AZURITE

a package to determine master integrals via computational algebraic geometry

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Based on work with Alessandro Georgoudis, Kasper Larsen

Loopfest XVI,
Argonne National Laboratory
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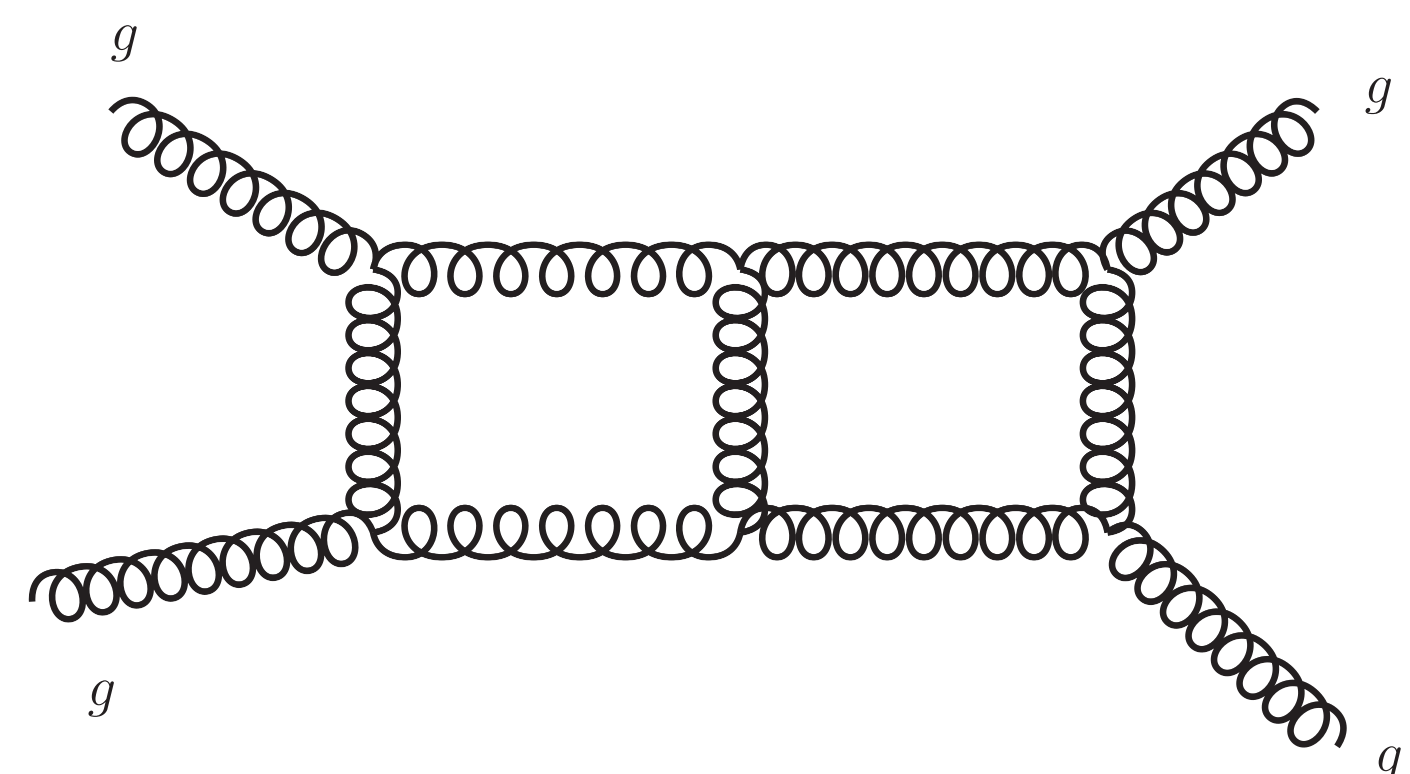
Multi-loop Feynman integrals

Feynman integrals are crucial for the precision collider physics



$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{N(l_i \cdot p_j)}{D_1 \ldots D_k}$$

→ Large number of terms, for non-supersymmetric theories

⇓

integrand basis

⇓ IBP

master integrals — Linear basis

Integration-by-parts (IBP)

$$\int \frac{dl_1^D}{i\pi^{D/2}} \cdots \int \frac{dl_L^D}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0$$

IBP: Chetyrkin, Tkachov 1981, Laporta 2001, …

IBP packages: FIRE (Smirnov), Reduze (von Manteuffel, Studerus), LiteRed (Lee), Kira (Maierhoefer, Usovitsch, Uwer)

Syzygy: Gluza, Kjada, Kosower 2010, Schabinger 2011

To find a list of master integrals basis

sector-by-sector searching ….

- MINT

  Lee, Pomeransky 1308.6676
  Based on Morse Theory

- Azurite

  Georgoudis, Larsen, YZ 1612.04252
  Based on Computational algebra geometry

quickly get a basis for integrals without using full IBP…

Targets

$$I[\alpha_1, \ldots \alpha_k] = \int \left(\prod_{i=1}^{L} \frac{d^D l_i}{i\pi^{D/2}} \right) \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}}$$

$$\alpha_i \in \mathbb{Z}$$

Feynman parametrization

When $\alpha_i > 0$,

$$I[\alpha_1, \dots \alpha_k] \propto \left(\prod_i^k \int_0^1 dz_i\right) \delta(1 - \sum_j z_j) z_1^{\alpha_1 - 1} \dots z_k^{\alpha_k - 1} \frac{F^{LD/2 - |\alpha|}}{U^{(L+1)D/2 - |\alpha|}}$$

$$\propto \left(\prod_i^k \int_0^\infty dz_i\right) z_1^{\alpha_1 - 1} \dots z_k^{\alpha_k - 1} G^{-D/2}$$

Good for

- unitarity cut
- integrand reduction and IBP reduction

Polynomials

$\deg U = L, \quad \deg F = L + 1$

$G = F + U$

Polynomial

$\deg P = 2L$

$\alpha_i \in \mathbb{Z}$

$$I[\alpha_1, \dots, \alpha_k] = \int_\Omega dz_1 \dots dz_k P(z)^{\frac{D - L - n_E}{2}} z_1^{-\alpha_1} \dots z_k^{-\alpha_k}$$

Baikov parametrization

IBPs in Feynman Para.

$$0 = \int_0^\infty dz_1 \dots dz_k \sum_{i=1}^{k} \partial_i \Big(a_i(z) G^{-D/2} \Big)$$

+surface term

IBPs in Baikov Para.

$$0 = \int_\Omega dz_1 \dots dz_k \sum_{i=1}^{k} \partial_i \Big(a_i(z) \frac{P^{(D-L-n_E)/2}}{z_1 \dots z_m} \Big)$$

no surface term

Sector-by-Sector Analysis for IBPs

integrals with the "same" Feynman diagram

Sector $(\overbrace{1,1,\dots,1}^{m},0,\dots 0)$ denotes all integrals with $a_1 > 0, \dots, a_m > 0$ and $a_{m+1} \le 0, \dots, a_k \le 0$

$$0 = \int_0^\infty dz_1 \dots dz_k \sum_{i=1}^{k} \partial_i \Big(a_i(z) G^{-D/2} \Big)$$

~~+surface term~~

Focus on one sector and neglect surface terms

$$0 = \int_\Omega dz_{m+1} \dots dz_k \sum_{i=m+1}^{k} \partial_i \Big(a_i(z) F^{(D-L-n_E)/2} \Big)$$

The residue of P at $z_1 = 0, \dots z_m = 0$

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Griffiths-Dwork reduction problem

$$0 = \int_0^\infty dz_1 \dots dz_k \sum_{i=1}^{k} \partial_i \left(a_i(z) G^{-D/2} \right)$$

