Numerical Unitarity Method for Two-loop Amplitudes in QCD

based on work with S. Abreu, F. Febres Cordero,
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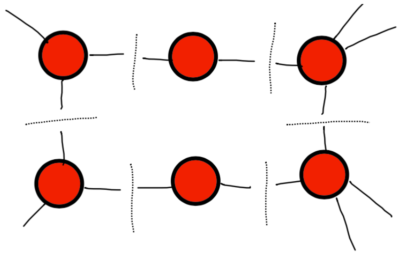
MWK

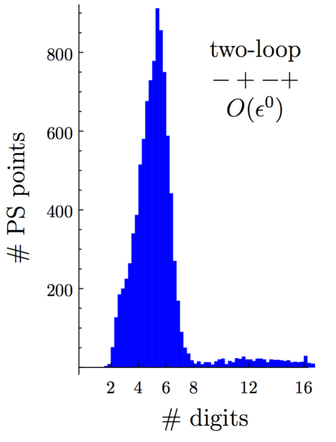
Content

Motivation

Unitarity Method @ 2-Loops

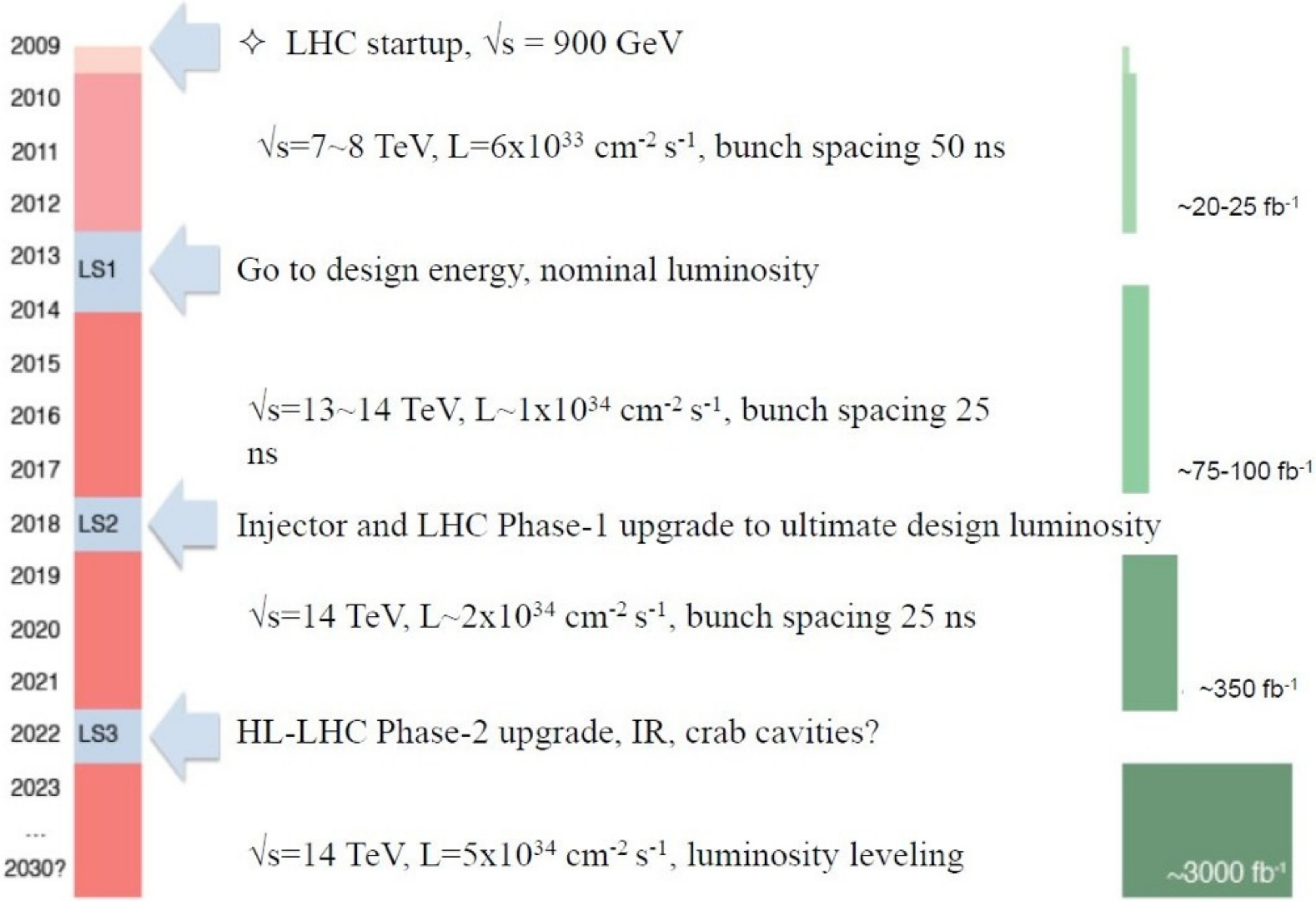


Geometric Properties

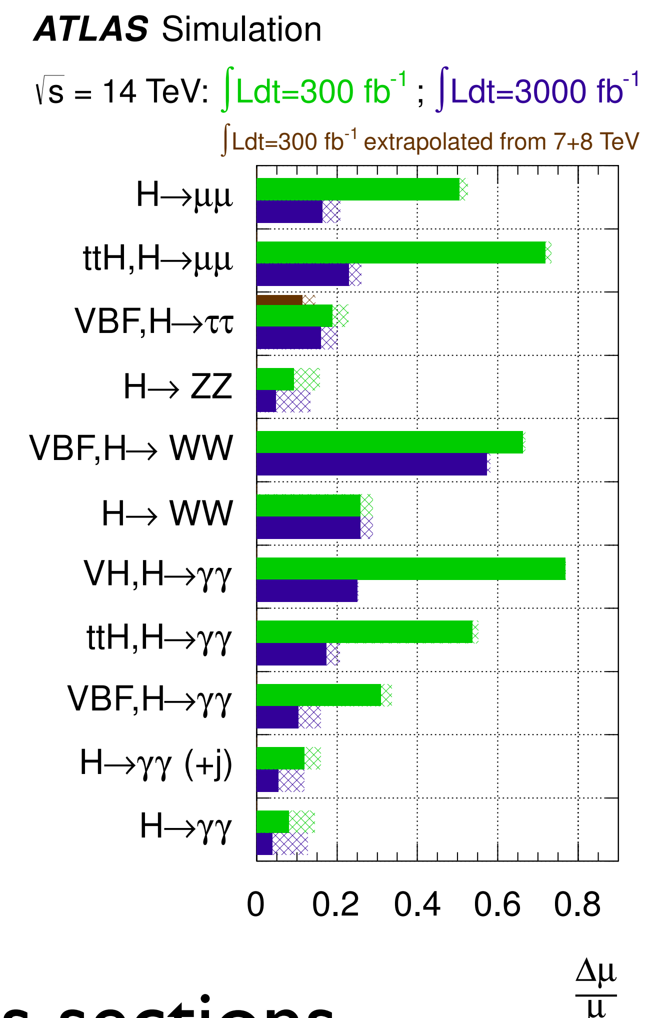


LHC Motivation

Taken from Rolf Heuer in CERN General Meeting January 2013.



(LS = Long Shutdown)



Potential: new physics & percent-level cross sections.

A Theory Aim

Percent-level precision target often requires NNLO in QCD.

Many 2-to-2 processes known @ NNLO.

Can we add recoiling jet to signature final states in order to access kinematic dependence? Can we add mass effects?

Benchmark processes:

