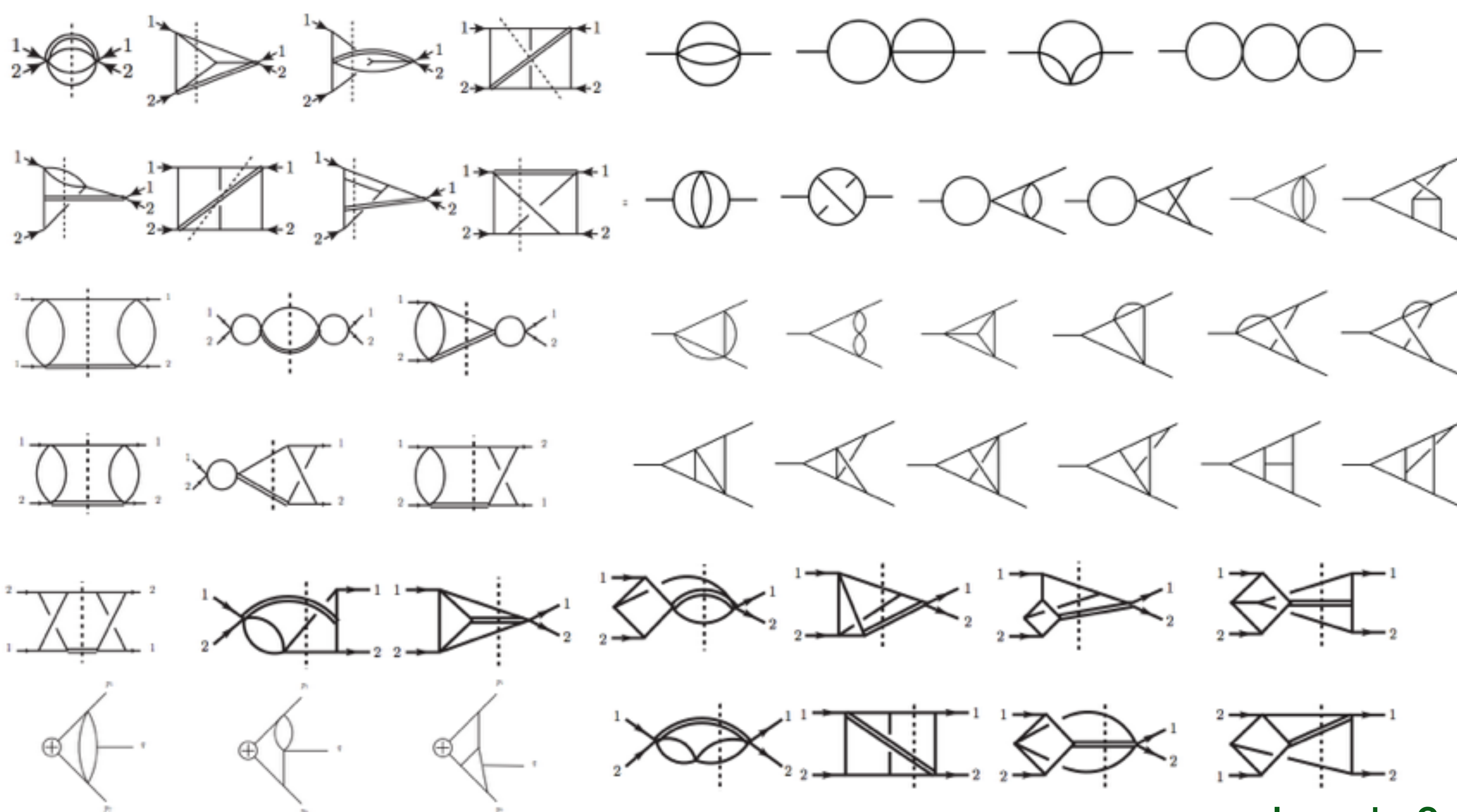


Image by C. Duhr

- All three-loop integrals for threshold soft function known

Anastasiou et al, 2015; Y. Li et al, 2014

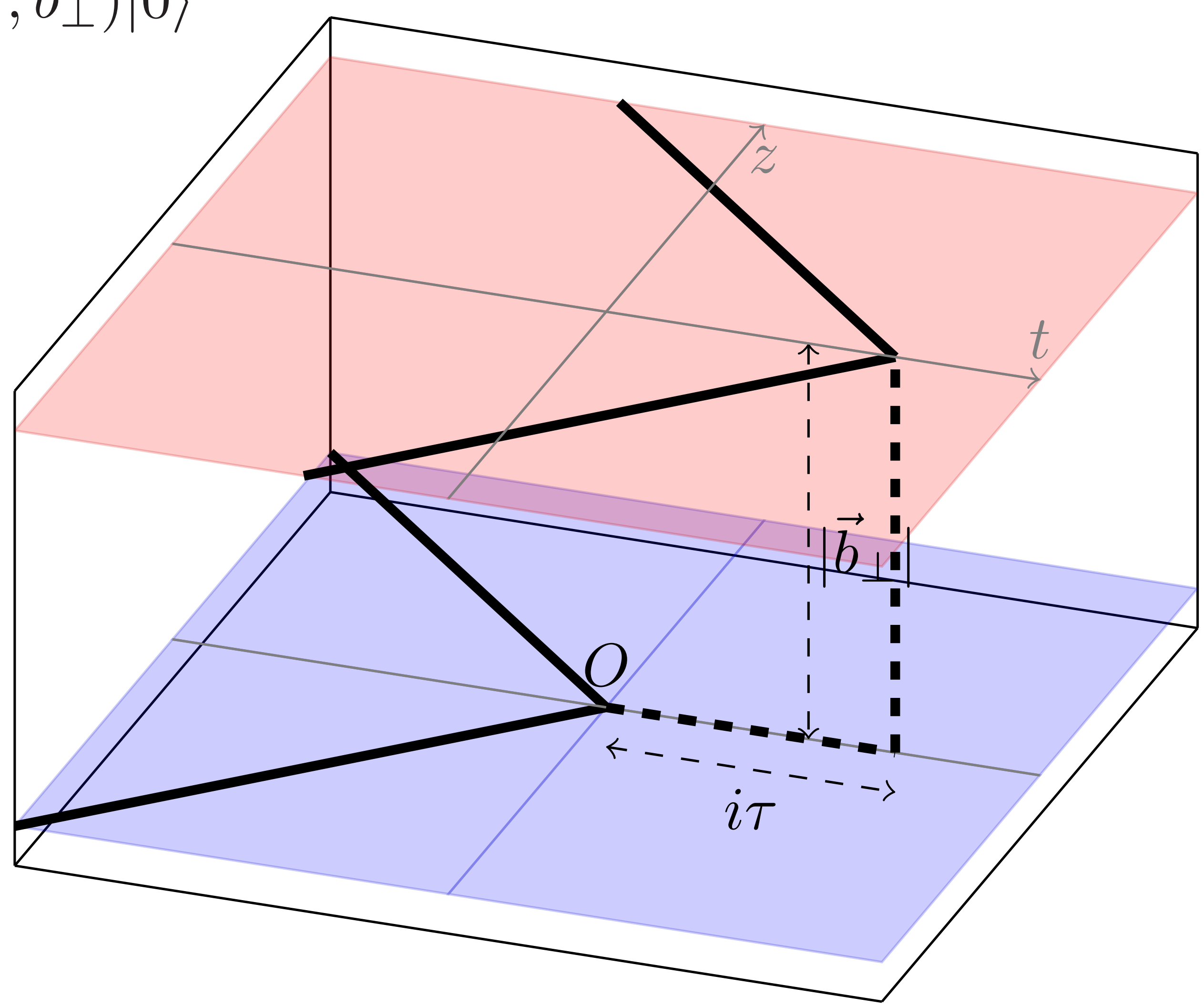
- Building block for Higgs production at N3LO

Anastasiou et al, 2015

Relation to other soft function: fully differential

- Light-like Wilson loop separated both in time and transverse spatial direction **Laenen, Sterman, Vogelsang, 2000; Mantry, Petriello, 2009**

$$S_{\text{F.D.}} = \frac{\text{Tr}}{C} \langle 0 | T\{S_{\bar{n}}^\dagger S_n(0,0,0)\} \overline{T}\{S_n^\dagger S_{\bar{n}}(i\tau, i\tau, \vec{b}_\perp) | 0 \rangle$$

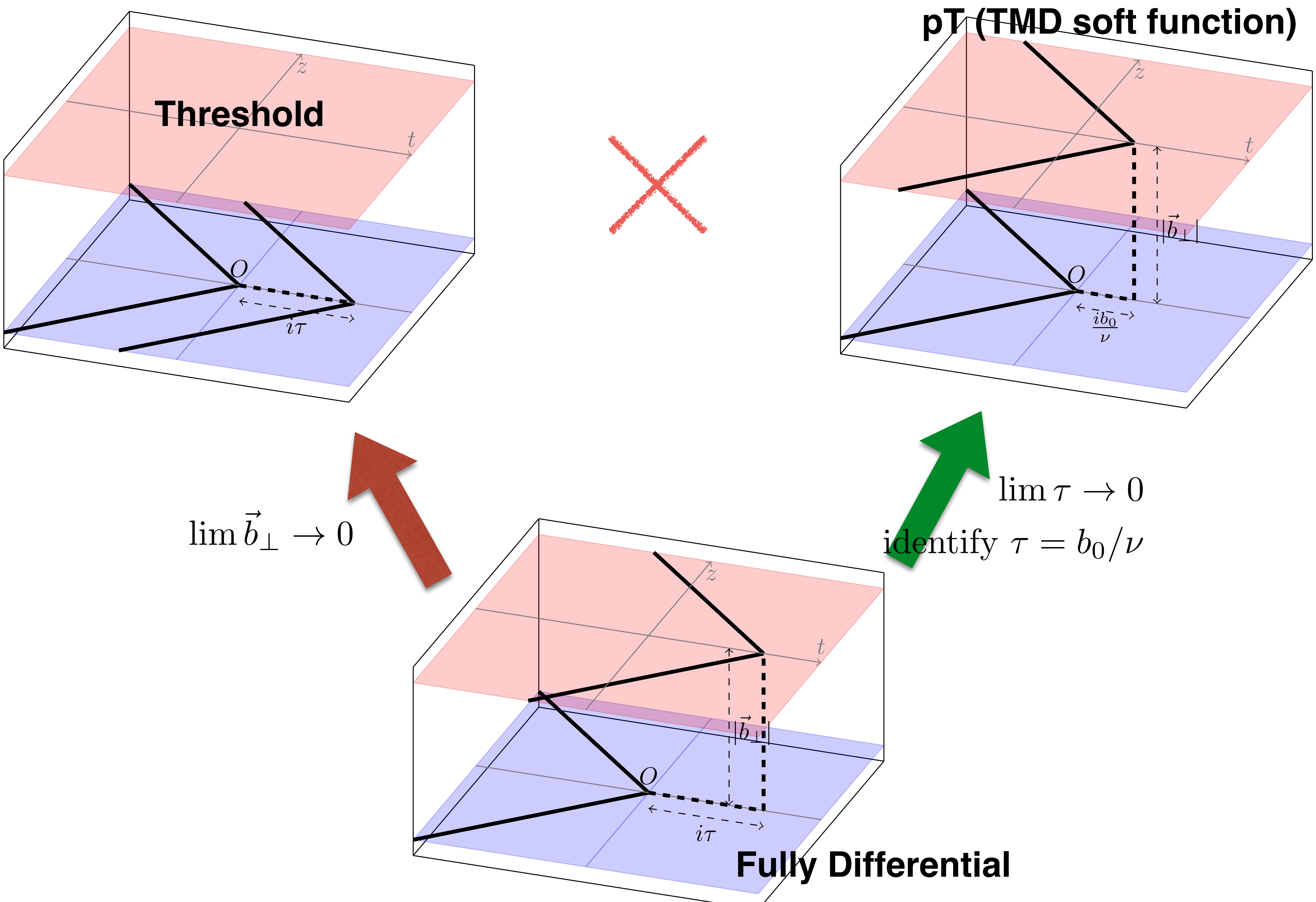


- Fully differential soft function free from rapidity divergence
- Useful for joint resummation **H.-n Li,98; Laenen, Sterman, Vogelsang, 2000; Lustermans, Waalewijn, Zeune, 05**
- Non-trivial dependence on dimensionless ratio