~~+surface term~~

Focus on one sector and neglect surface terms

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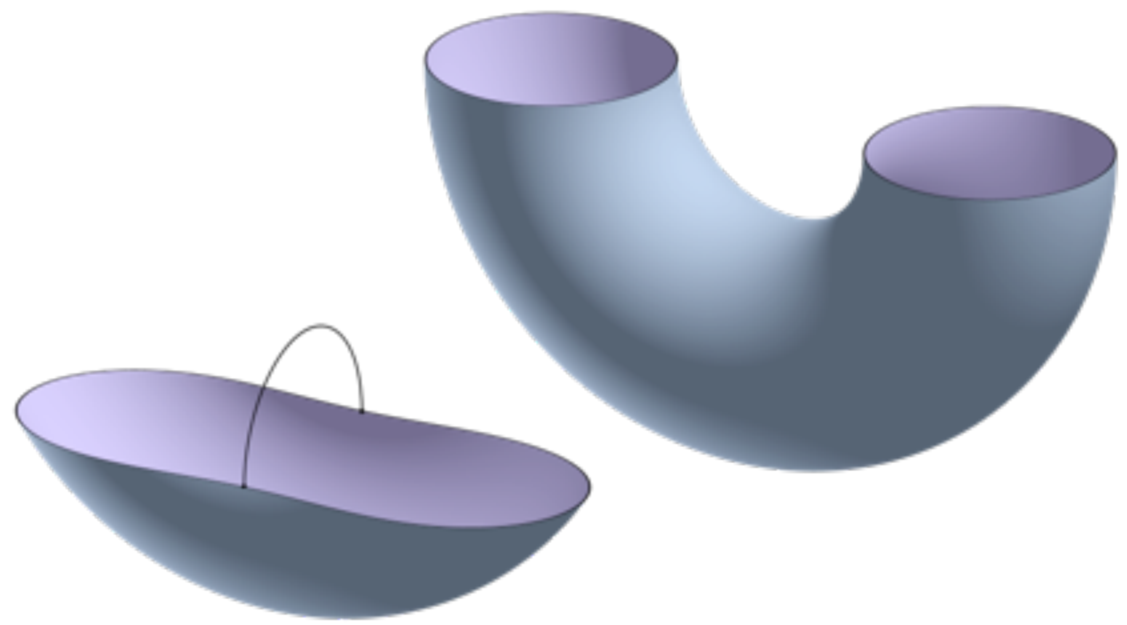
The residue of P at $z_1 = 0, \dots z_m = 0$

Sector-by-Sector searching for master integrals

MINT

Lee, Pomeransky 1308.6676

Morse theory



master integrals in one sector equals # critical points of the polynomial in Feynman/ Baikov representation.

Critical points: $$\frac{\partial F}{\partial z_1} = \ldots = \frac{\partial F}{\partial z_n} = 0, \quad F \neq 0$$

counting the number of solutions

Sector-by-Sector searching for master integrals

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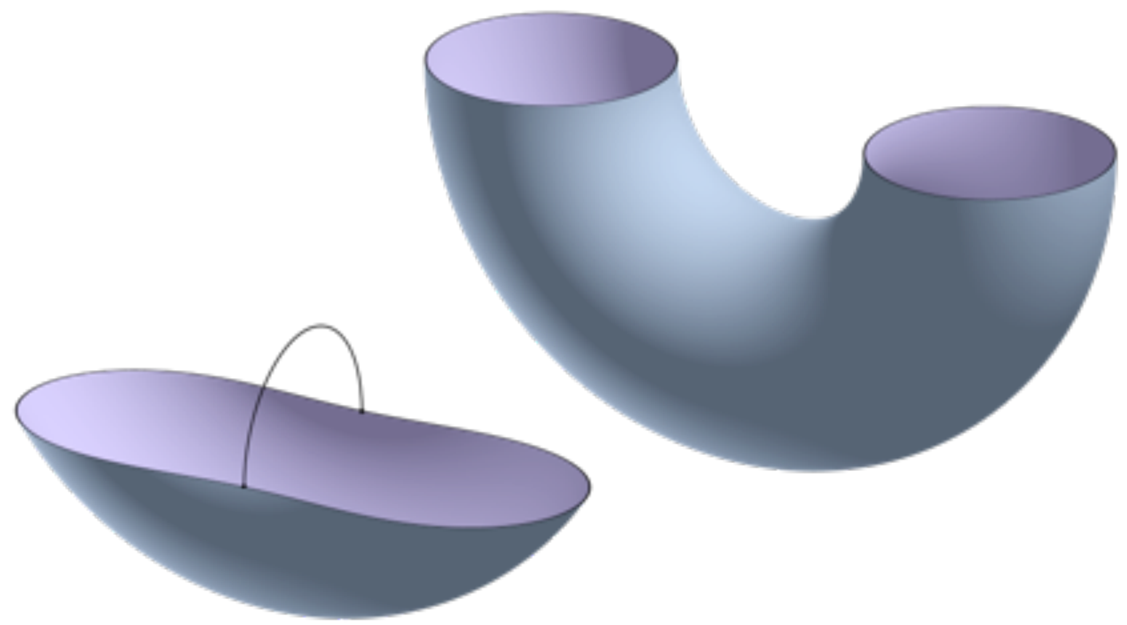
Morse theory



Lee, Pomeransky 1308.6676

master integrals in one sector equals # critical points of the polynomial in Feynman/Baikov representation.

if (1) there is no critical point at infinity and (2) the number of critical points is finite

Critical points: $$\frac{\partial F}{\partial z_1} = \ldots = \frac{\partial F}{\partial z_n} = 0, \quad F \neq 0$$

counting the number of solutions

However, these two conditions failed sometimes ….

Perturbations of the polynomials are not easy.

Azurite

Georgoudis, Larsen, YZ 1612.04252

A ZURich-bred InTEgral-determination method

Syzygy → IBPs within one sector → Master integrals

$$0 = \int dz_1 \ldots dz_k \sum_{i=1}^{k} \partial_i \left(a_i(z) G^{-D/2} \right)$$

If the IBP vector satisfies,

$$\sum_{i=1}^{k} a_i(z) \frac{\partial G}{\partial z_i} + b(z) G = 0$$

the resulting IBP has a simple form (without dimension shift)

Bases on

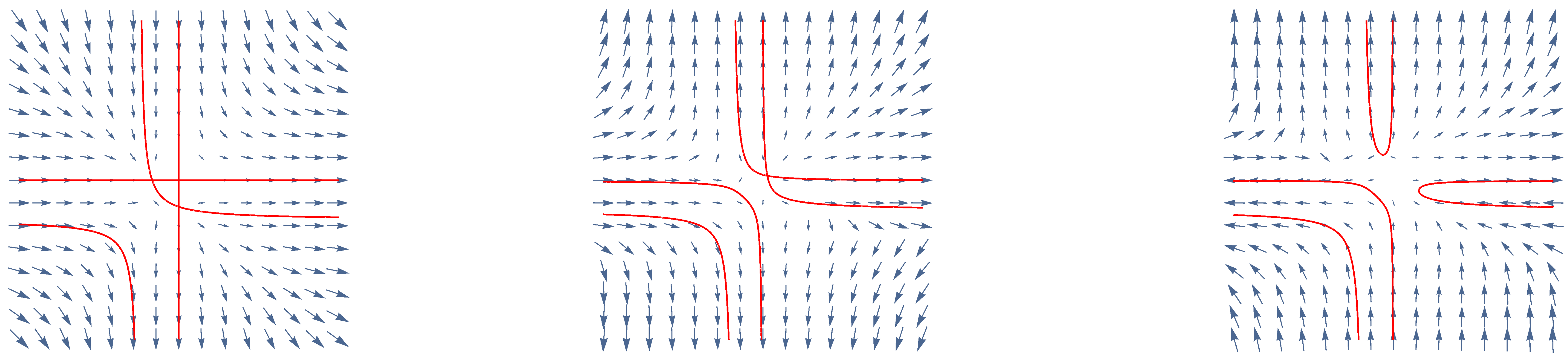
Syzygy for IBPs: Gluza, Kjada, Kosower 1009.0472

IBP with arbitrary cuts: Ita 1510.05626, Larsen, YZ 1511.01071

See Ita, Page and Zeng's talks for more applications on unitarity, integral reduction and differential equations

Syzygy and Geometry

Syzygy is equivalent to the tangent vector field of *G=0*



"EASY"

Case I, massless integral
three singular points

Case II, massive integral
one singular point

Case III, "fully" massive integral
no singular point

$$\frac{\partial G}{\partial z_1} = \ldots = \frac{\partial G}{\partial z_k} = G = 0$$ Singular Points

Quillen–Suslin: If *G=0* is smooth (no singular point), then trivial syzygies generate all syzygies.