- Pure QCD amplitudes; 4-point [Glover, Oleari Tejeda-Yeomans 01; Bern Freitas Dixon 02] — see Ben Page's talk
- 5/6-point pure QCD first results: [Badger, Frellesvig, Zhang 15; Gehrmann, Henn, Lo Presti 15; Badger, Mogull, Perabo 16; Dunbar, Jehu, Perkins 16].
- Loop induced processes, multi-scale processes

Complex dynamic of proton collisions:

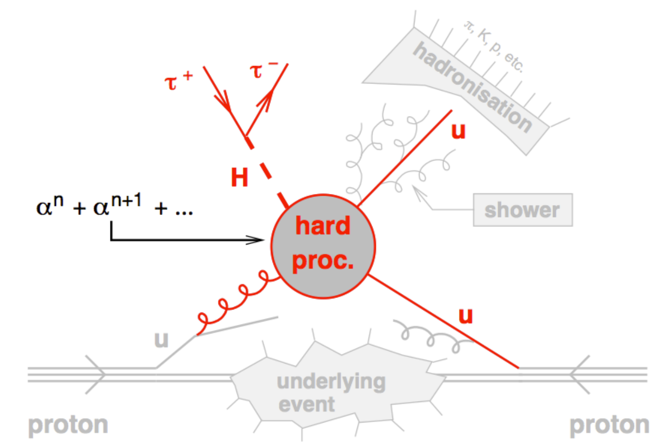


diagram from G. Salam

Precise predictions are a multi-layered problem:

$$d\hat{\sigma}_{ij,NNLO} = \int_{d\Phi_{n+2}} [d\hat{\sigma}^{RR}_{ij,NNLO} - d\hat{\sigma}^{S}_{ij,NNLO}] + \int_{d\Phi_{n+1}} [d\hat{\sigma}^{RV}_{ij,NNLO} - d\hat{\sigma}^{T}_{ij,NNLO}] + \int_{d\Phi_{n}} [d\hat{\sigma}^{VV}_{ij,NNLO} - d\hat{\sigma}^{U}_{ij,NNLO}]$$

[Antenna subtraction; Gehrmann-De Ridder, Gehrmann, Glover; Kosower]

Why Unitarity Approach?