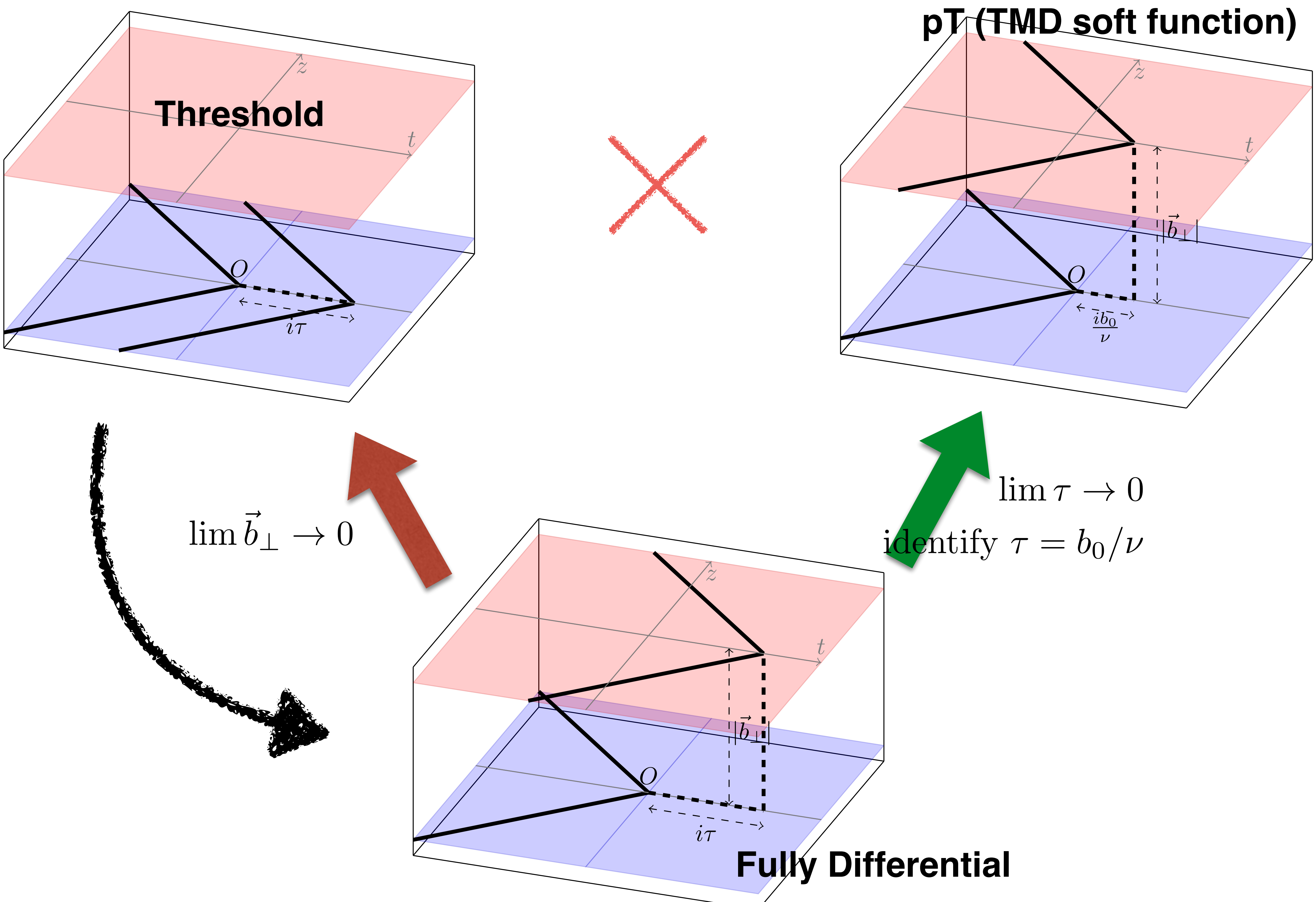
$$x = \frac{\vec{b}_\perp^2}{(i\tau)^2}$$

- Known to two loops **Y. Li, Mantry, Petriello, 2011**

An almost triangular relations



An almost triangular relations



Fully Differential soft function in N=4 SYM

$$S_{\text{F.D.}} = \exp\left\{\sum_{i=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{i+1}\left[\frac{\Gamma_i^{\text{cusp}}}{2}L_\tau^2 - \gamma_i^s L_\tau + c_{i+1}^{\text{F.D.}}(x)\right]\right\}$$

$$L_\tau = \ln\frac{\tau^2}{b_0^2\mu^2}$$

$$x = \frac{\vec{b}_\perp^2}{(i\tau)^2}$$

- The μ dependent part fixed by RG equation

$$c_1^{\text{F.D.}} = 4N_c H_{0,1}(x) + c_{1,\mathcal{N}=4}^s$$

$$c_2^{\text{F.D.}} = N_c^2\Big[-8\zeta_2 H_{0,1}(x) - 8H_{0,0,0,1}(x) - 8H_{0,1,0,1}(x) - 16H_{0,0,1,1}(x) - 16H_{0,1,1,1}(x)\Big] + c_{2,\mathcal{N}=4}^s$$

- Maximal transcendental weight at each order

- HPLs with 0 first entry, 1 last entry. Suggest a simple ansatz on three loops

- Constraint from single logarithmic rapidity divergence at each order

$$x \to -\infty$$

Using threshold soft function as boundary data

- Expanding around the zero-impact parameter limit (b=0)

$$S_{\rm F.D.} = \frac{\rm Tr}{C}\langle 0|T\{S_{\bar n}^\dagger S_n(0,0,0)\}\overline{T}\{S_n^\dagger S_{\bar n}(i\tau,i\tau,\vec b_\perp)|0\rangle$$
$$= \frac{\rm Tr}{C}\langle 0|T\{S_{\bar n}^\dagger S_n(0,0,0)\}\int d^{d_\perp}y_\perp \delta^{(d_\perp)}(y_\perp)\sum_{n=0}^{\infty}\frac{1}{n!}\left(b_\perp^\mu\cdot\frac{\partial}{\partial y_\perp^\mu}\right)^n\overline{T}\{S_n^\dagger S_{\bar n}(i\tau,i\tau,\vec y_\perp)|0\rangle$$

- Implement the expansion in momentum space

$$-i\frac{\partial}{\partial y_\perp^\mu}\to k_\perp^\mu = \sum_{i\in\text{on-shell parton}} k_{i,\perp}^\mu$$

- Rotational invariance in the transverse plane

$$(-i\vec b_\perp\cdot\vec k_\perp)^{2m} = f(2m)\left(\vec b_\perp^2\right)^m(k^+k^- - k^2)^m\ ; \quad f(2m) = (-1)^m\frac{1\cdot 3\cdot 5\ldots(2m-1)}{d_\perp\cdot(d_\perp+2)\cdot(d_\perp+4)\ldots(d_\perp+2m-2)}$$

- IBP reduction to known 3-loop integral. Obtain data up to

$$x^{17} = \left(\frac{\vec b_\perp^2}{(i\tau)^2}\right)^{17}$$

F.D. soft function at three loops in N=4 SYM

Y. Li, HXZ, 1604.01404

$$S_{\text{F.D.}} = \exp\left\{\sum_{i=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{i+1}\left[\frac{\Gamma_i^{\text{cusp}}}{2}L_\tau^2 - \gamma_i^s L_\tau + c_{i+1}^{\text{F.D.}}(x)\right]\right\}$$

$$c_1^{\text{F.D.}} = 4N_c H_{0,1}(x) + c^s_{1,\mathcal{N}=4}$$

one and two loops

$$c_2^{\text{F.D.}} = N_c^2\Big[-8\zeta_2 H_{0,1}(x) - 8H_{0,0,0,1}(x) - 8H_{0,1,0,1}(x) - 16H_{0,0,1,1}(x) - 16H_{0,1,1,1}(x)\Big] + c^s_{2,\mathcal{N}=4}$$

three-loop scale independent part

$$c^s_{3,\mathcal{N}=4} + N_c^3\Big(16\zeta_2 H_4 + 48\zeta_2 H_{2,2} + 64\zeta_2 H_{3,1} + 96\zeta_2 H_{2,1,1} + 120\zeta_4 H_2 + 48H_6 + 24H_{2,4} + 40H_{3,3}$$
$$+ 72H_{4,2} + 128H_{5,1} + 16H_{2,1,3} + 56H_{2,2,2} + 80H_{2,3,1} + 80H_{3,1,2} + 144H_{3,2,1} + 224H_{4,1,1}$$
$$+ 64H_{2,1,1,2} + 96H_{2,1,2,1} + 160H_{2,2,1,1} + 256H_{3,1,1,1} + 192H_{2,1,1,1,1}\Big)$$

- Uniform and maximal degree of transcendentality

  Anomalous dimension, form factor, momentum space Wilson loop

- Coefficients are integers

- Alternating/uniform sign and each loop order

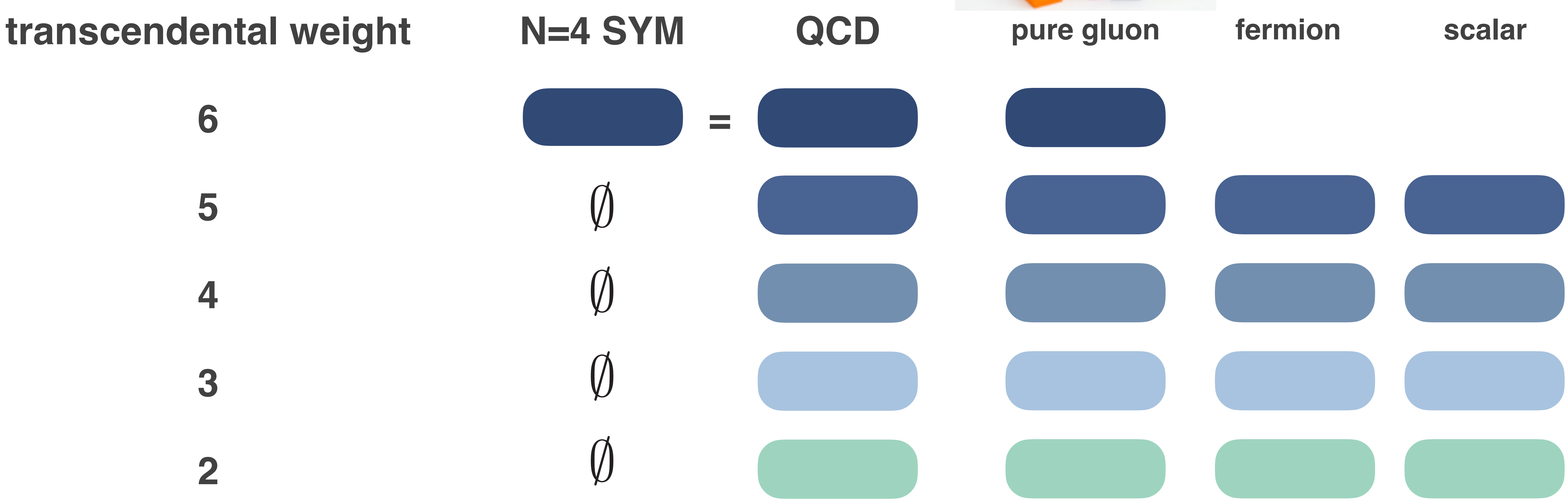
  also see cusp anomalous dimension, Henn, Huber, 2013