Counting singular points may provides # master integrals,
but there may be ∞ singular points …

Azurite version 1.x.x uses purely algebraic approach

Azurite

Determine all sub-sectors up to symmetries

↓

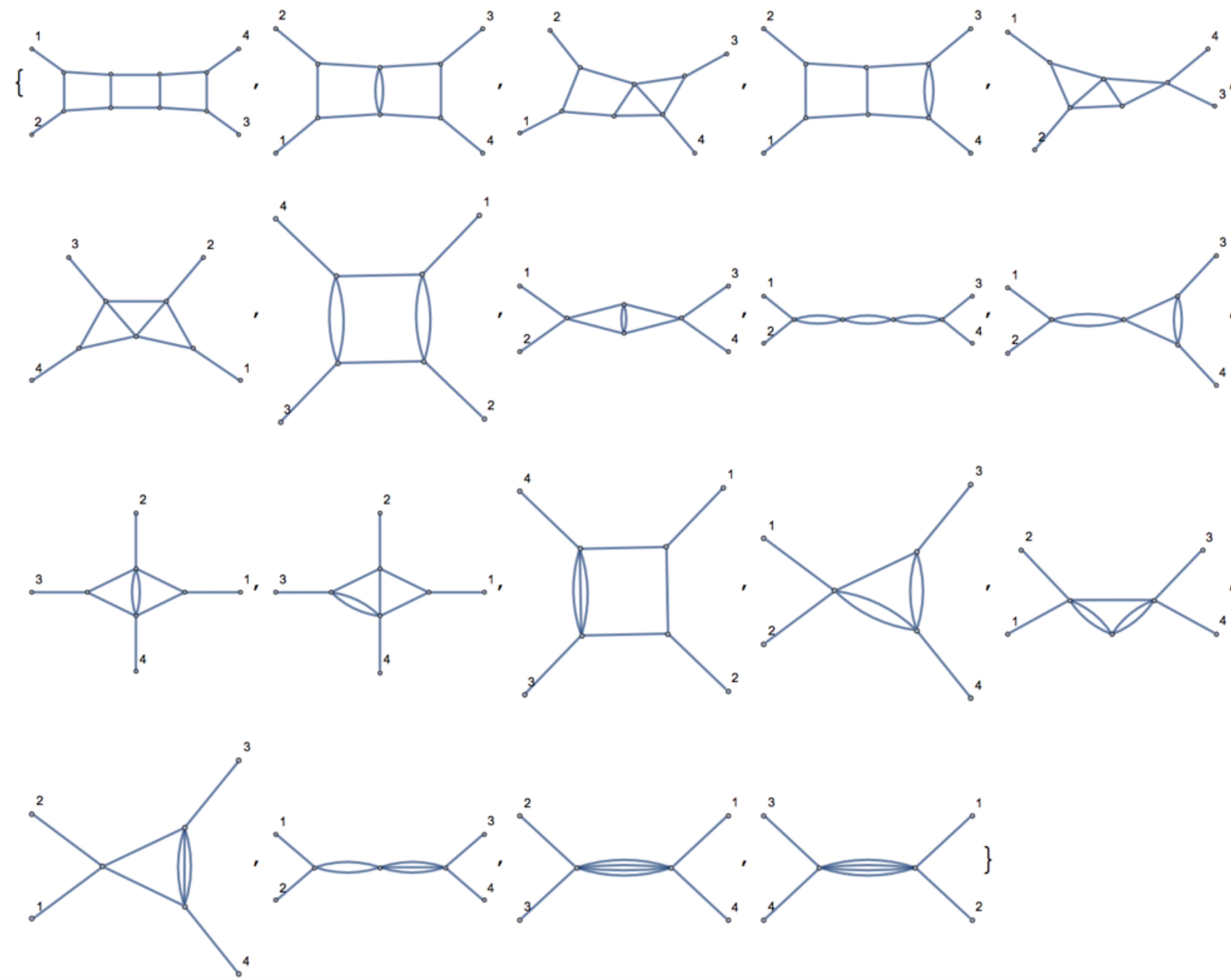
Find syzygies of each subsector in Z/p finite field (Parallelized)

↓

Linear reduction in Z/p (Parallelized)

↓

Master integral List

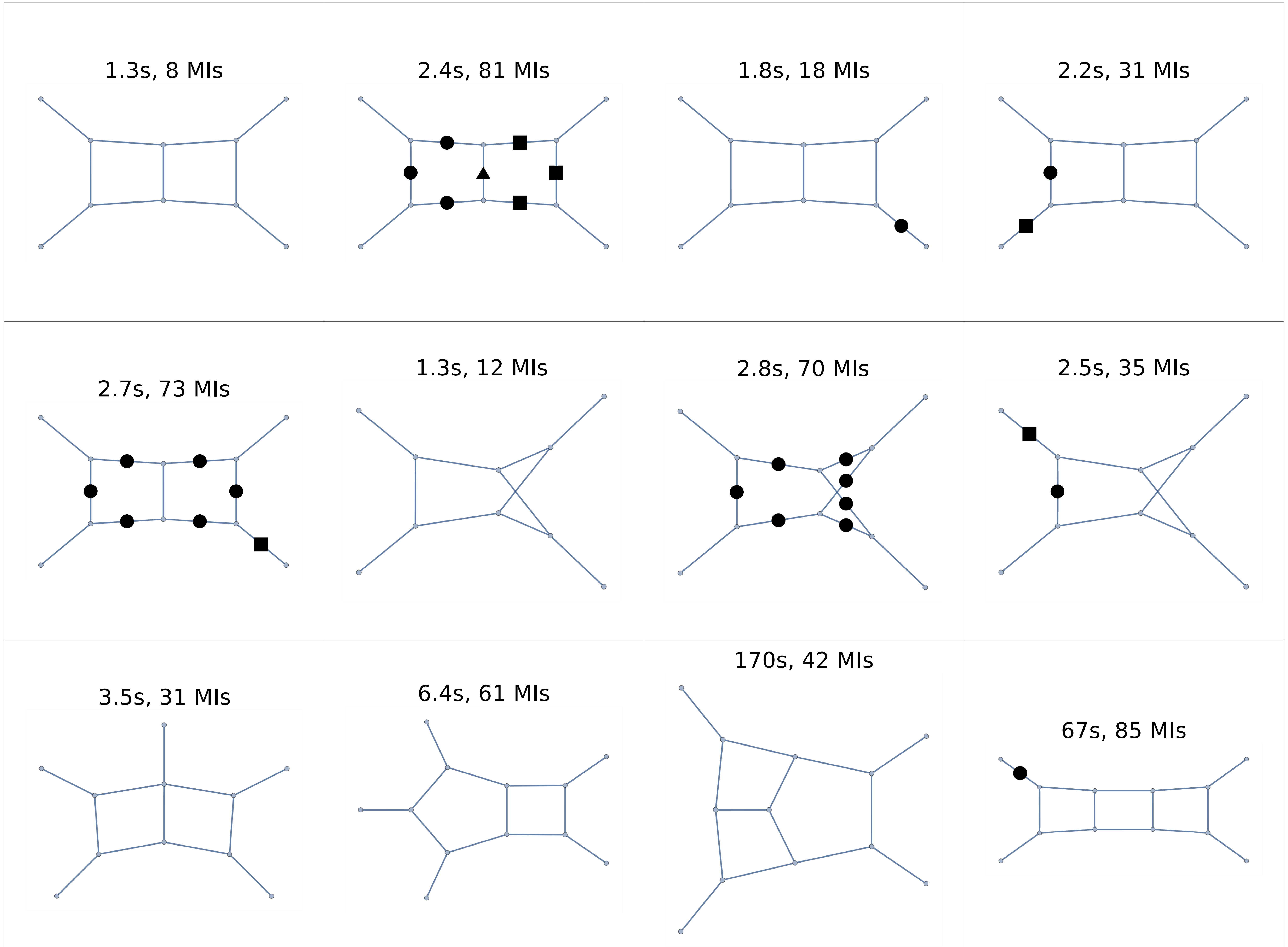


3-loop triple box, 19 irreducible diagrams, 26 master integrals determined by Azurite in 68 seconds