Amplitude computation are complex: Feynman diagrams — integral reduction — integration.

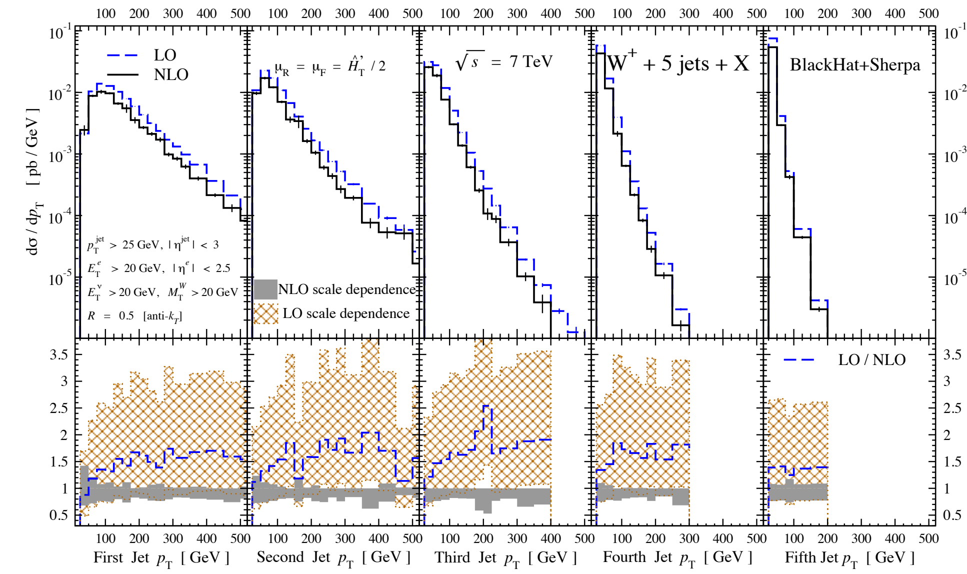
- Large intermediate expressions with compacter results.
- Can we find simplicity at intermediate steps?

Some key multi-loop methods: tensor reduction [Tarasov 96; Anastasiou, Glover, Oleari 99], Integration-by-parts identities [Tkachov, Chetyrkin 81], Lorentz invariance identities [Gehrmann, Remiddi 99], Laporta algorithm [Laporta 01]

Unitarity method:

- Exploits additional properties of amplitudes (on-shell & geometry)
- Numerical approach: suitable for multi-scale problems, e.g. W+5jets @ NLO.
- First spin off: classification of integral relations — see Yang Zhang's and Mao Zeng's talks

[BlackHat; Bern, Dixon, Febres Cordero, HI, Kosower, Maitre, '13]