QCD = ([N=4]) + (QCD - [N=4])

- N=4 SYM Also “predict” maximal transcendental part of QCD
  Kotikov, Lipatov, Velizhanin, 2003

- Knowing the maximal transcendental part significantly reduce the undetermined coefficient to be fixed

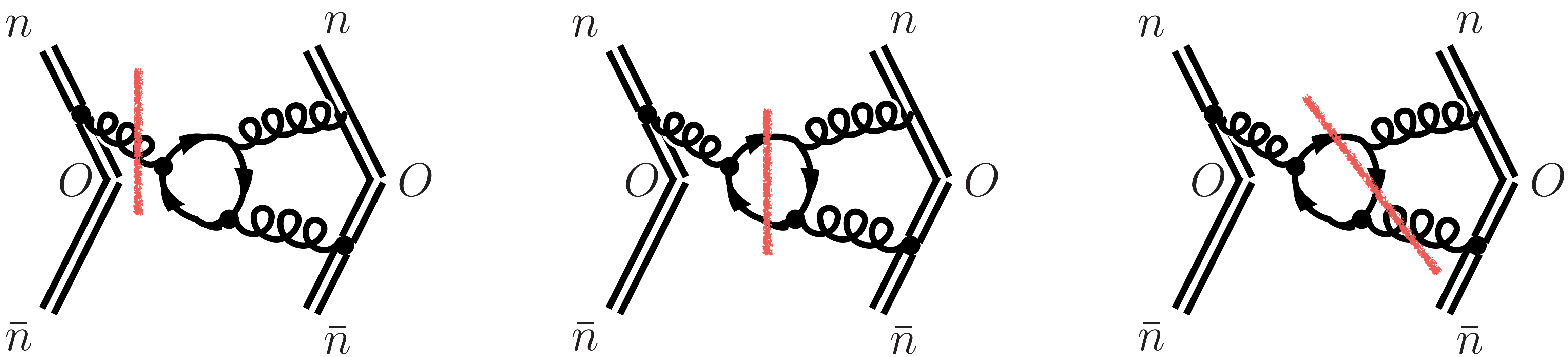
- QCD from deconstructing N=4 SYM



[N=4 SYM] = 1 gluon + 4 majorana fermion + 3 complex scalar

Directly integrating Nf matter part

$$S_{\mathrm{F.D.}} = \frac{\mathrm{Tr}}{C}\sum_X \langle 0|T\{S^\dagger_{\bar{n}} S_n(0)\}X|\rangle e^{-\tau k^0_X + i b_\perp \cdot k_{X,\perp}} \langle X|\overline{T}\{S^\dagger_n S_{\bar{n}}(0)|0\rangle$$



Example of the most complicated diagrams

- New functions appear in the double cut and triple cut contribution

$$H_1(x) - \frac{H_1(x)}{x} \qquad H_{11}(x) - \frac{H_{11}(x)}{x} \qquad \frac{H_{01}(x)}{x} \qquad \zeta_2 H_1(x) - H_{101}(x)$$

- Cancel in the sum of different cuts. Only one additional term survive in the final result

The QCD results to three loops

RG equation $$\frac{d\ln S_{\mathrm{F.D.}}(\vec{b}_\perp,\tau,\mu)}{d\ln\mu^2} = \Gamma_{\mathrm{cusp}}\left[\alpha_s(\mu)\right]\ln\frac{\tau^2\mu^2}{b_0^2} - \gamma_s\left[\alpha_s(\mu)\right]$$

$4C_aH_2$ one loop

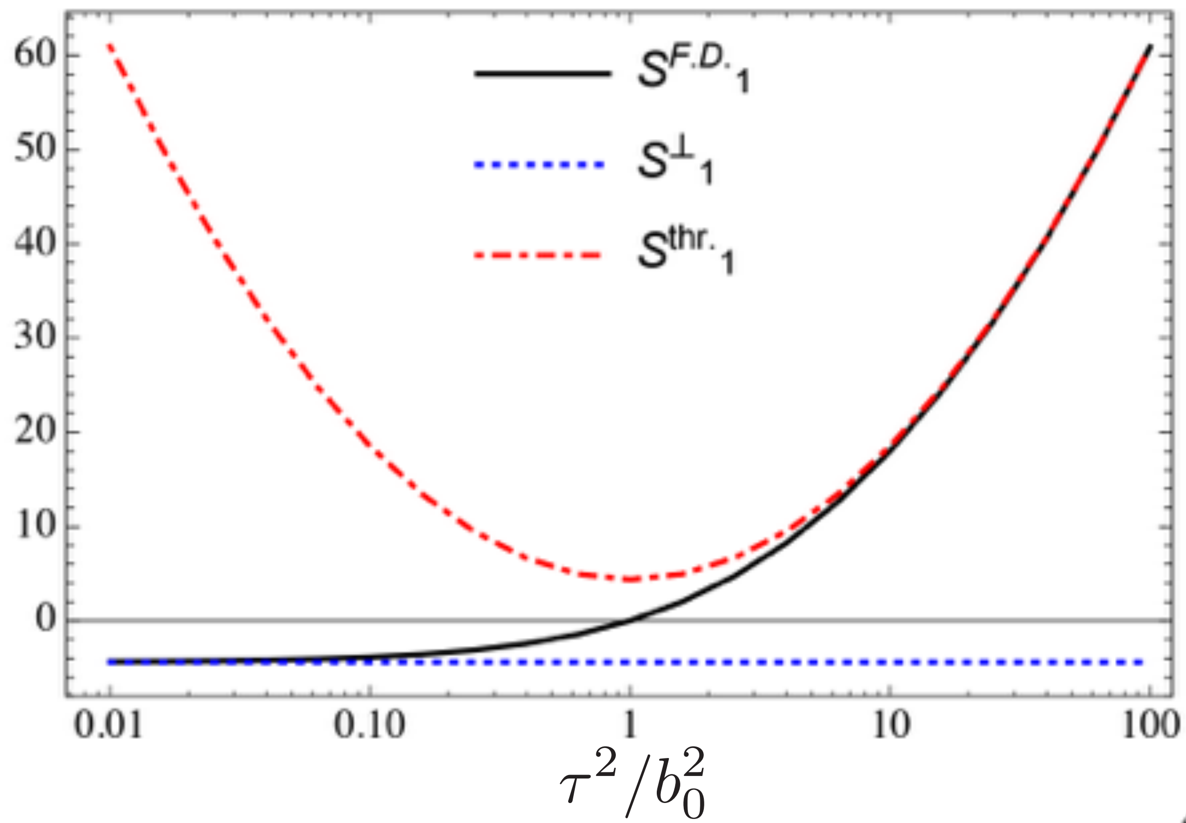
$$C_AC_a\left(-8\zeta_2H_2+\frac{268}{9}H_2+\frac{44}{3}H_3-8H_4-\frac{44}{3}H_{2,1}-8H_{2,2}-16H_{3,1}-16H_{2,1,1}\right)$$
$$+C_an_f\left(-\frac{40}{9}H_2-\frac{8}{3}H_3+\frac{8}{3}H_{2,1}\right)$$

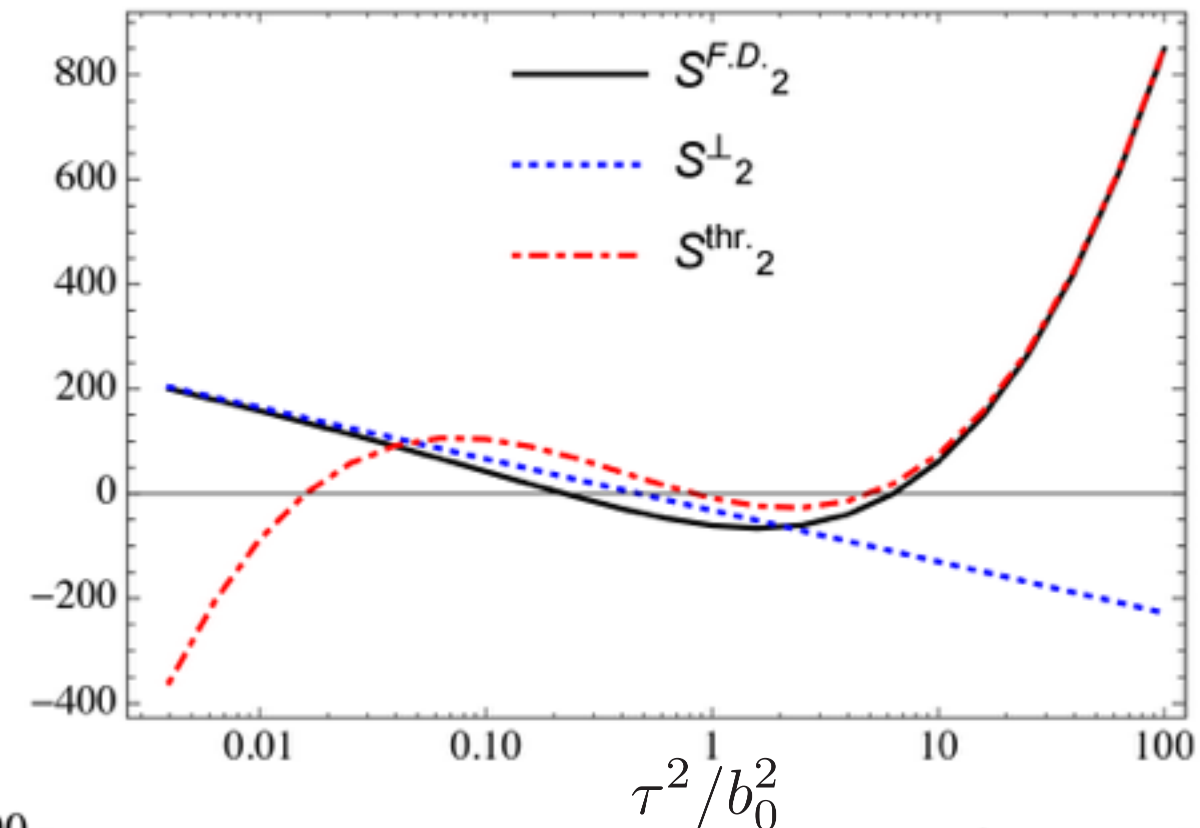
two loop

N=4SYM (maximal transcendental part)