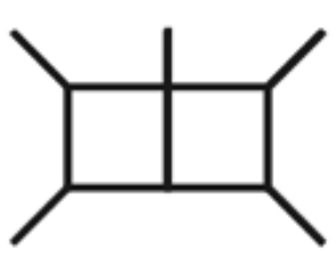
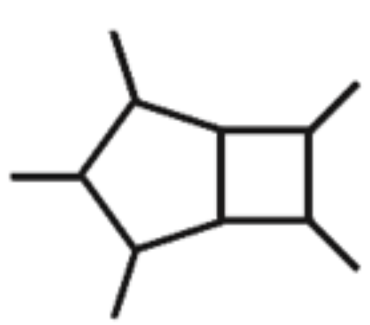
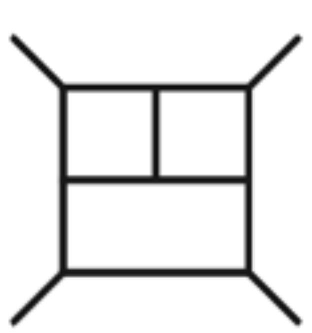
Using Finite Field

von Manteuffel, Schabinger 1406.4513
Peraro 1608.01902

Azurite, more examples



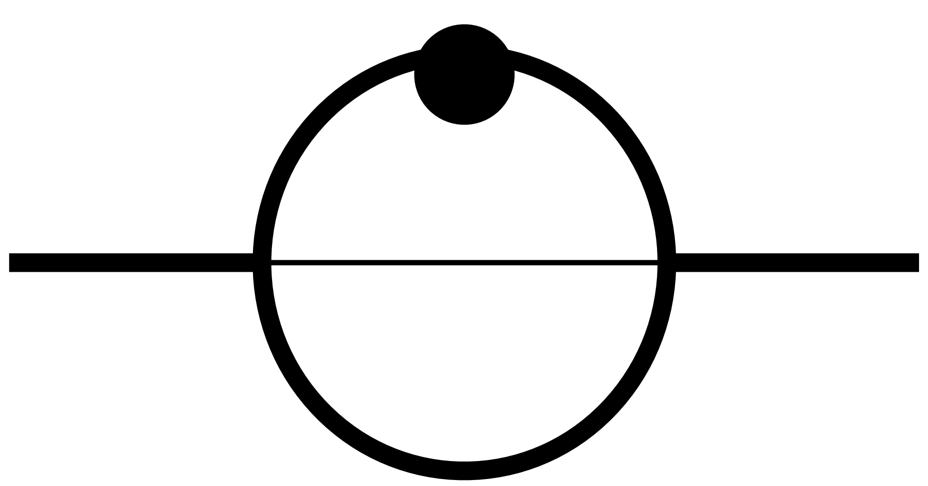
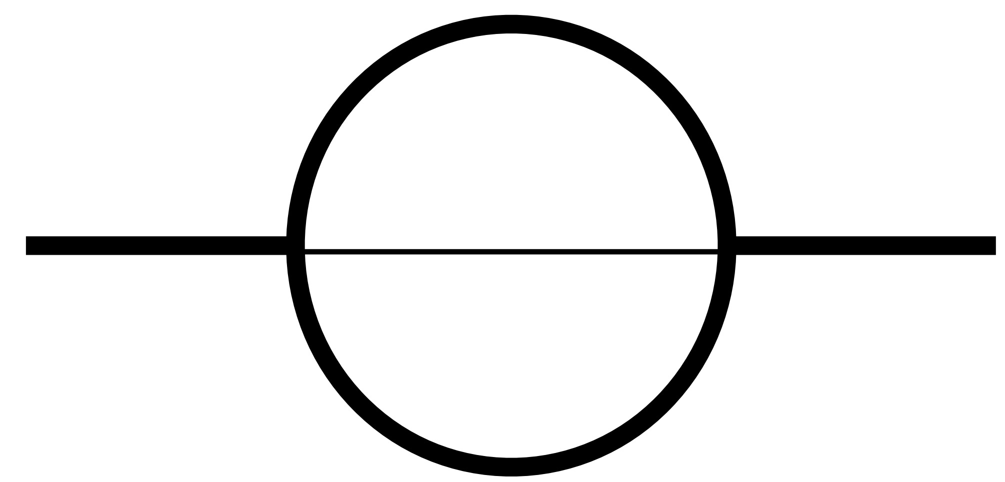
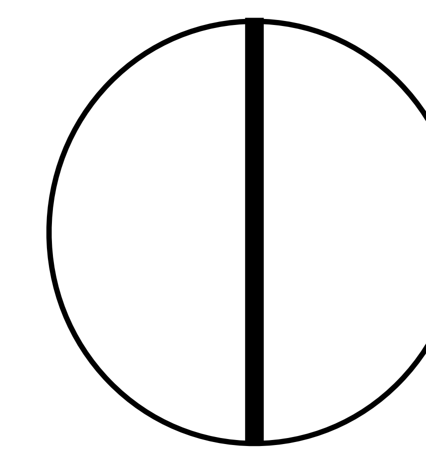
on a desktop with
32GB RAM

	Mint	Azurite
Symmetry	✓	✓
IBP within one sector	(1) Critical point at infinity (under-counting) (2) ∞ critical points (No output)	✓
supersector IBP	subtle	subtle
	6.9 seconds	3.5 seconds
	29.5 seconds	6.4 seconds
	>5 hours	170 seconds

Caveat in sector-by-sector searching

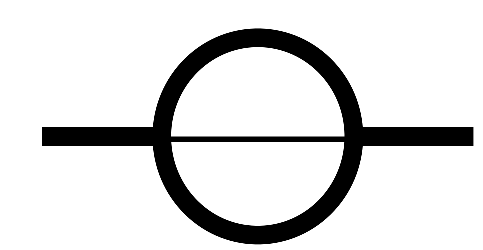
In rare cases, there are IBPs from super sectors

Czakon, Awramik, Freitas
hep-ph/0602029

 $= \frac{2}{3} - \epsilon$  $+ \frac{1}{3}(\epsilon - 1)$ 

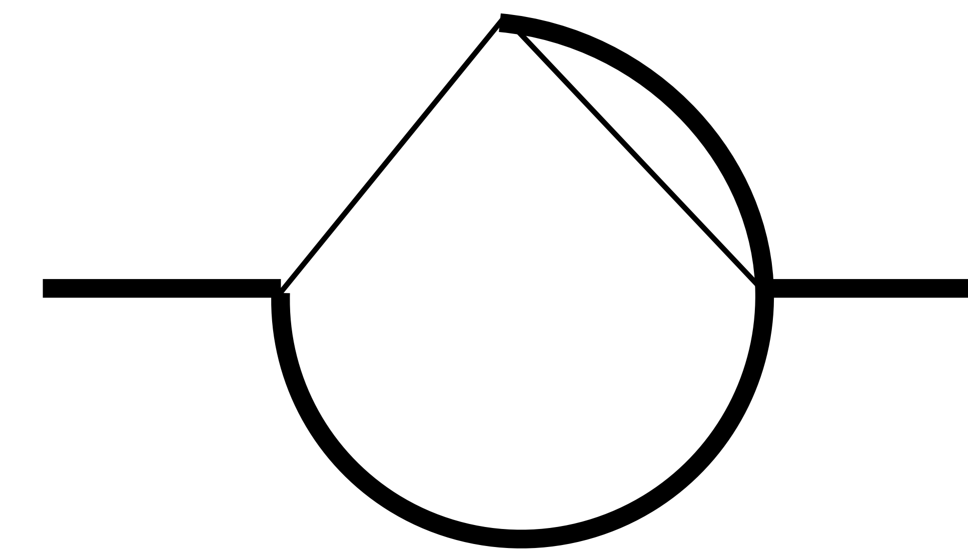
such an IBP cannot be derived from three-propagator integrals

not a sub-sector

All sector-by-sector searching method will find 2 MIs in the sector 
but actually there is 1 MI.