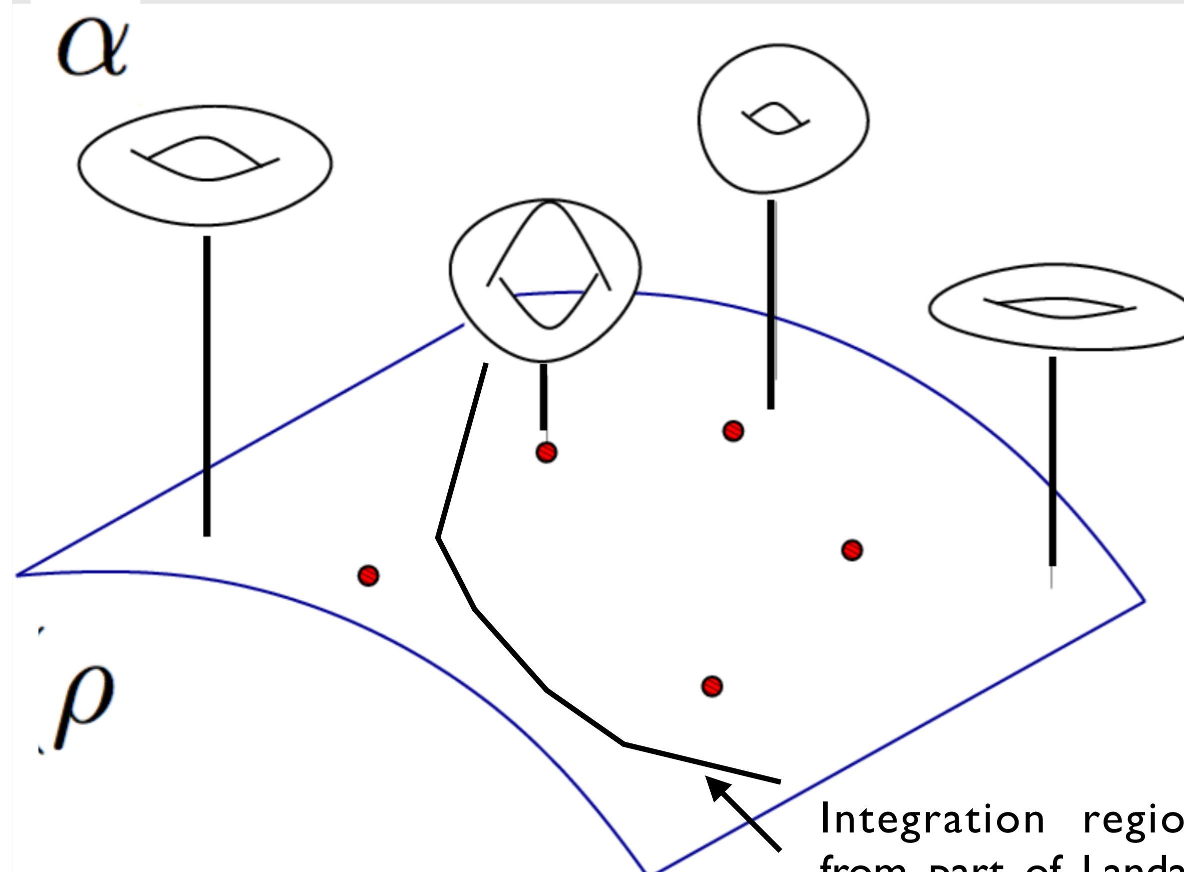


Hidden Geometry

See also: 'The Analytic S-Matrix', Eden, Landshoff, Olive, Polkinhorne; Baikov; HI; Zhang Larsen

Coordinate change exposes fiber structure:

$$\mathcal{I}[t] = \int \frac{[d\rho]}{\rho^0 \cdots \tilde{\rho}^{(\tilde{N}-1)}} \times t(\rho,\alpha)\,\mu(\rho,\alpha)[d\alpha]$$



Functional dependence on internal spaces important for full integral.

One loop example: internal spaces are spheres; all non-constant harmonic functions integrate to zero = IBP relations.

Geometry comes with natural structures:

- Function ring => irreducible numerators; tangent vectors => IBP relations
- Cohomology => master integrals; moduli spaces & connections => differential equations

The Unitarity Method

Amplitude master equation:

$$A(p_i) = \int [d^{(nD)}\ell]\ \tilde{A}(\ell, p_i) = \sum_{\text{integral basis}} c_j(p_i) \int [d^{(nD)}\ell]\ \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N}$$

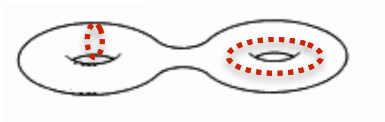
Bootstrap program: [Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; …]

Universal analytic properties of amplitudes imply:

Multi-loop pioneers [Bern, Dixon, Kosower, Dunbar 94; Bern, Dixon, Dunbar, Perelstein, Rozowsky 98; Bern, Dixon, Kosower 00]

$$\int_{\mathcal{C}} [\text{dLIPS}]\ \tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i)$$

$$= \sum_{\text{integrals with cuts}} c_j(p_i) \int_{\mathcal{C}} [\text{dLIPS}]\ \frac{m_j(\ell, p_i)}{(\text{uncut propagator terms})}$$

$\mathcal{C}$: 

Simpler equations & simpler on-shell input.

Recent work on duality of master integrals & contours [Kosower, Larsen 11; Caron-Huot, Larsen 12; Georgoudis, Zhang 15; Sogaard, Zhang 14; HI 15; Harley, Moriello, Schabinger; Bosma, Sogaard, Zhang; Primo, Tancredi 17]

Unitarity Approach - Integrands

Use integrand basis and remove integration:

$$\tilde{A}(\ell, p_i) = \sum_{\text{j in integrand basis}} c_j(p_i) \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N}$$

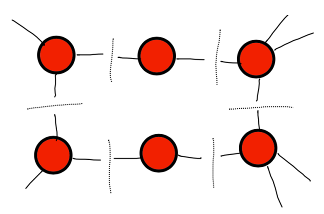
Factorisation in loop momenta [Ellis, Giele, Kunszt]:

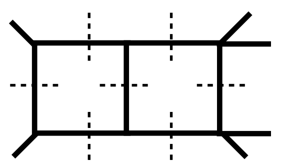
$$\lim_{\{\rho^i\} \to 0} \tilde{A}(\ell, p_i) \to \tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) \frac{1}{(\text{large propagator terms})}$$

Algebraic equations using tree-level data:

$$\tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) = \sum_{\text{j in large integrands}} c_j(p_i)\, m_j(\ell, p_i) + \text{previously computed topologies}$$

Classification of integrands [Badger, Frellesvig, Zhang. 13; Mastrolia, Mirabella, Ossola, Peraro 12]





Properties:

- Universal and numerical
- But: additional integral reduction required

Numerical Unitarity Method

Integrand decomposition into masters integrands and vanishing integrals:

$$\tilde{A}(\ell, p_i) = \sum_{\text{j in master integrands}} c_j(p_i) \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N} + \sum_{\text{j in surface terms}} c_j(p_i) \frac{m_j(\ell, p_i)}{\rho^1 \cdots \rho^N}$$

@ one-loop [Ossola Papadopoulos, Pittau 07; Ellis, Giele Kunszt 07; Giele Kunszt, Melnikov 08]

@ two/multi-loop [HI15]

use **IBPs**

Algebraic `cut equations' suitable for numerics (& analytic computations):

$$\tilde{A}_1(\ell, p_i) \times \cdots \times \tilde{A}_m(\ell, p_i) = \sum_{\text{j in large integrands}} c_j(p_i)\, m_j(\ell, p_i) + \text{previously computed topologies}$$

Two-in-one approach: obtain coefficients & reduction. No additional integral reduction required — just drop surface terms.