$$+C_aC_A^2\left[-\frac{1072}{9}\zeta_2H_2-176\zeta_3H_2-\frac{88}{3}\zeta_2H_3+88\zeta_2H_{2,1}\right.$$
$$+\frac{30790}{81}H_2+\frac{7120}{27}H_3-\frac{104}{9}H_4-\frac{440}{3}H_5-\frac{8}{3}\left(H_{1,1}-\frac{H_{1,1}}{x}\right)-\frac{7120}{27}H_{2,1}-\frac{1072}{9}H_{2,2}-\frac{88}{3}H_{2,3}$$
$$\left.-\frac{3112}{9}H_{3,1}-88H_{3,2}-\frac{352}{3}H_{4,1}-\frac{392}{3}H_{2,1,1}+\frac{88}{3}H_{2,1,2}+\frac{352}{3}H_{2,2,1}+\frac{352}{3}H_{3,1,1}+352H_{2,1,1,1}\right]$$
$$+C_aC_An_f\left[\frac{160}{9}\zeta_2H_2+\frac{16}{3}\zeta_2H_3-16\zeta_2H_{2,1}-\frac{7988}{81}H_2-\frac{2312}{27}H_3-\frac{64}{3}H_4+\frac{80}{3}H_5+\frac{8}{3}\left(H_{1,1}-\frac{H_{1,1}}{x}\right)\right.$$
$$+\frac{2312}{27}H_{2,1}+\frac{160}{9}H_{2,2}+\frac{16}{3}H_{2,3}+\frac{224}{3}H_{3,1}+16H_{3,2}+\frac{64}{3}H_{4,1}-\frac{32}{9}H_{2,1,1}-\frac{16}{3}H_{2,1,2}-\frac{64}{3}H_{2,2,1}$$
$$\left.-\frac{64}{3}H_{3,1,1}-64H_{2,1,1,1}\right]+C_an_f^2\left(\frac{400}{81}H_2+\frac{160}{27}H_3+\frac{32}{9}H_4-\frac{160}{27}H_{2,1}-\frac{32}{9}H_{3,1}+\frac{32}{9}H_{2,1,1}\right)$$
$$+C_aC_Fn_f\left(32\zeta_3H_2-\frac{110}{3}H_2-8H_3+8H_{2,1}\right) \quad (8)$$

Three loop

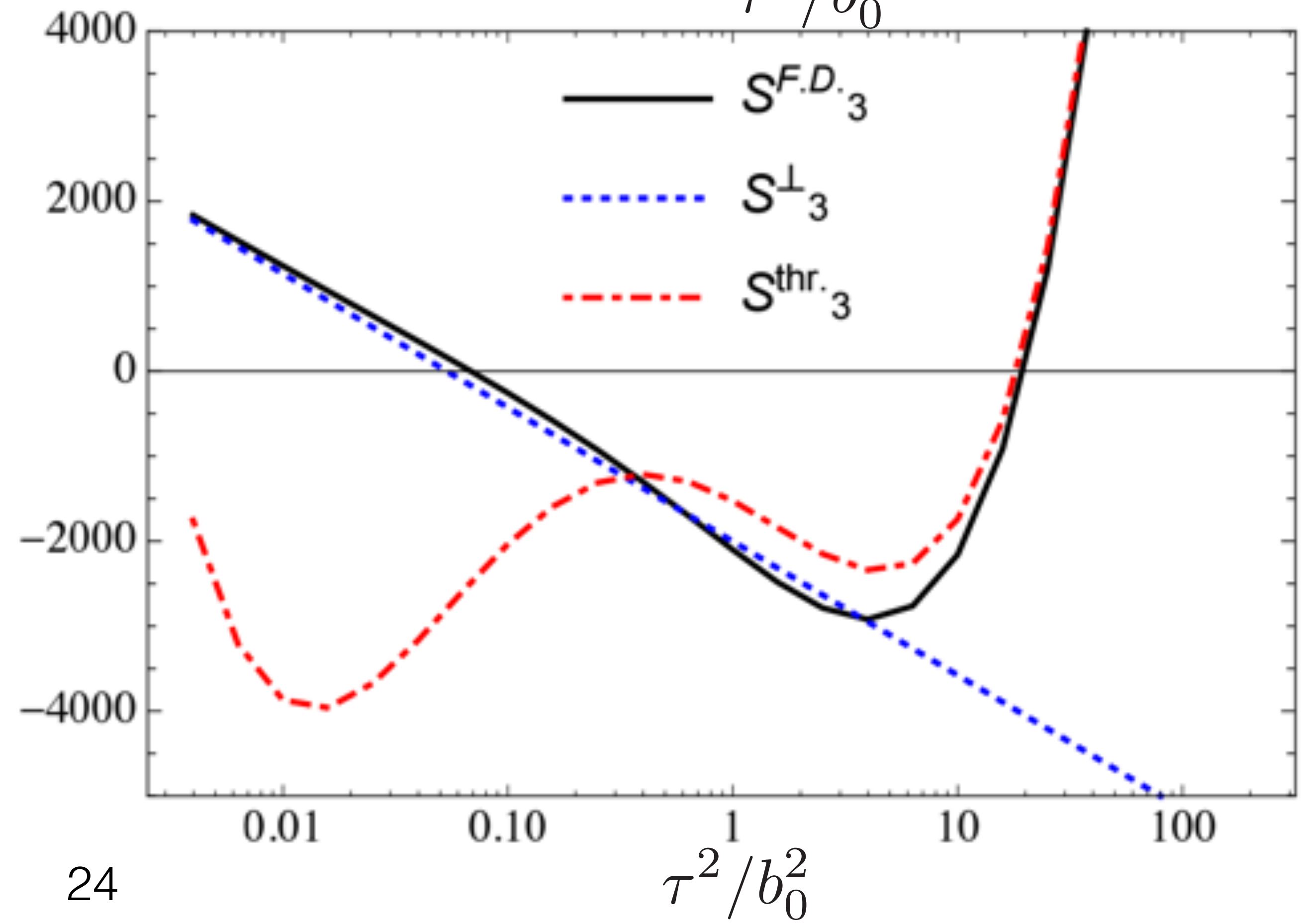




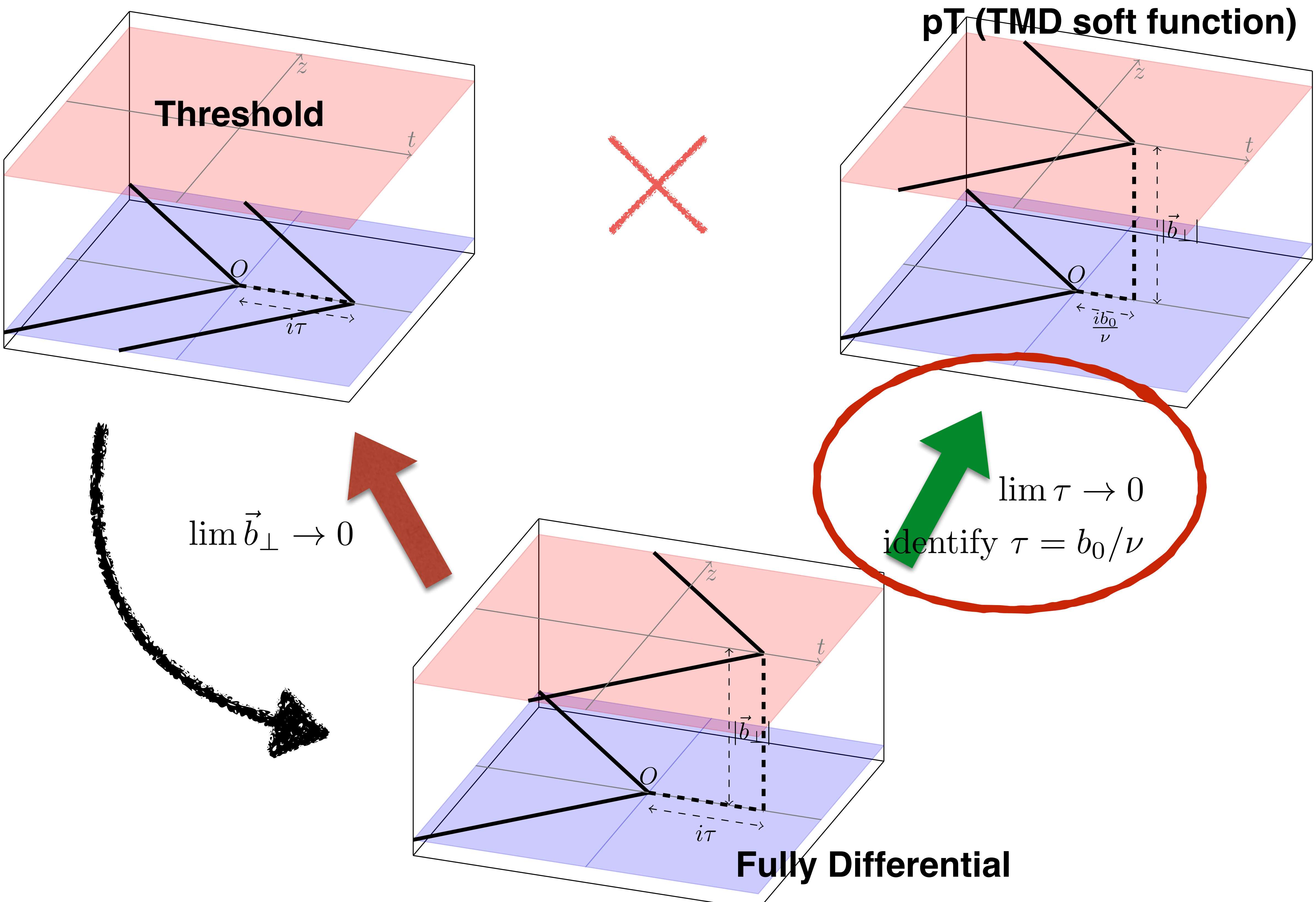
- Fix renormalization scale

$$\mu^2 = b_0^2/\vec{b}_\perp^{\,2}$$

- Use the color factor for Drell-Yan process



An almost triangular relations



Rapidity anomalous dimension @ 3 loop

The rapidity renormalization group

Chiu, Jain, Neill, Rothstein, 2012

$$\frac{\mathrm{d}\ln S_\perp(\vec{b}_\perp,\mu,\nu)}{\mathrm{d}\ln\nu^2} = \int_{\mu^2}^{b_0^2/\vec{b}_\perp^2}\frac{\mathrm{d}\bar{\mu}^2}{\bar{\mu}^2}\Gamma_{\mathrm{cusp}}[\alpha_s(\bar{\mu})] + \gamma_r[\alpha_s(b_0/|\vec{b}_\perp|)]$$

$$\frac{d\sigma_{\mathrm{DY}}}{dQ^2 dY d^2\vec{q}_\perp} = \sigma_0\int\frac{d^2\vec{b}_\perp}{(2\pi)^2}e^{i\vec{b}_\perp\cdot\vec{q}_\perp}H(Q,\mu_h)B_q(x_A,Q,\vec{b}_\perp,\mu_b,\nu_b)B_{\bar{q}}(x_B,Q,\vec{b}_\perp,\mu_b,\nu_b)S_\perp(\vec{b}_\perp,\mu_s,\nu_s)$$
$$\cdot\exp\left\{-\int_{b_0^2/\vec{b}_\perp^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\left(\Gamma_{\mathrm{cusp}}[\alpha_s(\bar{\mu})]+\frac{d\gamma^r[\alpha_s(\bar{\mu})]}{d\ln\bar{\mu}^2}\right)\ln\frac{Q^2}{\bar{\mu}^2}+\left(\gamma^V[\alpha_s(\bar{\mu})]-\gamma^r[\alpha_s(\bar{\mu})]\right)\right]\right\}$$

$$\gamma_0^R = 0$$
$$\gamma_1^R = C_a C_A\left(28\zeta_3-\frac{808}{27}\right)+\frac{112 C_a n_f}{27}$$
$$\gamma_2^R = C_a C_A^2\left(-\frac{176}{3}\zeta_3\zeta_2+\frac{6392\zeta_2}{81}+\frac{12328\zeta_3}{27}+44\zeta_4-192\zeta_5-\frac{297029}{729}\right)$$
$$+C_a C_A n_f\left(-\frac{824\zeta_2}{81}-\frac{904\zeta_3}{27}+8\zeta_4+\frac{62626}{729}\right)+c\beta_0$$
$$+C_a n_f^2\left(-\frac{32\zeta_3}{9}-\frac{1856}{729}\right)+C_a C_F N_f\left(-\frac{304\zeta_3}{9}-16\zeta_4+\frac{1711}{27}\right)$$