Azurite may give a slightly redundant basis

Caveat in sector-by-sector searching



super-sector

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Czakon, Awramik, Freitas
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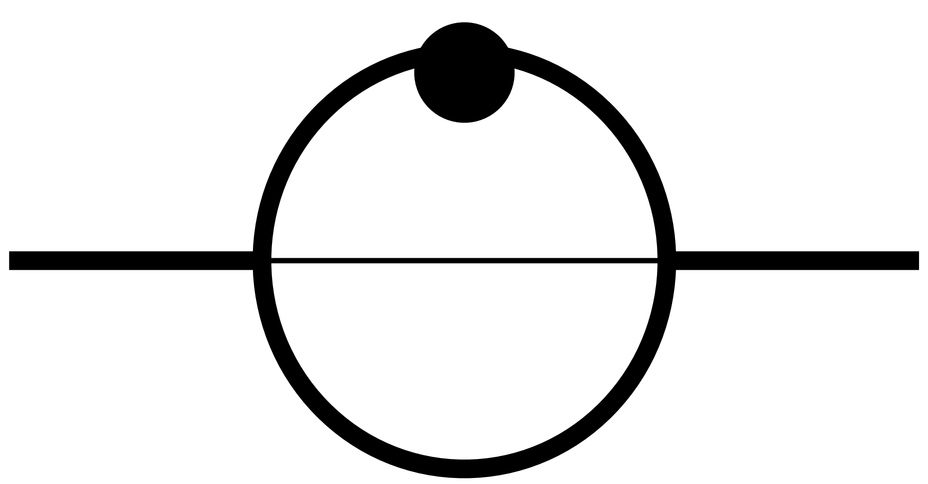
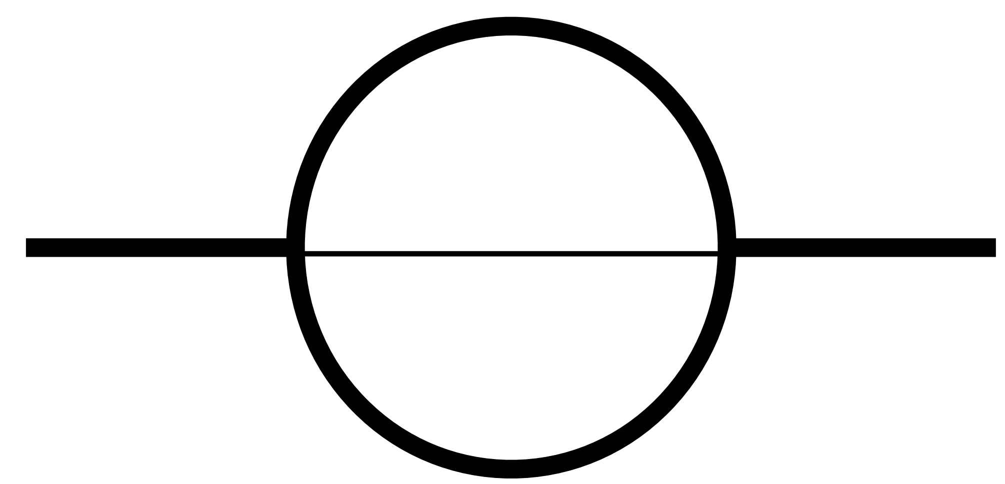
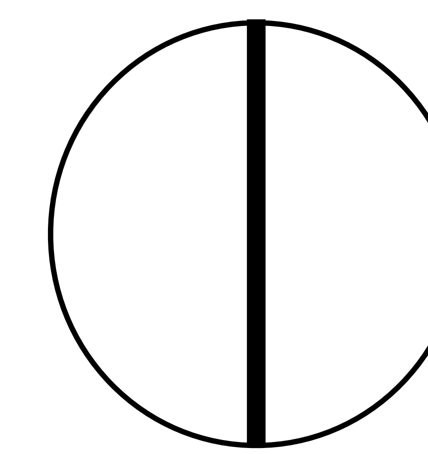


$= \frac{2}{3} - \epsilon$



$+ \frac{1}{3}(\epsilon - 1)$



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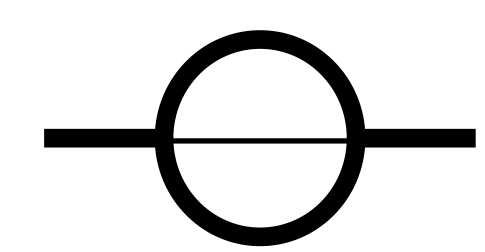

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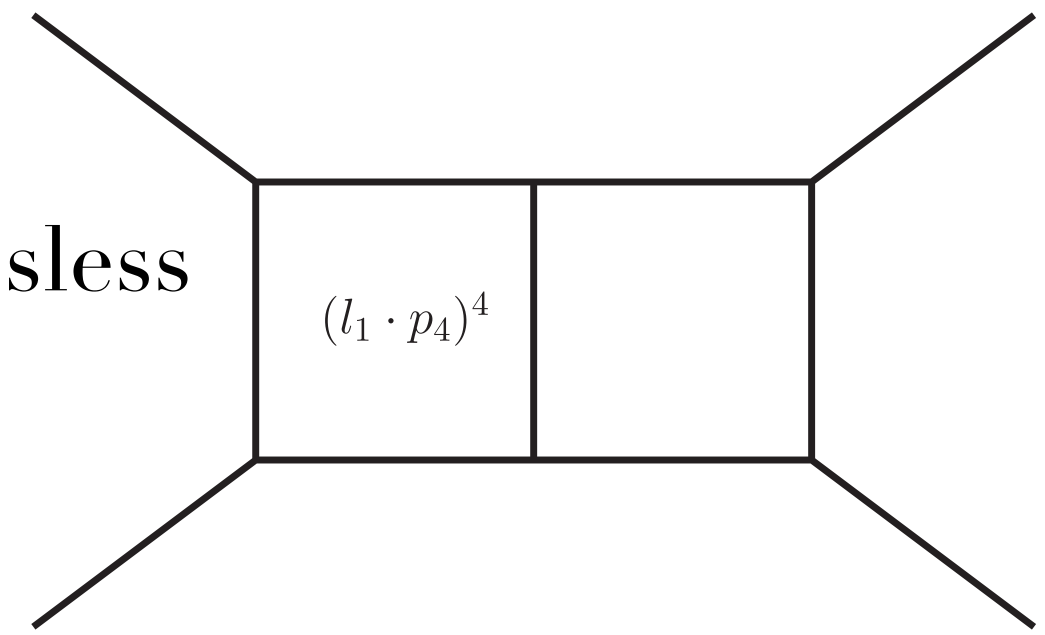
Azurite may give a slightly redundant basis