Surface Terms

Vanishing integrals given as integration-by-parts identities [HI 15]. A generalisation of [Aguila, Ossola, Papadopoulos, Pittau 04; Ellis, Giele, Kunszt 07]

[Tkachov, Chetyrkin 81]

$$0 = \int \prod_{l=1,2} d^D \ell_l \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right]$$

Need to control propagator powers for compatibility with unitarity equations => IBP-generating vectors u (see eqns).

[Gluza, Kajda, Kosower 10]

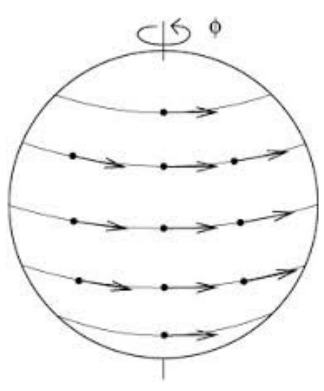
$$\partial_\mu \left(\frac{u^\mu}{\rho^i} \right) = \frac{1}{\rho^i} \partial_\mu u^\mu - \frac{1}{(\rho^i)^2} u^\mu \partial_\mu \rho^i$$

$$\longrightarrow \quad u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j$$

Geometric interpretation [HI 15, see similar Zhang 14]:

- Particular vector fields in momentum space which become tangent to unitarity-cut surface — see Mao Zeng's talk

Unitarity-cut surface and IBP-generating vector field:



Natural Coordinates

Work at individual propagator structures.

General coordinate transformation [HI 15; Larsen, Zhang 15].

Additional constraint for polynomial vector fields => syzygy equations.

Explicit solutions for planar topologies with generic mass assignments [HI 15].

Special cases solved with Singular-program. Other approaches using computer algebra [Gluza, Kajda, Kosower 10; Schabinger 11].

(1) $$u_i^\nu \frac{\partial}{\partial \ell_i^\nu} \rho_j = f_j \rho_j$$

inverse propagators

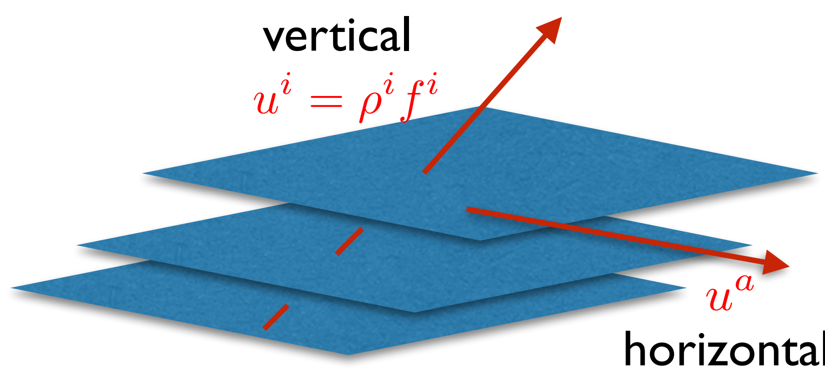
(2) $$\{\ell_1, \ell_2, \ldots\} \quad \to \quad \{\rho_k, \alpha^j\}$$

auxiliary coordinates

(3) $$\left(u_j \frac{\partial}{\partial \rho_j} + u^a \frac{\partial}{\partial \alpha^a} \right) \rho_k = u_j \delta_k^j = f_k \rho_k$$

→ $$\{u_i, u^a\} = \{f_i \rho_i, u^a\}$$

Classification of IBP relations



foliation of momentum space
in ρ^i = const. slices

Classification of IBP-generating vectors:

- Horizontal:

$$(u^i, u^a) = (0, u^a)$$ → relations within integral topology

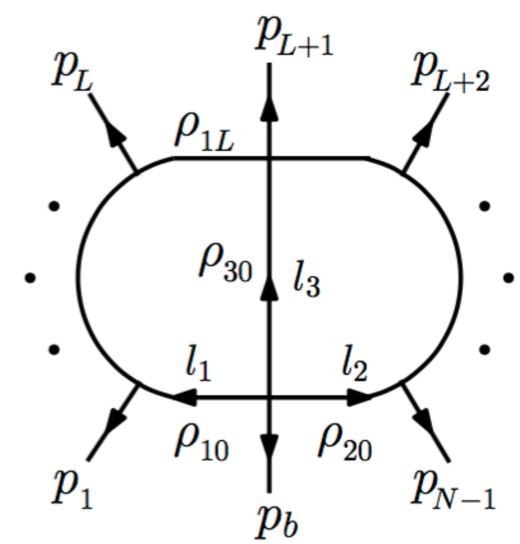
- Vertical:

$$(u^i, u^a) = (f^i \rho^i, 0)$$ → relations between distinct integral topologies

- Mixed:

$$(u^i, u^a) = (f^i \rho^i, u^a)$$ → massless integrals, D-dependent relations

Tracking Polynomials



Are vectors polynomial when transformed back to momentum space?