New three loop results!

one and two loops known. Direct calculation:
Luebbert, Oredsson, Stahlhofen (2016)
also extractable from:

- Davies, Webber, Stirling (1985)
- Grazzini, de Florian (2000)
- Gehrmann, Lubbert, Yang (2012,2014)
- Echevarria, Scimemi, Vladimirov (2015)

Intriguing relation between rapidity anomalous dimension and threshold anomalous dimension

Control $\left[\frac{1}{1-z}\right]_+$ **of threshold logarithms**

constant term in threshold soft function

$$\gamma_0^r = \gamma_0^s$$

$$\gamma_1^r = \gamma_1^s - \beta_0 c_1^s$$

$$\gamma_2^r = \gamma_2^s - 2\beta_0 c_2^s - \beta_1 c_1^s + 2C_a C_A \beta_0 \zeta_4$$

Implication for non-perturbative parameterization?

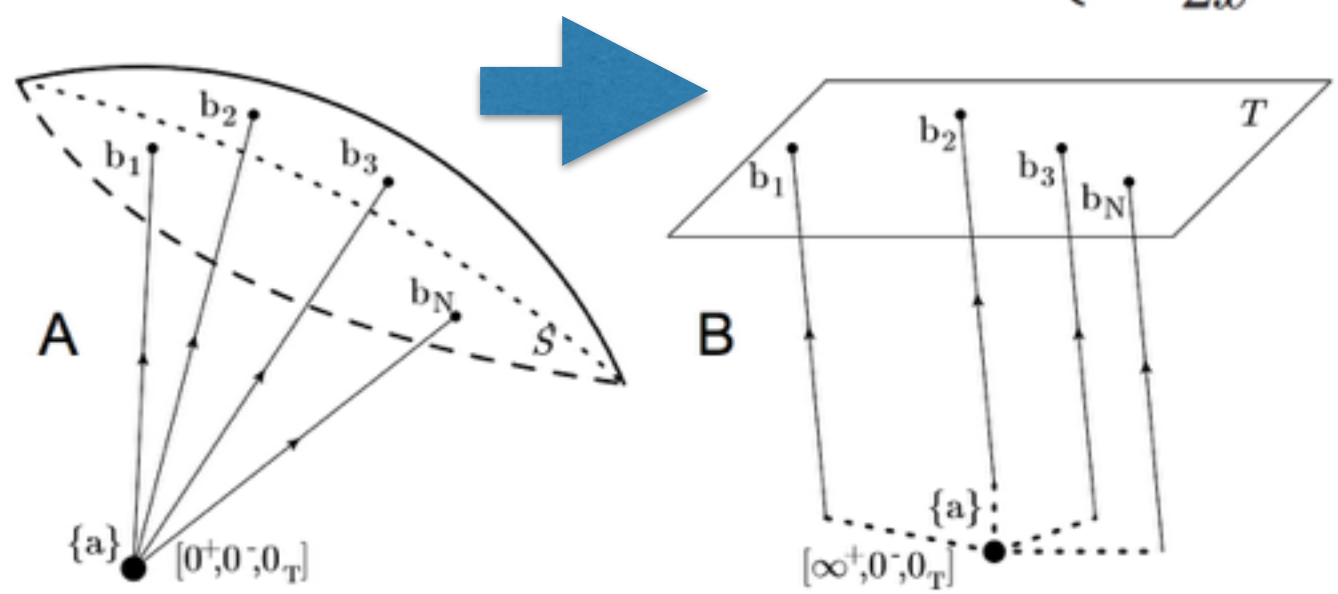
Control $\left[\frac{1}{P_T^2}\right]_*$ **of pT distribution**

Conformal Symmetry and Soft/Rapidity A.D.

A. Validimirov, 1610.05791

- Mapping of hard scattering configuration to TMD config.

stereographic projection $\mathcal{C}:\{x^+,x^-,x_T\}\to\left\{-\frac{1}{2x^+},x^--\frac{x_T^2}{2x^+},\frac{x_T}{\sqrt{2}x^+}\right\}$



- Establish $\gamma_s=\gamma_r$ to all orders in P.T. for any CFT

- Obtain γ_r for QCD in critical dimension with beta function vanishing, confirm the result of direct three calculation

$$\beta(g)=g(-\epsilon-a_s\beta_0-a_s^2\beta_1-...)$$

Small pT cross section for Higgs production

- **There are many different ways to perform pT resummation for Higgs production. We follow** Neill, Rothstein, Vaidya (2015)

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}^2\vec{Q}_T} = \int x_a \int x_b \, \delta\Big(x_a x_b - \frac{m_H^2}{S}\Big)\sigma_0 \int \frac{\mathrm{d}^2\vec{b}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{Q}_T} W\big(x_a, x_b, m_H, \vec{b}, \mu, \nu\big) + \frac{\mathrm{d}^2\sigma}{\mathrm{d}^2\vec{Q}_T}\bigg|_{\mathrm{n.s.}}$$

$$W(x_a, x_b, m_H, \vec{b}, \mu, \nu) = \Big|C_V\big(m_t, m_H, \mu\big)\Big|^2 S(\vec{b}, \mu, \nu) B^{\alpha\beta}_{g/N_1}(x_a, Q, \vec{b}, \mu, \nu) B^{\alpha\beta}_{g/N_2}(x_b, Q, \vec{b}, \mu, \nu)$$

$$C_V(m_t, m_H, \mu) = C_V(m_t, m_H, \mu_H)\exp\left[\frac{1}{2}\int_{\mu_H^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\Gamma_{\mathrm{cusp}}[\alpha_s(\bar{\mu})] \ln\frac{M_H^2}{\bar{\mu}^2} + \gamma^V[\alpha_s(\bar{\mu})]\right)\right]$$

$$B^{\alpha\beta}_{g/N}(x, \vec{b}, Q, \mu, \nu) = \frac{g_\perp^{\alpha\beta}}{d-2} B_{g/N}(x, b, Q, \mu, \nu) + \left(\frac{g_\perp^{\alpha\beta}}{d-2} + \frac{b^\alpha b^\beta}{b^2}\right) B'_{g/N}(x, b, Q, \mu, \nu)$$

$$B_{g/N}(x, b, Q, \mu, \nu) = \sum_j \int_x^1 \frac{\mathrm{d}z}{z} I_{gj}(z, b, Q, \mu, \nu) f_{j/N}(x/z, \mu) + \ldots$$

$$S_\perp(b, \mu, \nu) = S_\perp(b, \mu_s, \nu_s)\exp\left[\int_{\mu_s^2}^{\mu^2} \frac{\mathrm{d}\bar{\mu}^2}{\bar{\mu}^2}\left(\Gamma_{\mathrm{cusp}}[\alpha_s(\bar{\mu})] \ln\frac{b^2\bar{\mu}^2}{b_0^2} + \gamma^s[\alpha_s(\bar{\mu})]\right)\right.$$
$$\left. + \ln\frac{\nu^2}{\nu_s^2}\left(-\int_{b_0^2/b^2}^{\mu^2} \frac{\mathrm{d}\bar{\mu}^2}{\bar{\mu}^2}\Gamma_{\mathrm{cusp}}[\alpha_s(\bar{\mu})] + \gamma^r[\alpha_s(b_0/b)]\right)\right]$$

pT resummation for Higgs production at N3LL

Y. Li, Neill, Schulze, Stewart, HXZ, work in progress

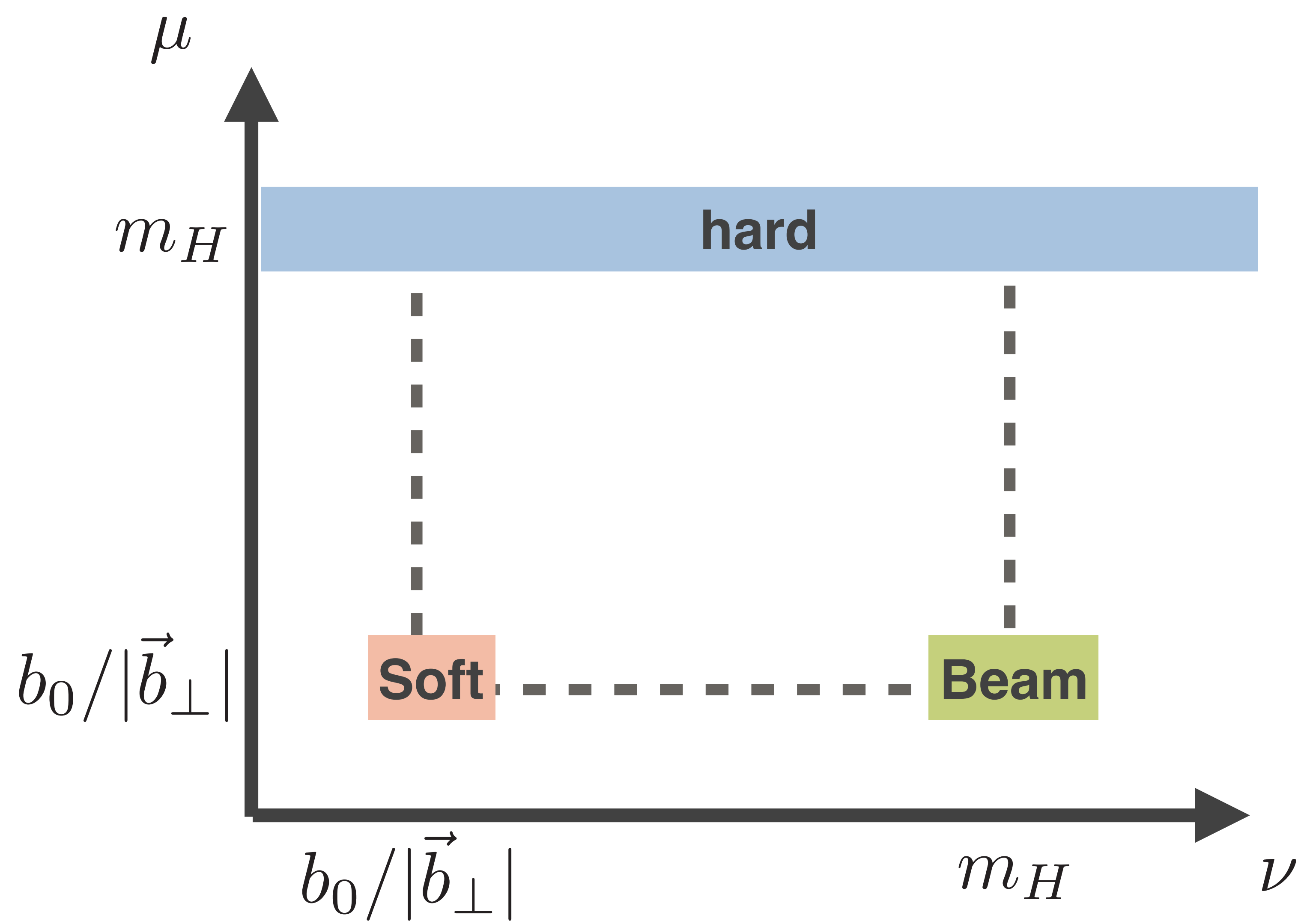
- **Perturbative order of various ingredients:**
  - Two-loop hard function, beam function, soft function
  - Three-loop normal anomalous dimension
  - Three-loop splitting amplitude
  - Three-loop rapidity anomalous dimension (new)
  - Four-loop cusp anomalous dimension (Pade approximation)
- **Resummation performed in b space**
- **Simple b* scheme for non-perturbative effects** $b^* = \frac{b}{\sqrt{1 + b^2/b_{\rm max}^2}}$
- **Light quark mass effects included at fixed order**