Full IBPs

Working progress with syzygy+ Finite field/FFLU approach
Georgoudis, Larsen, YZ,
based on 1511.01071 Larsen, YZ

massless



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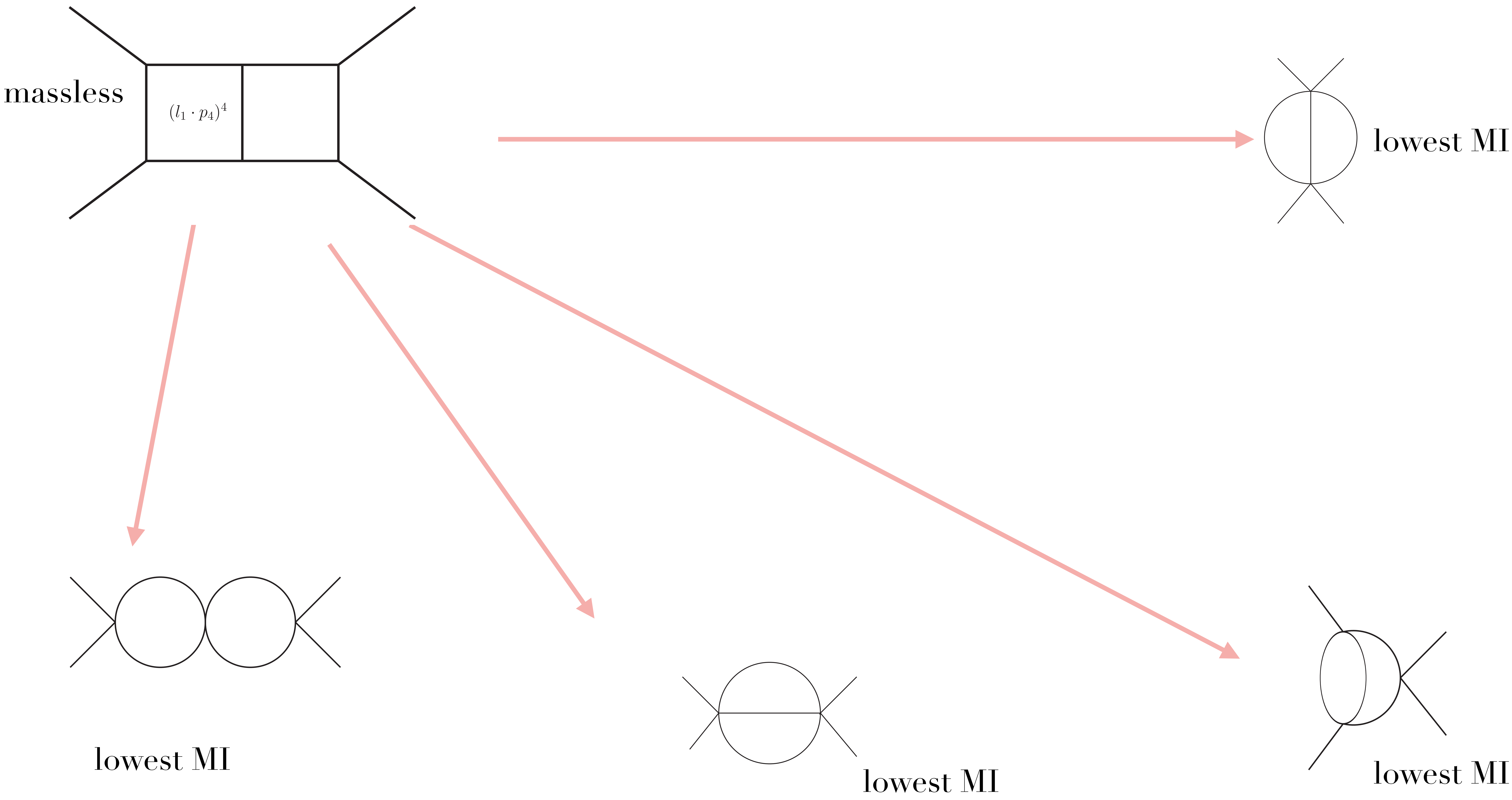

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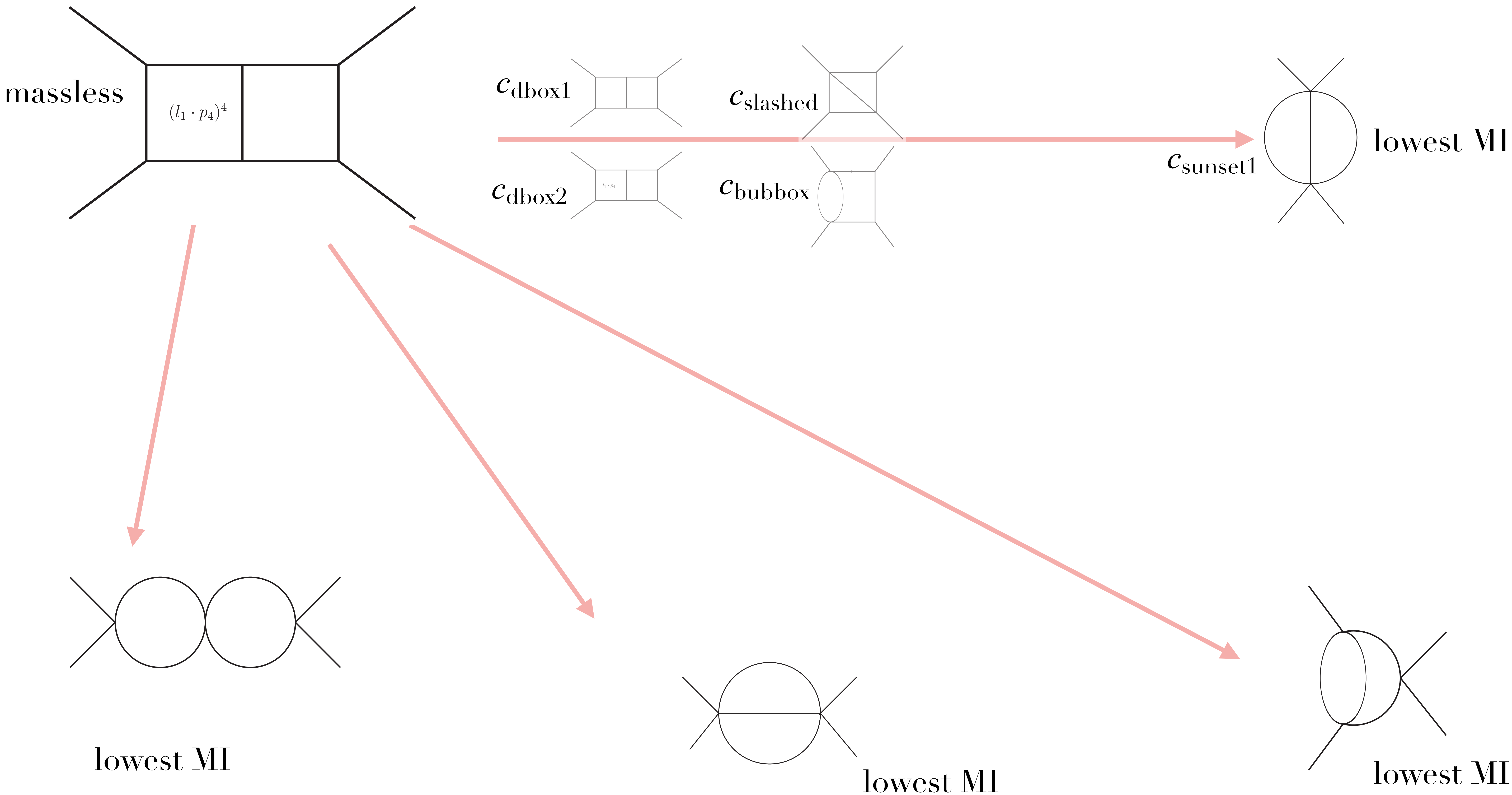

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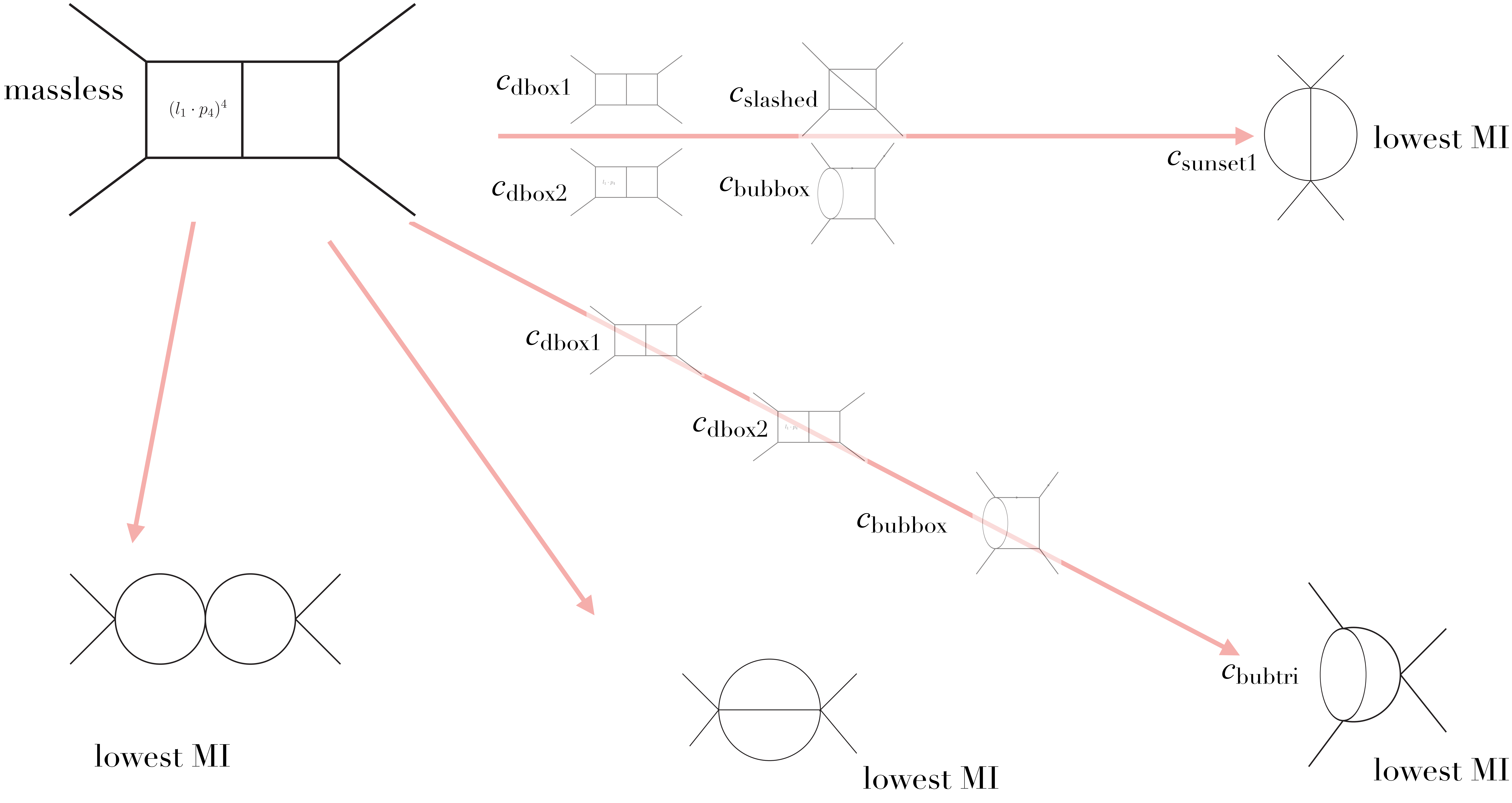

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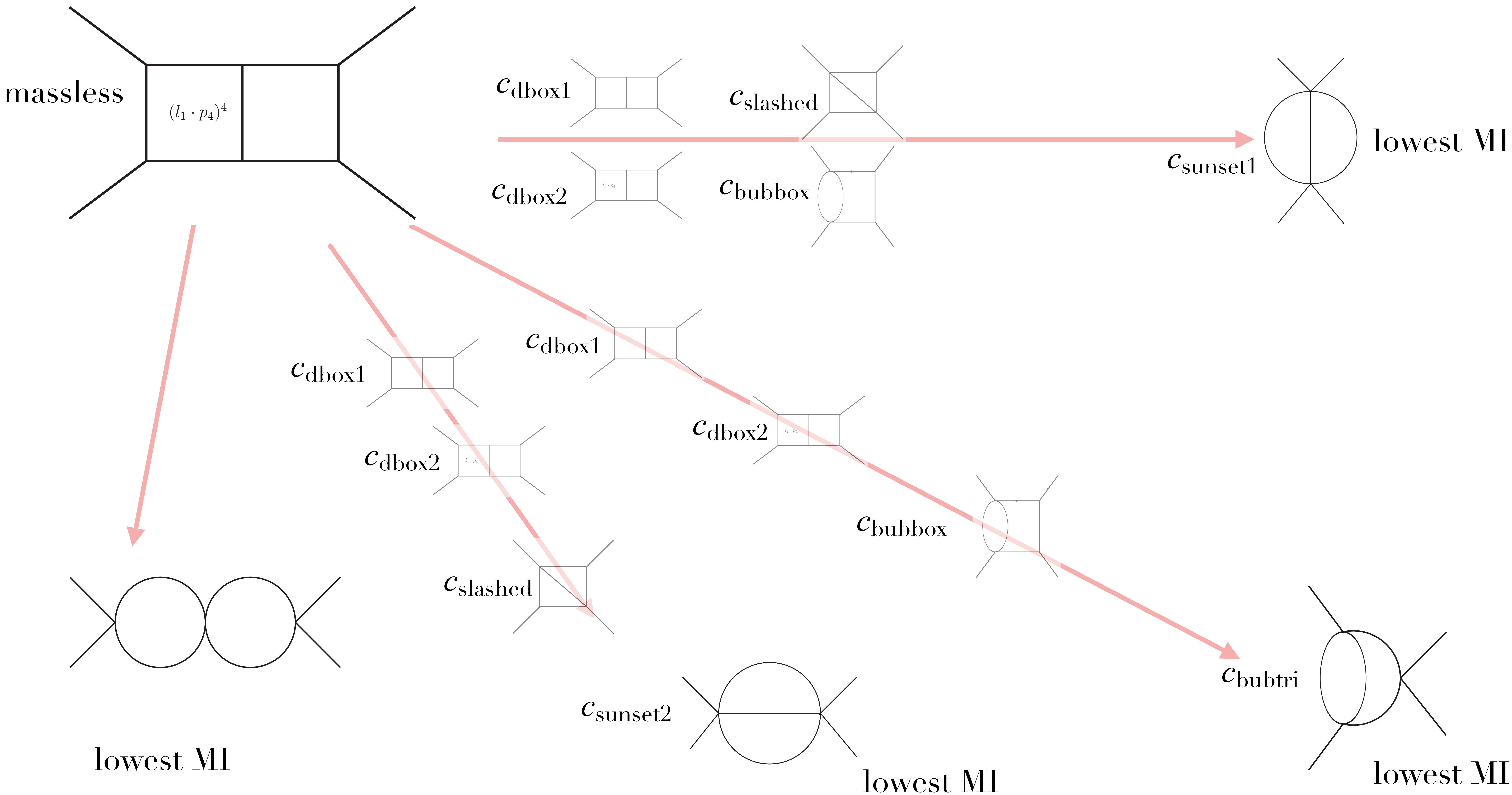

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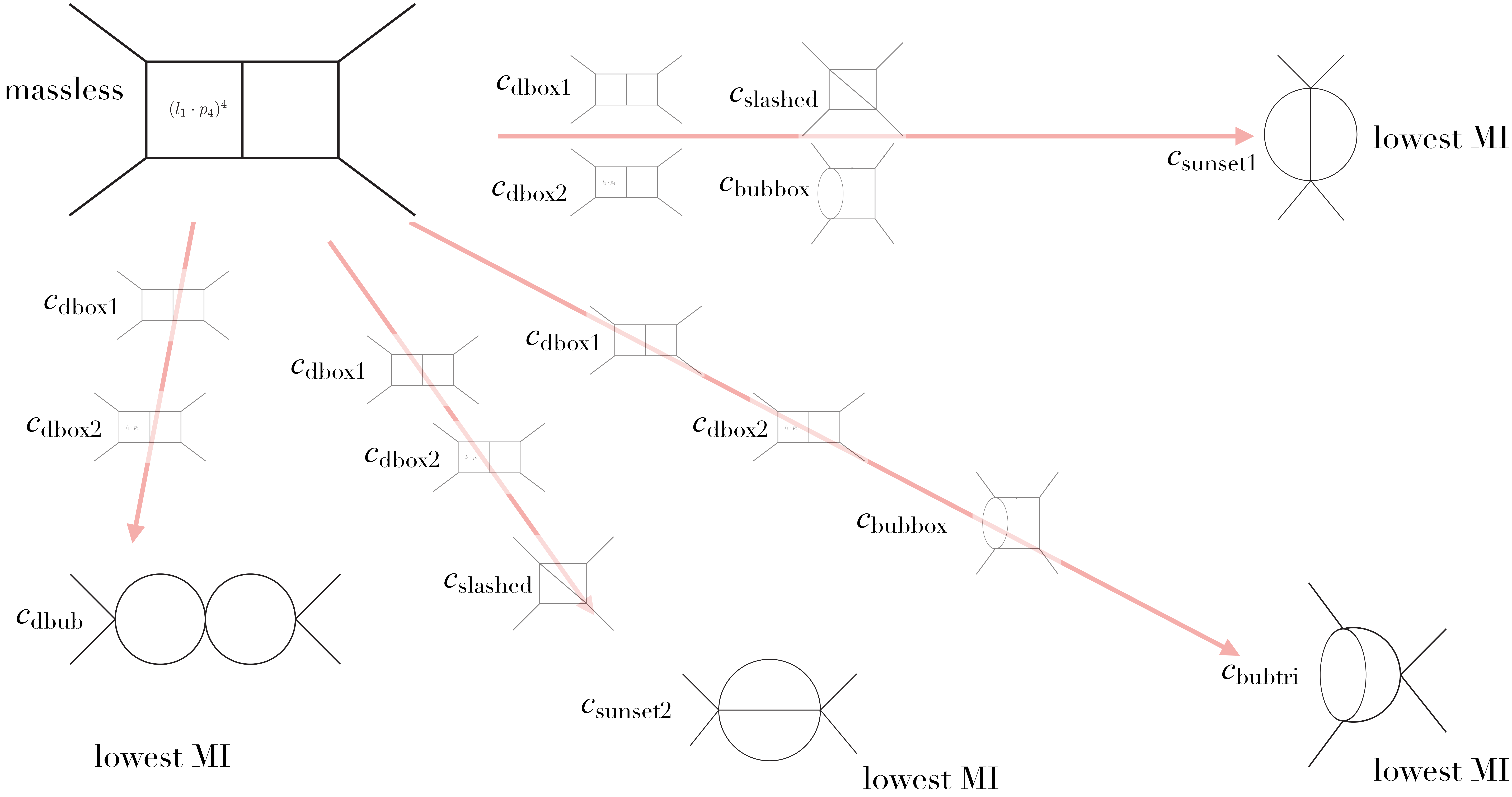

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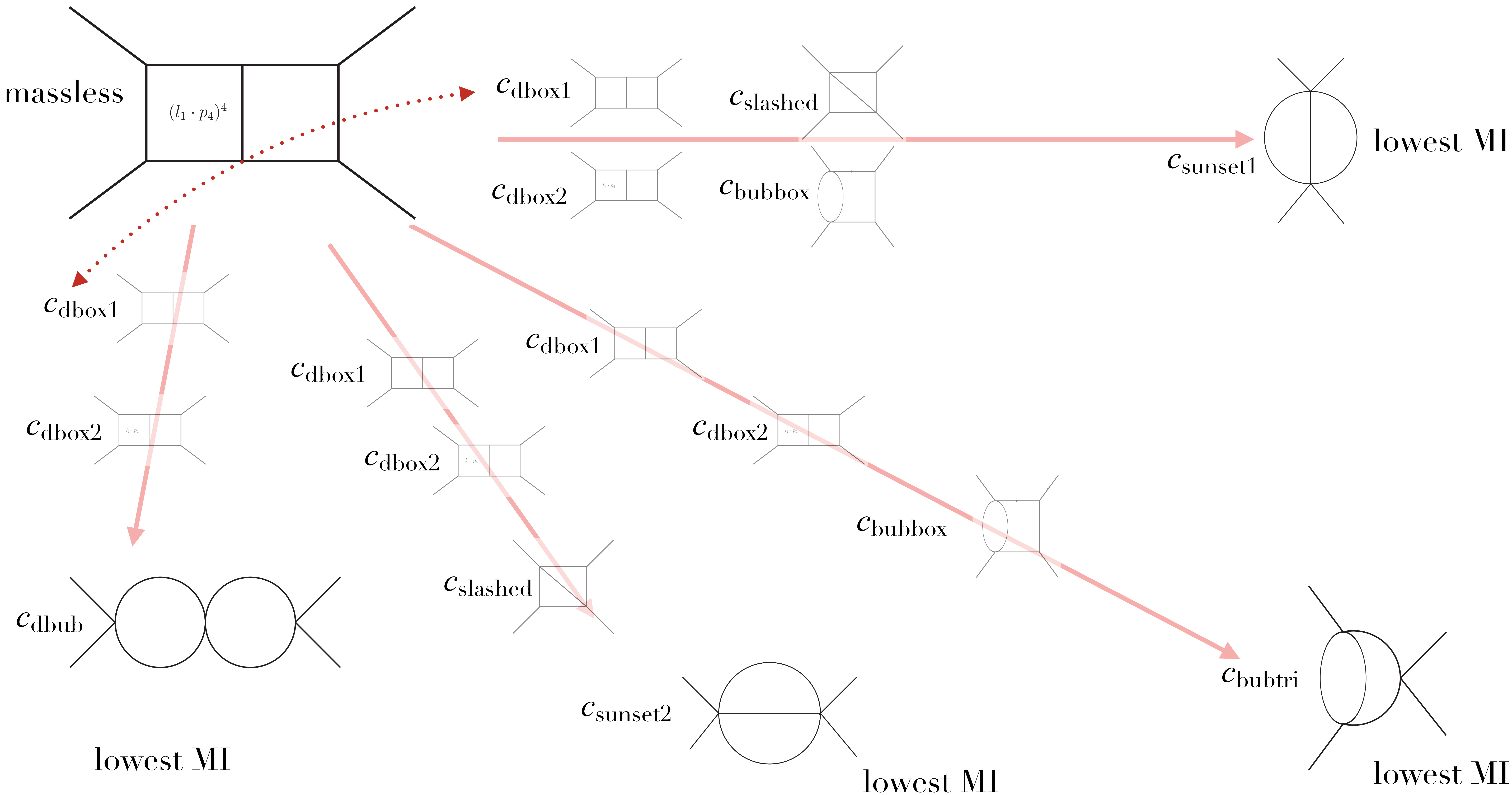

Summary

- Algebraic geometry approach for finding master integrals
- highly efficient for examples tested

Future directions

- Weyl algebra and D-modules approach
- dlog integrand form
- A fully automatic program for IBP reduction with cut reconstruction