Use redundant set of coordinates with constraints => simple to keep track of polynomial structure.

Loop-momentum polynomials are polynomials in new coordinates.

Irreducible scalar products given by polynomials in alpha-coordinates.

(1) $$\ell_l = \sum_{j \in B_l^p} v_l^j r^{lj} + \sum_{j \in B_l^t} v_l^j \alpha^{lj} + \sum_{i \in B^{ct}} n^i \alpha^{li} + \sum_{i \in B^{\epsilon}} n^i \mu_l^i$$

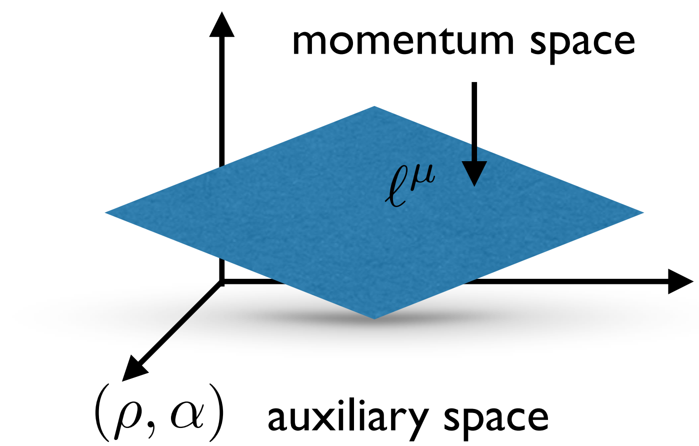
(2) l.p: $$r^{li} = -\frac{1}{2}(\rho_{li} - (q_{li})^2 - \rho_{l(i-1)} + (q_{l(i-1)})^2)$$

(3) $$c_l := (\ell_l)^2 - \rho_{l0} = (\ell_l)^2|_{4D} - \mu_l^2 - \rho_{l0} = 0$$

(4) $$t^{\mu_1 \cdots \mu_k} \ell_{\mu_1} \cdots \ell_{\mu_k} \quad \sim$$

$$(\rho, \alpha) - \text{polynomials}$$

Constraints on Vectors



Vectors have to map polynomials to polynomials => polynomial components.

Symmetric in (D-4) dimensional part.

Vectors have to point along physical momentum space.

Simplest form when solving for $\mu_{ll'}$'s.

=> quadratic equations for polynomial vector components. Solve for u's and f's.

(1) Symmetric IBP generating vector:

$$u = \underbrace{f_i \rho_i \frac{\partial}{\partial \rho_i} + u_j \frac{\partial}{\partial \alpha^j}}_{\bar{u}} + \sum_{l,l'=1,2} f^l_{l'} \mu^k_l \frac{\partial}{\partial \mu^k_{l'}}$$

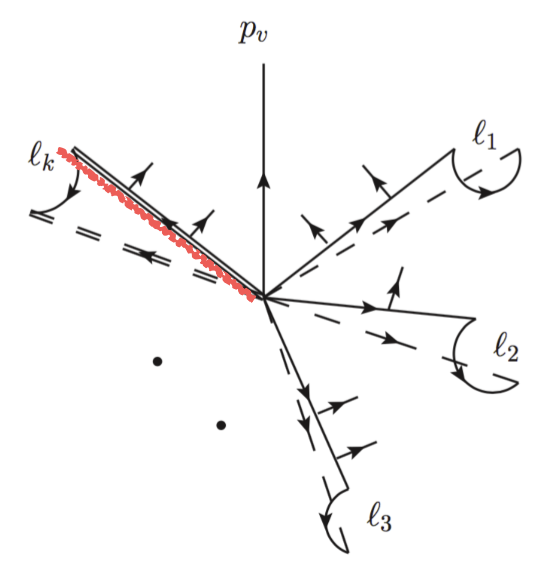
(2) Syzygy equations from $u(c_l) \sim 0$

$$\begin{aligned}
\bar{u}(\mu_{11}) &= 2\mu_{11} f^1_1 + 2\mu_{12} f^2_1 \,, \\
\bar{u}(\mu_{22}) &= 2\mu_{22} f^2_2 + 2\mu_{12} f^1_2 \,, \\
\bar{u}(\mu_{12}) &= \mu_{12}(f^1_1 + f^2_2) + \mu_{11} f^1_2 + \mu_{22} f^2_1 \,.
\end{aligned}$$

Solve IBP Reduction?