Scale setting in the resumed regime

- **Three independent scale variation:**
    - hard μ scale, beam and soft μ scale, soft ν (rapidity) scale

$$\mu_H \in [\frac{1}{4} m_H, m_H]$$

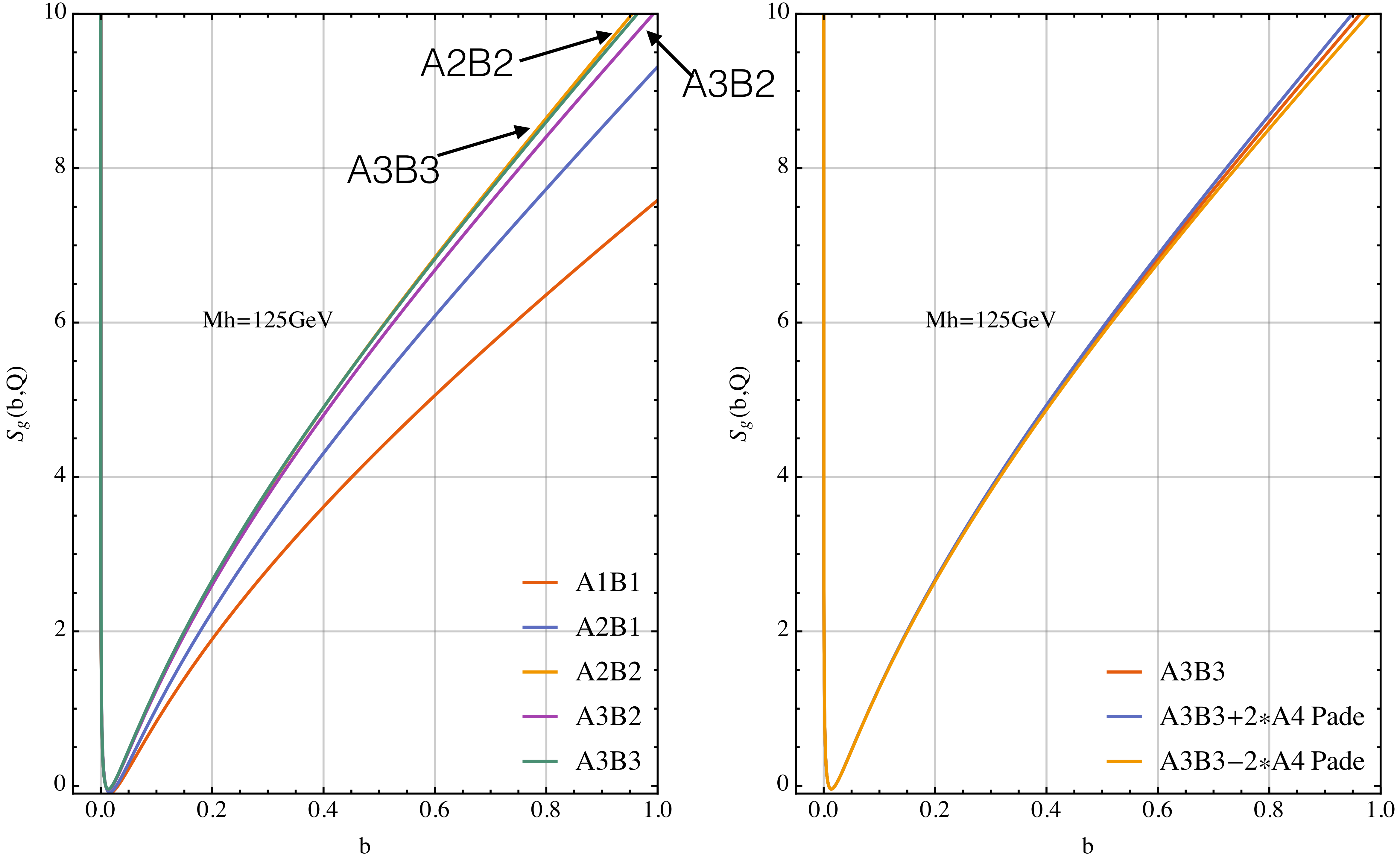
$$\mu_B = \mu_S \in [\frac{1}{2}\frac{b_0}{b_\perp}, 2\frac{b_0}{b_\perp}]$$