IBP-generating vectors:

- Rotation/scaling/translation generators for each rung, consistent with momentum conservation at vertices.
- Can we write down all solutions and solve integral reduction?



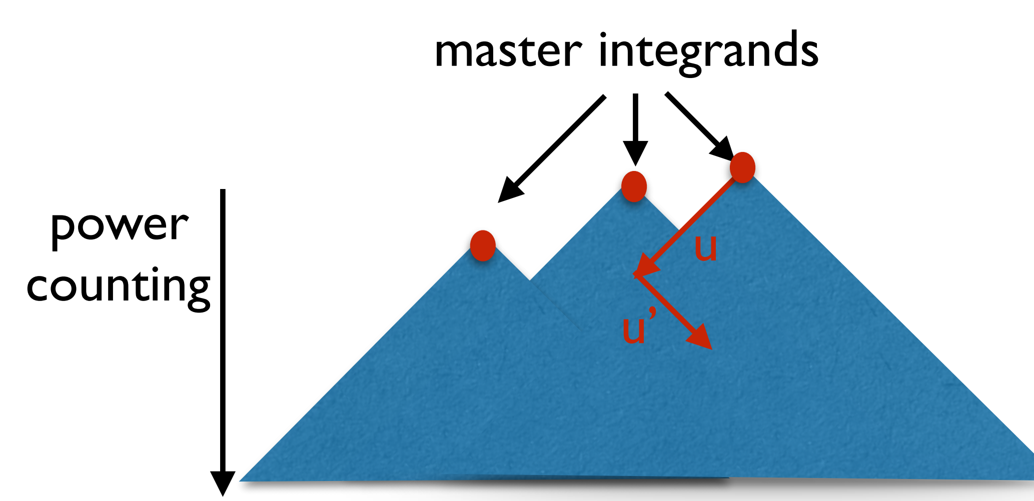
Geometric structures:

- Lie-algebra & representation theory:

$$[u_a, u_b] = f^c_{ab}(\ell, p_i) u_c$$

- Numerators are representations space.
- 'Highers weight' representation are master integrals

Numerator polynomials:



On-shell Linear Algebra

Cutting loop integral:

holomorphic form

$$\int \frac{t}{\rho^1 \cdots \rho^N} J[d\alpha d\rho] \xrightarrow{\text{cut}} \int t\, J[d\alpha]$$

IBP-relations give exact forms on shell [HI15]:

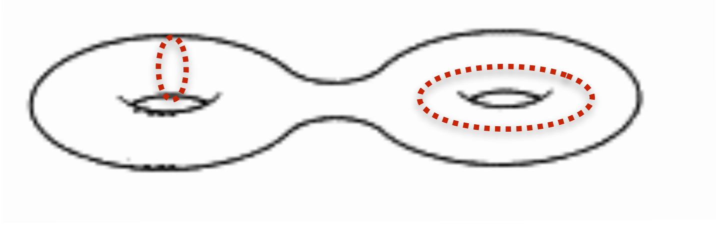
exact holomorphic form

$$\int \left[\left(\sum_i \frac{\rho^i \partial_i (f^i\, J)}{\rho^1 \cdots \rho^N} \right) + \left(\frac{\partial_a (u^a\, J)}{\rho^1 \cdots \rho^N} \right) \right] [d\alpha d\rho] \xrightarrow{\text{cut}} \int \partial_a (u^a\, J)[d\alpha]$$

Master integrands:

- Relevant IBPs don't vanish on cut — can check linear dependence on-shell
- (holomorphic forms) modulo (exact forms) = (cohomology of unitarity cut variety) [HI 15; Larsen, Zhang 15]. — see talk by Yang Zhang

Topological count gives master integrands



Surface-Term Numerators

Given the IBP-generating vectors we obtain all surfaces terms.

Large part of surface terms given by rotations transverse to full scattering plane or sub loops (can be written down by hand).

Singularities of surface give conformal scaling vectors; global continuation hard — role syzygy equations.

D-dependence very explicit.

Transverse relations:

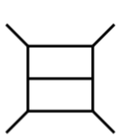
$$\alpha^{lj}\alpha^{l'j} - b_1 \frac{\mu_{ll'}}{D-4}\,, \quad l, l' \in \{1, 2\},$$

$$(\alpha^{lj})^2(\alpha^{l'j})^2 - b_1 \frac{\mu_{ll}}{D-4}(\alpha^{l'j})^2 - b_2 \frac{\mu_{ll'}}{D-4}\alpha^{lj}\alpha^{l'j}\,, \; l \neq l'$$

Generic & scaling relations:

$$m_{\Gamma,u}(\ell_l) = \left[-(\nu_i - 1) f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha^j} + \left(D - \frac{n_\alpha + 1}{2} \right) (f_1^1 + f_2^2) \right]$$