$$\nu_S \in [\frac{1}{2}\frac{b_0}{b_\perp}, 2\frac{b_0}{b_\perp}]$$

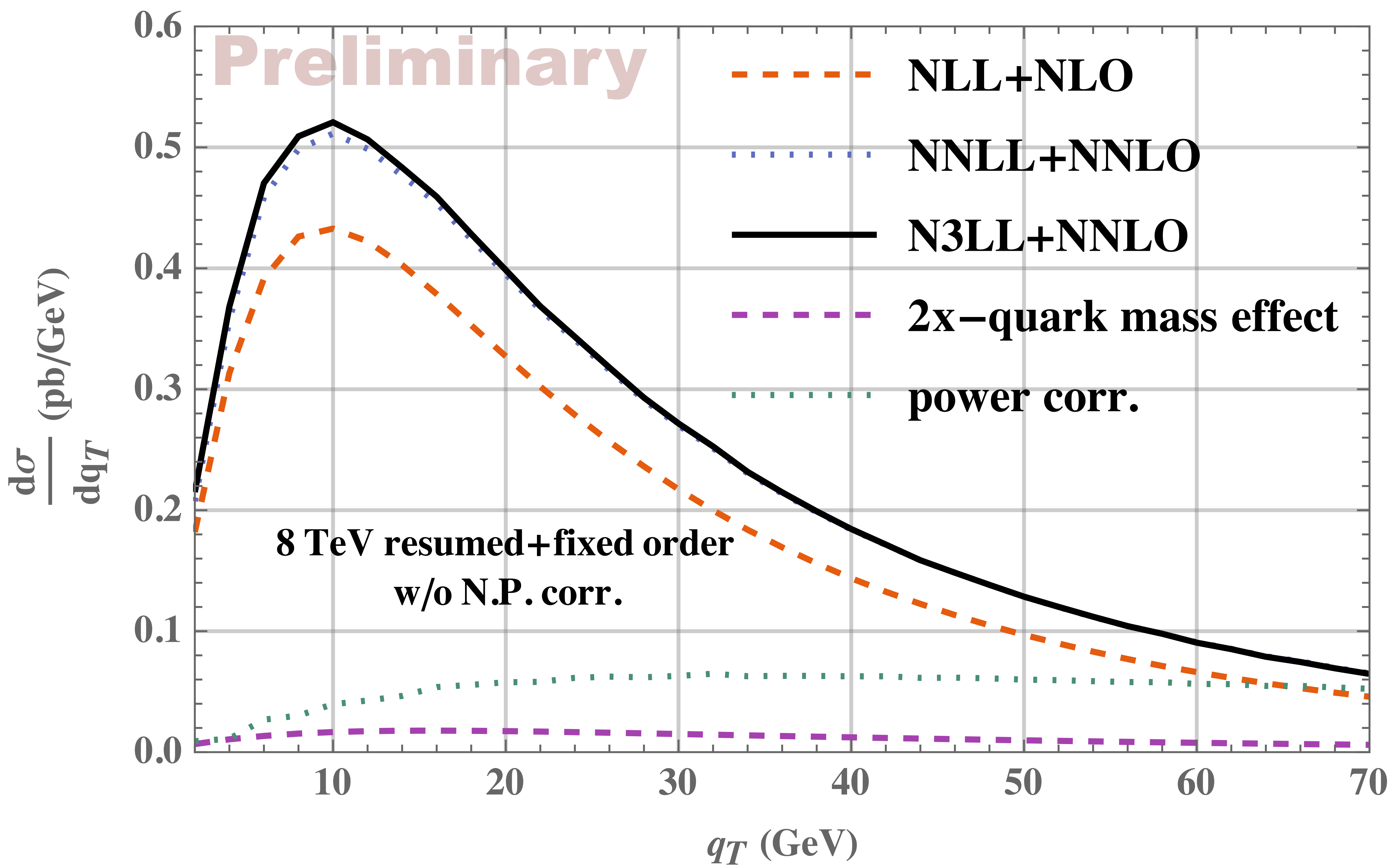
$$\nu_B = m_H$$

$$\sigma \sim \sigma_0 H(f \otimes C) \otimes (f \otimes C) e^{-S_g(b,Q)}$$

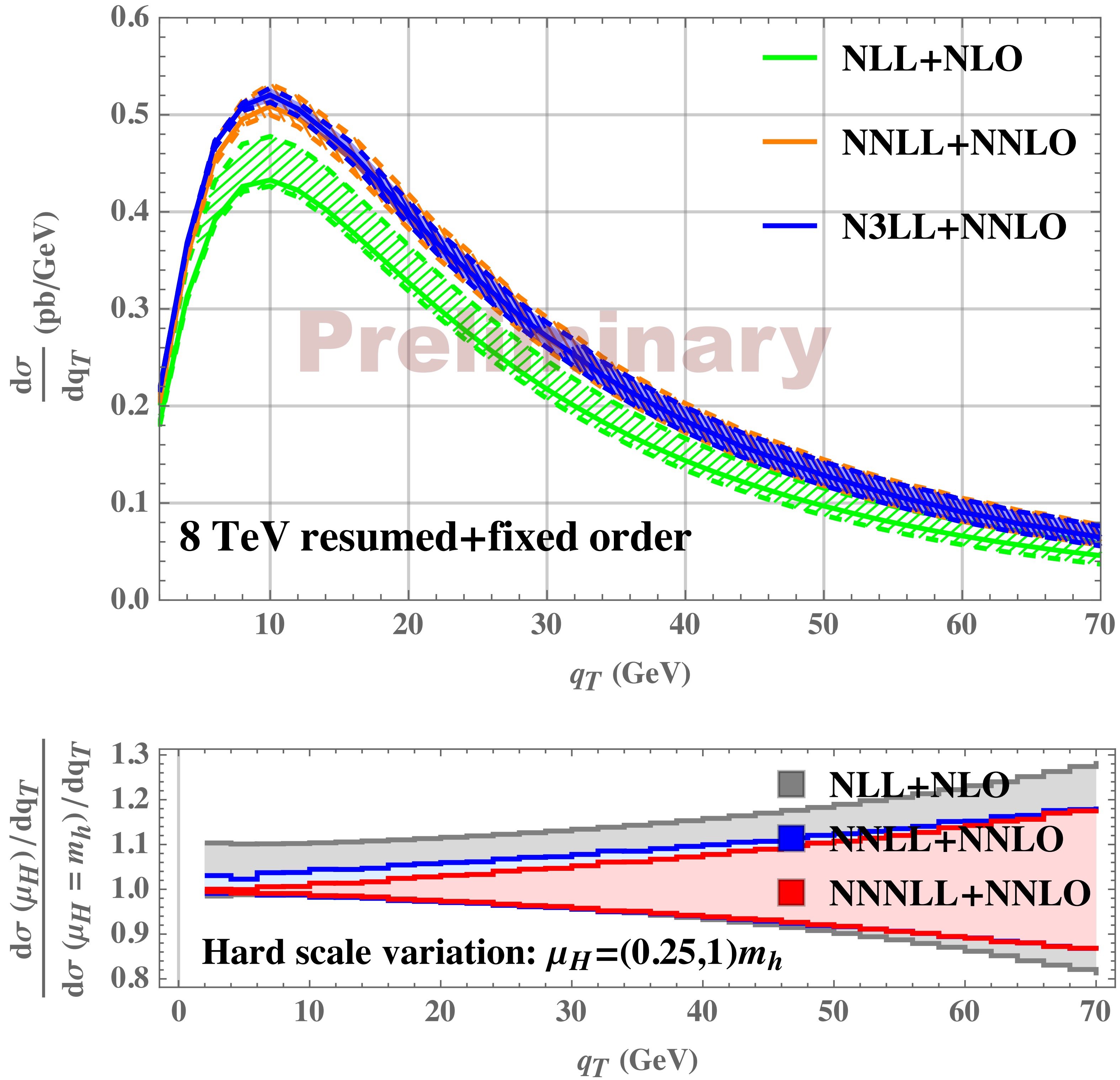


- Large Sudakov suppression at large b. Small N.P. effects. See also [Berger, Qiu, 2002]

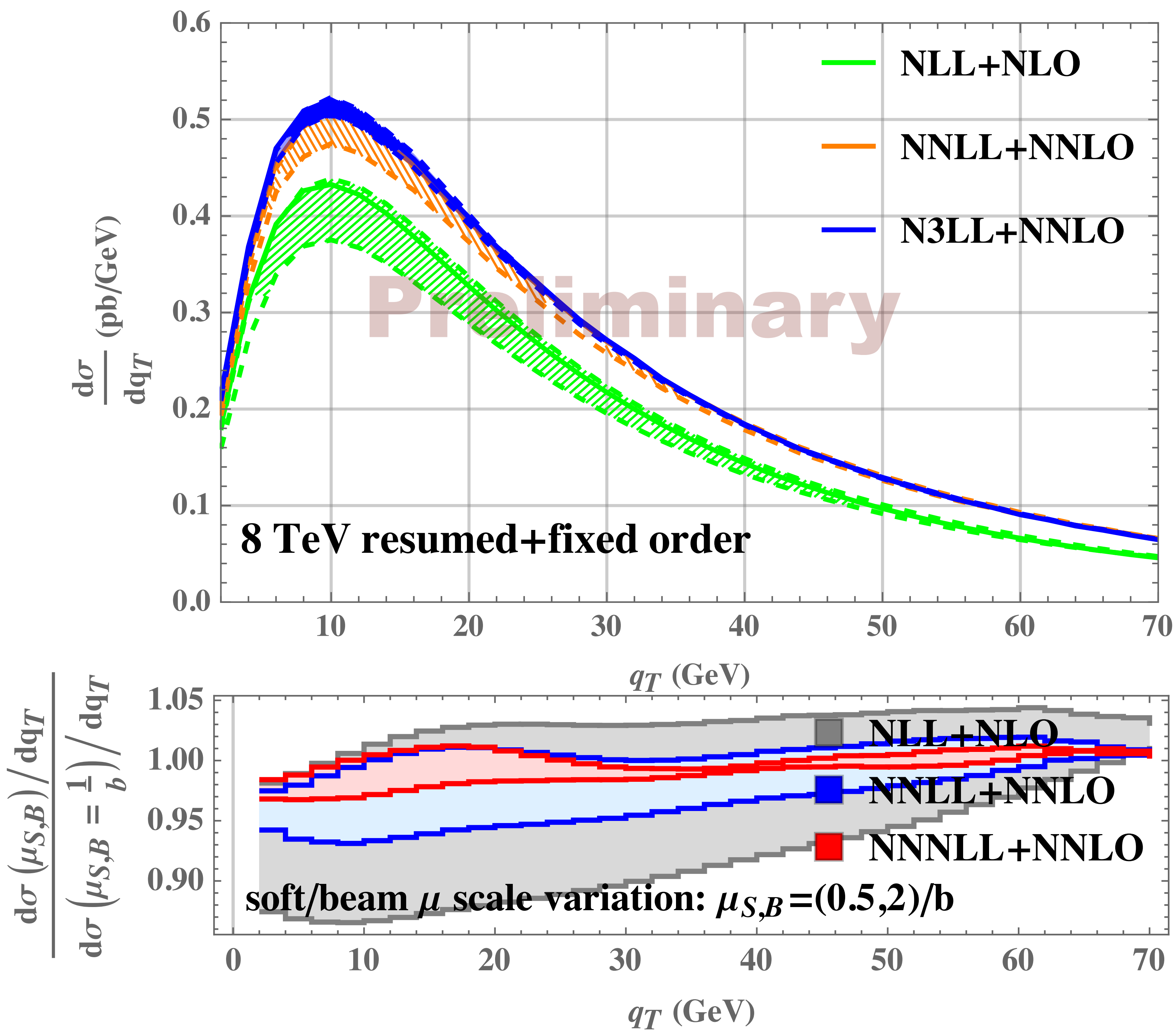
Quark mass effects and perturbative power corrections



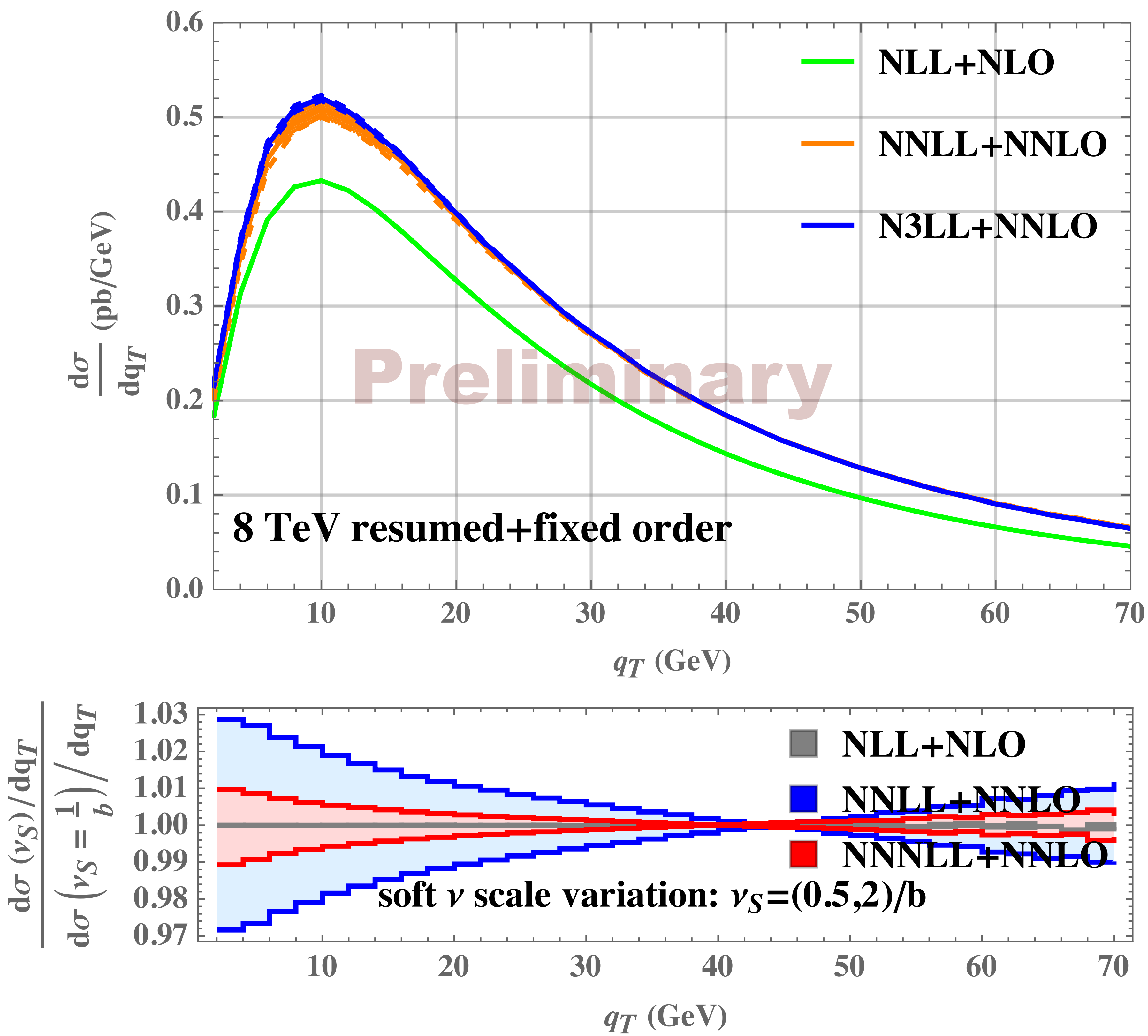
Hard scale variation



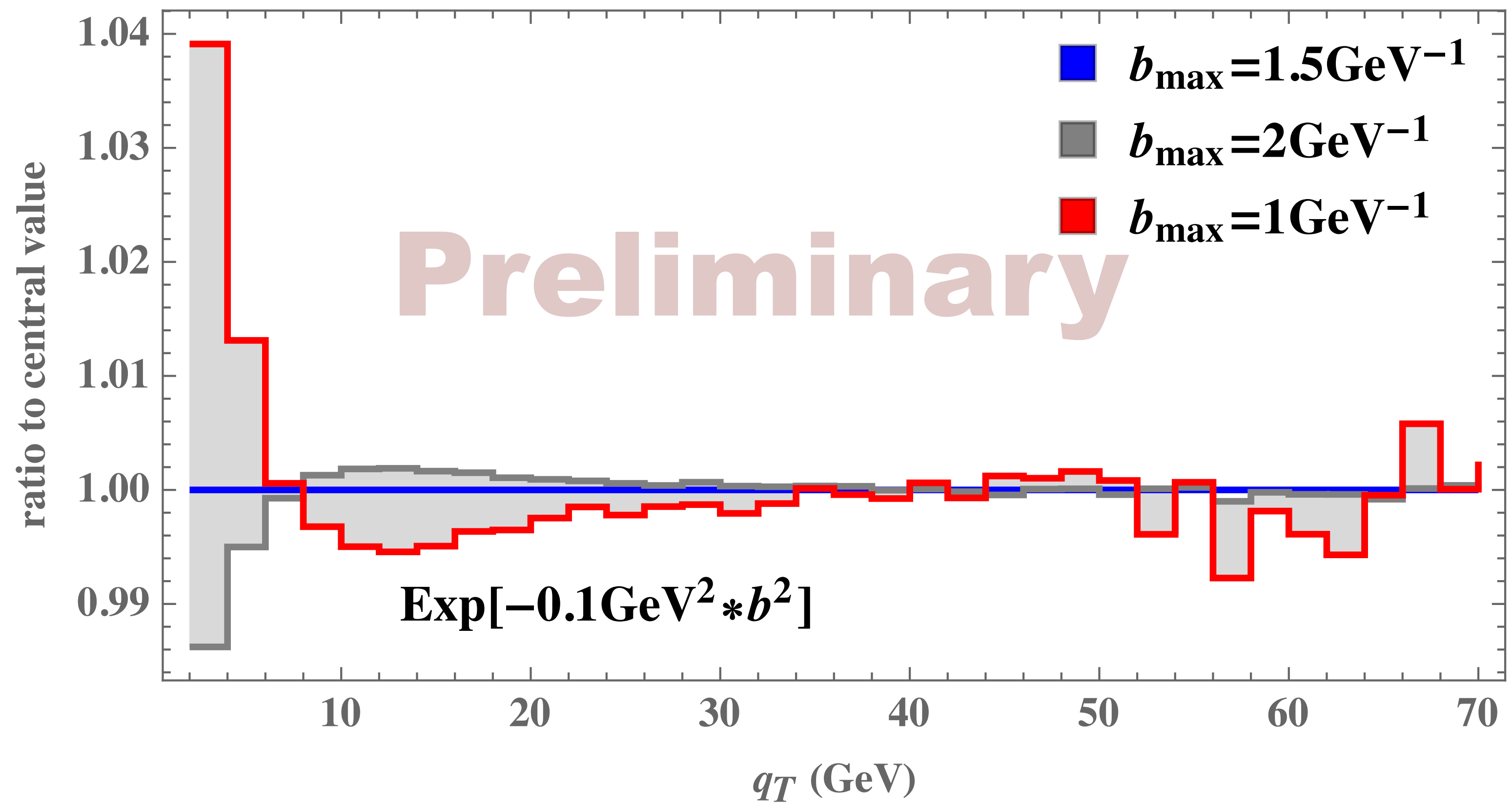
soft/beam renormalization scale variation

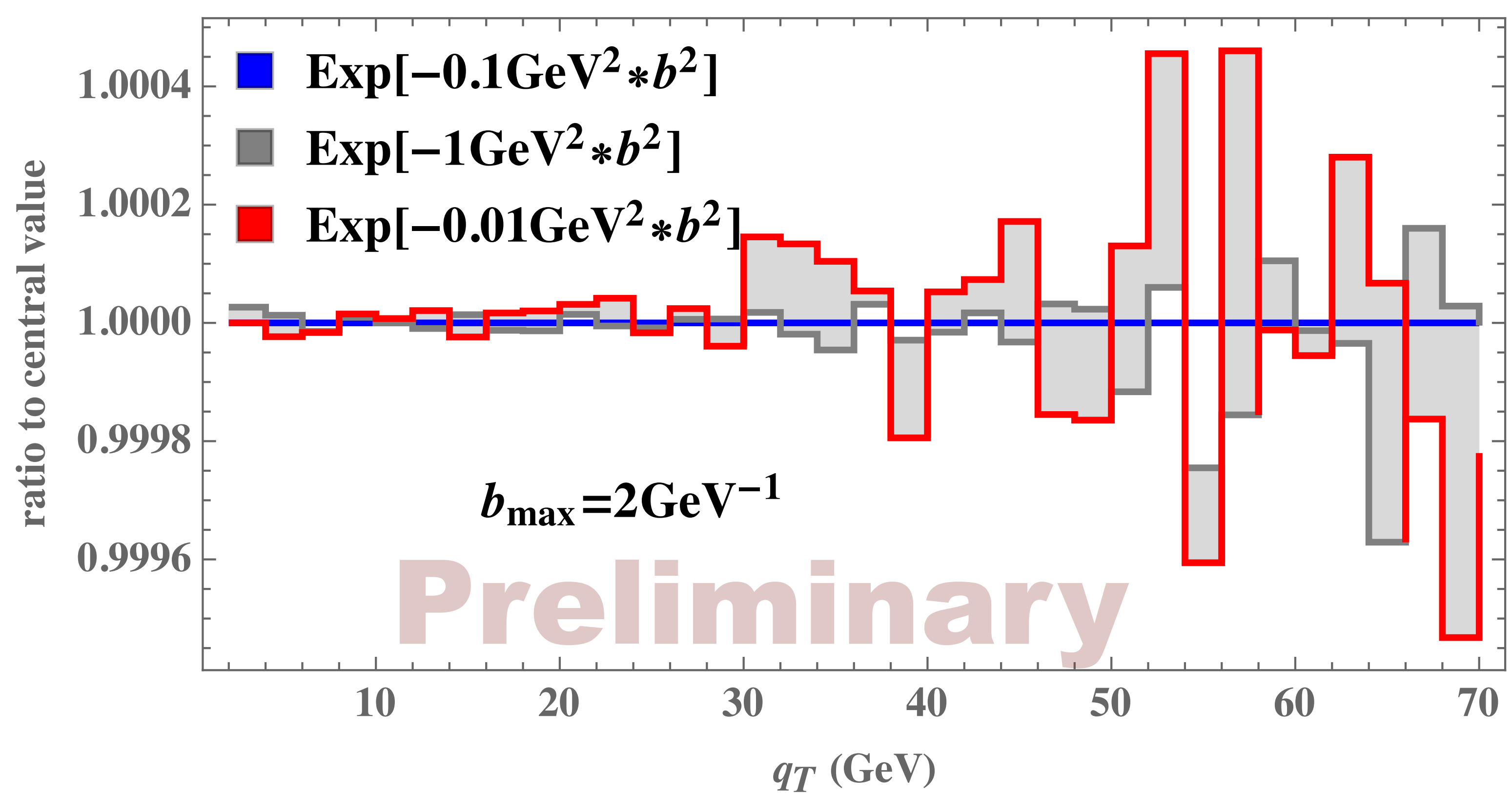


rapidity scale variation

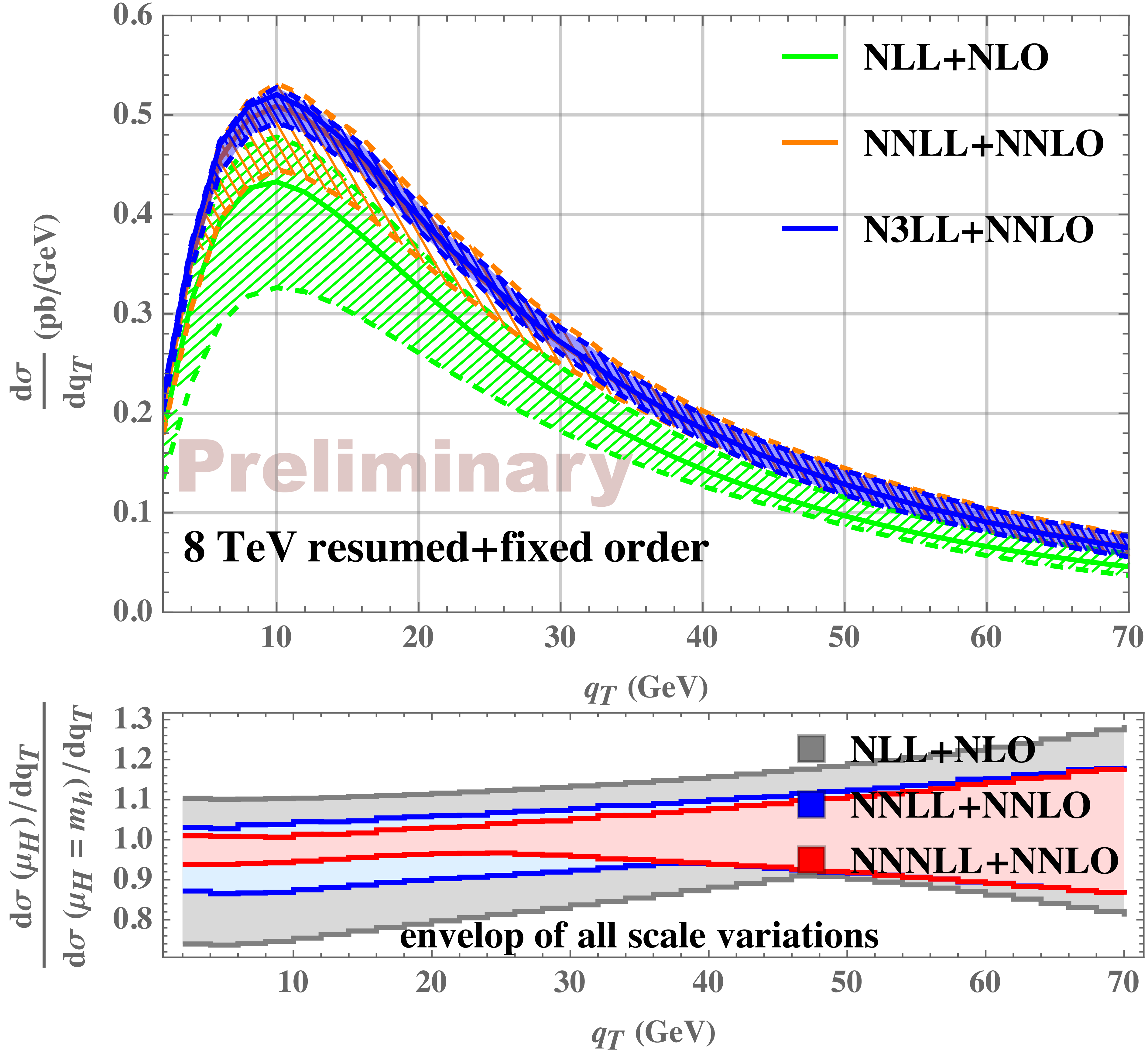


Non-Perturbative uncertainties





Total scale uncertainties



Conclusion

- **Introduce a new regulator for rapidity divergence in SCET description of transverse-momentum distribution.**
- **Analytic calculation of the resulting three-loop soft function through three-loops for the first time, extracting the rapidity anomalous dimension (also known as collinear anomaly d2)**
  - Lifting the rapidity regulator as an dynamical variable: double differential soft function
  - Compute the double differential soft function (the N=4 part) by making an ansatz, and then fixing the coefficient using expansion around b=0. Two different method for the remaining QCD part.
  - Intriguing relation between rapidity anomalous dimension and soft anomalous dimension.
- **N3LL pT resummation for Higgs production (except for four-loop cusp)**
  - Significant reduction of uncertainties. About 10% total uncertainties in the resumed region.