Example count for double box:



masters	tensors	transverse relations	generic relations	scaling relations
5-s	160	138	17	s
	easy	easy	easy	hard

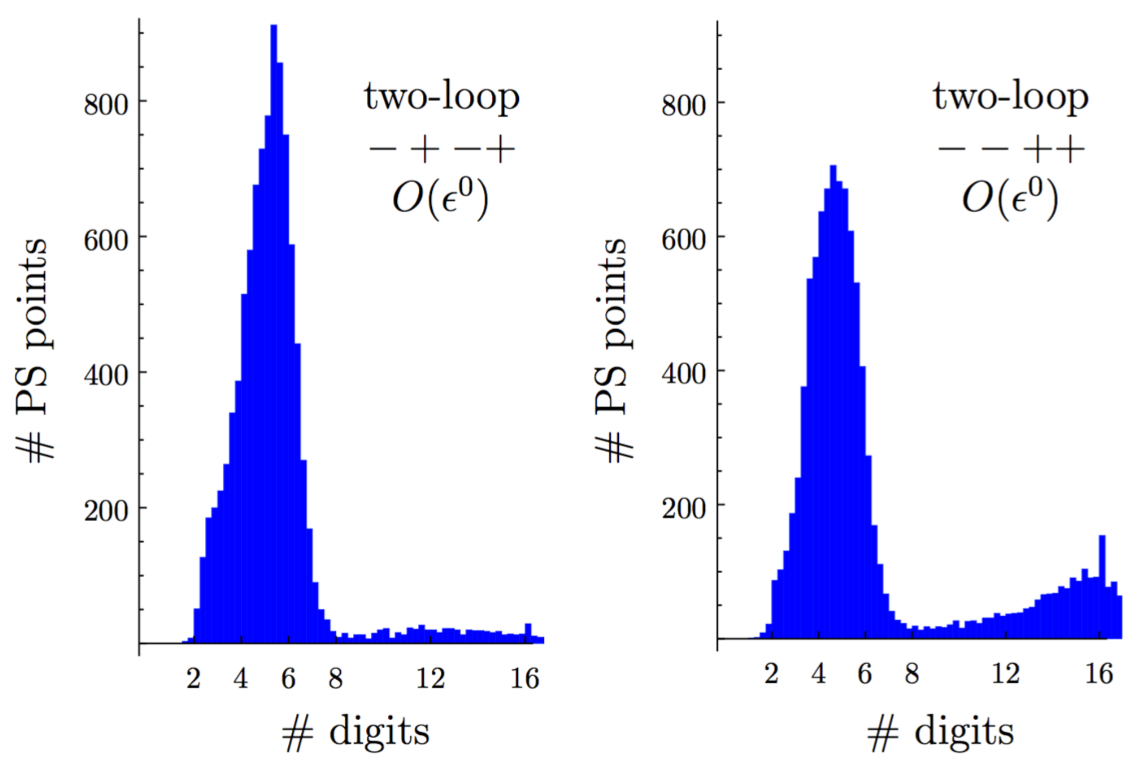
see Ben Page's talk

Proof of Principle

Proof of principle: 4-gluon 2-loop amplitudes:

First computed by [Glover, Oleari, Tejeda-Yeomans 01; Bern, Dixon, de Freitas 02];

- Much of QCD complexity turned on.
- Analytic results reconstructed from numerics at high precision.



Conclusions

Presented first two-loop computation with numerical unitarity approach:

- Discussed central ingredient the surface terms
- Numerical approach: computer-cluster compatible
- Exact: analytic reconstruction

Potential:

- Current status suggests good for adding scales (legs/masses)

Geometry backbone to `unitarity approach':

- Unitarity-cut phase spaces
- Natural coordinates adapted to integrals, log-vector fields
- Classification of IBP-generating vectors

Geometry & Unitarity

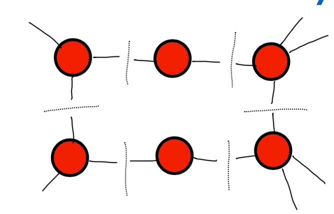
Unitarity approach:

- Discontinuities of loop amplitudes localise integral on (cycles of) on-shell phase spaces.
- Similarly, integrands factorise for on-shell loop momenta.

Integral perspective:

- Questions modulo pinching lines, defines polynomial ideal of irrelevant terms and thus algebraic varieties.

Ongoing amplitudes field [Chew, Mandelstam; Eden, Landshoff, Olive, Polkinghorne; Veneziano; Virasoro, Shapiro; …Bern, Dixon, Dunbar, Kosover; Arkani-Hammed, Cachazo, … amplitudes community]



Recent years [Badger, Frellesvig, Zhang; Mastrolia, Mirabella, Ossola, Peraro]

Recently [Abreu, Britto, Caron-Huot, Duhr, Gardi, Georgoudis, HI, Kosower, Larsen, Sogaard, Zeng, Zhang]

$\mathcal{C}$: