# Baikov-Lee Representations Of Cut Feynman Integrals

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with Mark Harley and Francesco Moriello (arXiv:1705.03478)

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# Outline

1. Motivation
2. Background
   - The Euclidean Baikov-Lee Formula
   - A Simple One-Loop Example
3. The Baikov-Lee Representation For Cuts
   - Analytically Continuing The Baikov-Lee Formula
   - Implementing The Cuts With Residues
4. A Two-Loop Example
   - Triple Cut Of A Two-Loop Double Triangle
5. Outlook

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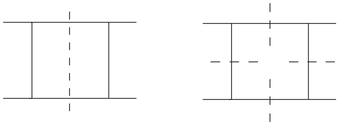

H. Ita, Phys. Rev. **D94** (2016) 116015; K. J. Larsen and Y. Zhang, Phys. Rev. **D93** (2016) 041701

# Why Not Just Use The Reverse Unitarity Method?

Reverse unitarity is a useful indirect approach to the evaluation of cut Feynman integrals which relies on differential equations.

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A. Primo and L. Tancredi, Nucl. Phys. **B916** (2017) 94;

H. Frellesvig and C. G. Papadopoulos, JHEP **1704** (2017) 083;

Bosma *et. al.*, arXiv:1704.05465; A. Primo and L. Tancredi, arXiv:1704.05465

# Integrate With Respect To The Set Of Scalar Products

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$$Q = L + n - 1 \qquad q_\ell \in \{k_1, \ldots, k_L, P_1, \ldots, P_{n-1}\}$$

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$$\prod_{j=i+1}^{Q} \mathrm{d}\left(q_i \cdot q_j\right) = G\left(q_{i+1}, \ldots, q_Q\right) \prod_{j=i+1}^{Q} \mathrm{d}c_j$$

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$$\Longrightarrow \mathrm{d}^{Q-i} q_i^{\parallel} = \sqrt{G\left(q_{i+1}, \ldots, q_Q\right)} \prod_{j=i+1}^{Q} \mathrm{d}c_j = \frac{\prod_{j=i+1}^{Q} \mathrm{d}\left(q_i \cdot q_j\right)}{\sqrt{G\left(q_{i+1}, \ldots, q_Q\right)}}$$

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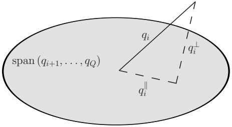

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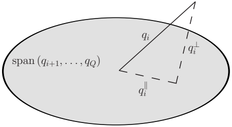


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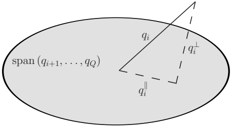


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$$= \frac{\pi^{\frac{d-Q+i}{2}}}{\Gamma\left(\frac{d-Q+i}{2}\right)} \left( \frac{\sqrt{G\left(q_i, \ldots, q_Q\right)}}{\sqrt{G\left(q_{i+1}, \ldots, q_Q\right)}} \right)^{d-Q+i-2} \mathrm{d}\left(q_i^2\right)$$

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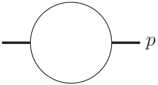
$$I_{\mathrm{E}} = \frac{\pi^{L(3+2d-2n-L)/4}}{\prod_{r=0}^{L-1} \Gamma\left(\frac{d-n-r+1}{2}\right) [G(P_1,\ldots,P_{n-1})]^{(d-n)/2}} \int_{\mathcal{D}} \cdots \int \left(\prod_{i=1}^{L} \prod_{j=i}^{L+n-1} \mathrm{d}(q_i \cdot q_j)\right)$$
$$[G(q_1, \ldots, q_{n+L-1})]^{(d-n-L)/2} \prod_{\ell=1}^{N} \left(Q_\ell^2(q_i \cdot q_j) + m_\ell^2\right)^{-\nu_\ell}$$

P. Baikov, Nucl. Instrum. Meth. **A389** (1997) 347; R. N. Lee, Nucl. Phys. Proc. Suppl. **205-206** (2010) 135

# The Virtual One-Loop Bubble In Baikov-Lee




$$= \int_{\mathcal{D}} \int \mathrm{d}(q_1^2)\mathrm{d}(q_1 \cdot q_2) \frac{\pi^{\frac{3}{2}-\epsilon}\left(p^2 q_1^2 - (q_1 \cdot q_2)^2\right)^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)\left(p^2\right)^{1-\epsilon} q_1^2 \left(q_1^2 - 2q_1 \cdot q_2 + p^2\right)}$$

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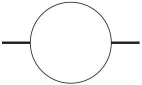


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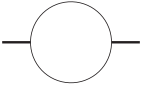


$$= \int_0^\infty \mathrm{d}(q_1^2)\int_{-\sqrt{p^2q_1^2}}^{\sqrt{p^2q_1^2}}\mathrm{d}(q_1\cdot q_2)\frac{\pi^{\frac{3}{2}-\epsilon}\left(p^2q_1^2-(q_1\cdot q_2)^2\right)^{\frac{1}{2}-\epsilon}}{\Gamma\left(\frac{3}{2}-\epsilon\right)\left(p^2\right)^{1-\epsilon}q_1^2\left(q_1^2-2q_1\cdot q_2+p^2\right)}$$

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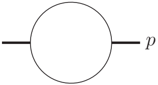


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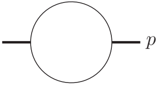


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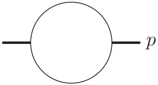
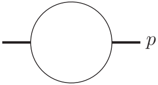



$$= \int_0^\infty \mathrm{d}(q_1^2)\frac{\pi^{2-\epsilon}\left(q_1^2\right)^{-\epsilon}}{\Gamma(2-\epsilon)(q_1^2+p^2)}\,{}_2F_1\left(1,\frac{1}{2};2-\epsilon;\frac{4q_1^2p^2}{(q_1^2+p^2)^2}\right)$$




$$= \frac{\pi^{2-\epsilon}\left(p^2\right)^{-\epsilon}\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}$$

# The Connection Between The Euclidean And Minkowski Feynman Parametric Formulae

$$I_{\mathrm{E}} = \frac{\pi^{\frac{Ld}{2}}\Gamma\left(\nu - \frac{Ld}{2}\right)}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left[\prod_{j=1}^{N}\int_0^\infty \mathrm{d}x_j\right]\delta(1-x_N)\,\mathcal{U}_{\mathrm{E}}^{\nu-(L+1)d/2}\mathcal{F}_{\mathrm{E}}^{Ld/2-\nu}\prod_{k=1}^{N}x_k^{\nu_k-1}$$

$$I_{\mathrm{M}} = \frac{i^L\pi^{\frac{Ld}{2}}e^{-i\pi\nu}\Gamma\left(\nu - \frac{Ld}{2}\right)}{\prod_{i=1}^{N}\Gamma(\nu_i)}\left[\prod_{j=1}^{N}\int_0^\infty \mathrm{d}x_j\right]\delta(1-x_N)\,\mathcal{U}_{\mathrm{M}}^{\nu-(L+1)d/2}\mathcal{F}_{\mathrm{M}}^{Ld/2-\nu}\prod_{k=1}^{N}x_k^{\nu_k-1}$$

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Given external kinematic invariants $\{\omega_1, \ldots, \omega_{n(n-1)/2}\}$,

$$I_{\mathrm{M}} = i^L e^{-i\pi\nu} I_{\mathrm{E}}\Big|_{\omega_i \to -\omega_i,\, \{P_j^*\}\to -\{P_j^*\}}$$

# Analytically Continuing The Baikov-Lee Formula

$$I_{\mathrm{M}} = \frac{i^L \pi^{L(3+2d-2n-L)/4} e^{-i\pi\nu}}{\prod_{r=0}^{L-1}\Gamma\left(\frac{d-n-r+1}{2}\right)\left[G(P_1,\ldots,P_{n-1})\right]^{(d-n)/2}} \int_{\mathcal{D}}\cdots\int \left(\prod_{i=1}^{L}\prod_{j=i}^{n+L-1}\mathrm{d}(q_i\cdot q_j)\right)$$

$$\left[G(q_1,\ldots,q_{n+L-1})\right]^{(d-n-L)/2}\prod_{\ell=1}^{N}\left(Q_\ell^2(q_i\cdot q_j)+m_\ell^2\right)^{-\nu_\ell}\Bigg|_{\omega_i\to-\omega_i,\,\{P_j^*\}\to-\{P_j^*\}}$$

$$= \frac{i^L \pi^{L(3+2d-2n-L)/4} e^{-i\pi\nu}}{\prod_{r=0}^{L-1}\Gamma\left(\frac{d-n-r+1}{2}\right)\left[\bar{G}(P_1,\ldots,P_{n-1})\right]^{(d-n)/2}} \int_{\bar{\mathcal{D}}}\cdots\int \left(\prod_{i=1}^{L}\prod_{j=i}^{n+L-1}\mathrm{d}(q_i\cdot q_j)\right)$$

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# Cut Propagators From Cauchy's Theorem

R. N. Lee and V. A. Smirnov, JHEP **1212** (2012) 104

Higher multiplicity cut propagators can be straightforwardly treated using the generalized Cutkosky cutting rule

$$\frac{1}{(k^2+i0)^\nu} - \frac{1}{(k^2-i0)^\nu} = -2\pi i\theta(k^0)\frac{(-1)^{\nu-1}}{\Gamma(\nu)}\delta^{(\nu-1)}\left(k^2\right)$$

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The Baikov-Lee representation of the $j$-th integral cut in $\omega_i$:

$$I^{(j)}_{\omega_i-\mathrm{cut}} = \frac{(-2\pi i)^{n_j}\pi^{L(3-2n-L)/4}e^{-i\pi\nu}}{\prod_{r=0}^{L-1}\Gamma\left(\frac{d-n-r+1}{2}\right)\left[\bar{G}(P_1,\ldots,P_{n-1})\right]^{(d-n)/2}}\,\mathbf{sgn}\left(\left|\frac{\partial\{\bar{Q}^2_\ell\}^{(j)}_{\omega_i-\mathrm{cut}}}{\partial\{\bar{s}_i\}}\right|\right)$$
$$\int\cdots\int_{\bar{\mathcal{D}}^{(j)}_{\omega_i-\mathrm{cut}}}\left(\prod_{k=1}^{L(L-1)/2+nL-n_j}\mathrm{d}s_k\right)\underset{\{\bar{s}_i\}}{\mathbf{Res}}\left\{\frac{\left[\bar{G}(q_1,\ldots,q_{n+L-1})\right]^{(d-n-L)/2}}{\prod_{\ell=1}^{N}\left(\bar{Q}^2_\ell(\bar{s}_i,s_k)+m^2_\ell\right)^{\nu_\ell}}\right\}$$

# Why sgn?

The **sgn** function is necessary because

$$\underset{\{a\}}{\mathbf{Res}}\left\{\frac{f(z)}{a-z}\right\} = -f(a)$$

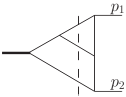
but

$$\int_{-\infty}^{\infty} \mathrm{d}z\, \delta(a-z) f(z) = \int_{-\infty}^{\infty} \mathrm{d}z\, \delta(z-a) f(z) = f(a)$$

for arbitrary test functions $f(z)$ regular at $z = a$

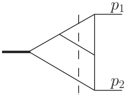
# Triple Cut Of A Two-Loop Double Triangle

$$s_1 = k_1 \cdot p_1 \quad s_2 = k_1^2 \quad s_3 = k_2 \cdot p_1 \quad s_4 = k_2 \cdot p_2$$
$$\bar{s}_1 = k_2^2 \quad \bar{s}_2 = k_1 \cdot k_2 \quad \bar{s}_3 = k_1 \cdot p_2 \quad s = (p_1 + p_2)^2$$




$$= -\frac{2^{2-4\epsilon} s^{-1+2\epsilon} i}{\Gamma(1-2\epsilon)} \int\int\int\int_{\bar{\mathcal{D}}^{(1)}_{s-\mathrm{cut}}} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4 \underset{\{\bar{s}_1,\bar{s}_2,\bar{s}_3\}}{\mathbf{Res}} \Bigg\{ \frac{\Big(-s^2(\bar{s}_2^2 - \bar{s}_1 s_2)/4 + s(\bar{s}_1\bar{s}_3 s_1 - \bar{s}_2(\bar{s}_3 s_3 + s_1 s_4) + s_2 s_3 s_4) - (\bar{s}_3 s_3 - s_1 s_4)^2\Big)^{-1/2-\epsilon}}{(s_2 + 2s_1)(\bar{s}_1 + 2s_3)(\bar{s}_2 - s_2/2 - \bar{s}_1/2)\bar{s}_1 s_2 (\bar{s}_3 - s_2/2)} \Bigg\}$$

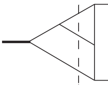
# Localize The $\bar{s}_i$ Variables With Cauchy's Theorem, Find Branch Points Of The Integrand In The $s_i$




$$= \frac{2^{1-4\epsilon} s^{-1+2\epsilon} i}{\Gamma(1-2\epsilon)} \int\int_{\bar{\mathcal{D}}^{(1)}_{s-\mathrm{cut}}}\int\int \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_2 s_3 (s_2 + 2s_1)} \Big( -(s_2 s_3 - 2 s_1 s_4)^2/4$$

$$+ s(s_2 s_3 s_4 - (s_2 - 2s_3)(s_2 s_3 + 2 s_1 s_4)/4 - s_1 s_2 s_3) - s^2 (s_2 + 2s_3)^2/16\Big)^{-1/2-\epsilon}$$

# Localize The $\bar{s}_i$ Variables With Cauchy's Theorem, Find Branch Points Of The Integrand In The $s_i$

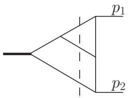



$$= \frac{2^{1-4\epsilon} s^{-1+2\epsilon} i}{\Gamma(1-2\epsilon)} \int\int\int\int_{\bar{\mathcal{D}}^{(1)}_{s-\mathrm{cut}}} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_2 s_3 \left(s_2 + 2 s_1\right)} \Big( -\left(s_2 s_3 - 2 s_1 s_4\right)^2/4$$

$$+ s(s_2 s_3 s_4 - (s_2 - 2 s_3)(s_2 s_3 + 2 s_1 s_4)/4 - s_1 s_2 s_3) - s^2 (s_2 + 2 s_3)^2/16 \Big)^{-1/2-\epsilon}$$

$$\Longrightarrow s_4^{\pm} = \frac{(s_1 s_2 + s(s_1 + s_2)) s_3 - s_1 s_2 s/2 \pm \sqrt{s}\sqrt{s_2(2 s_1 + s_2) s_3 (s_3 - s_1)(2 s_1 + s)}}{2 s_1^2}$$

# Explicitly Evaluate The Resulting Definite Integrals

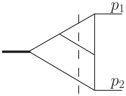



$$= \frac{2^{1-4\epsilon} s^{-1+2\epsilon} i}{\Gamma(1-2\epsilon)} \int_{-s/2}^{0} \mathrm{d}s_1 \int_{0}^{-2s_1} \mathrm{d}s_2 \int_{s_1}^{0} \mathrm{d}s_3 \int_{s_4^-}^{s_4^+} \mathrm{d}s_4$$

$$\frac{1}{s_2 s_3 (s_2 + 2s_1)} \Big( -(s_2 s_3 - 2 s_1 s_4)^2/4 - s^2 (s_2 + 2s_3)^2/16$$

$$+ s(s_2 s_3 s_4 - (s_2 - 2s_3)(s_2 s_3 + 2 s_1 s_4)/4 - s_1 s_2 s_3) \Big)^{-1/2-\epsilon}$$

# Explicitly Evaluate The Resulting Definite Integrals



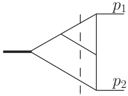

$$= \frac{2^{1-4\epsilon}s^{-1+2\epsilon}i}{\Gamma(1-2\epsilon)}\int_{-s/2}^{0} \mathrm{d}s_1 \int_0^{-2s_1} \mathrm{d}s_2 \int_{s_1}^{0} \mathrm{d}s_3 \int_{s_4^-}^{s_4^+} \mathrm{d}s_4$$

$$\frac{1}{s_2 s_3 \left(s_2+2s_1\right)}\Big( -\left(s_2 s_3 - 2 s_1 s_4\right)^2/4 - s^2\left(s_2+2s_3\right)^2/16$$

$$+s\left(s_2 s_3 s_4 - (s_2 - 2s_3)(s_2 s_3 + 2 s_1 s_4)/4 - s_1 s_2 s_3\right)\Big)^{-1/2-\epsilon}$$

Mathematica $\Longrightarrow$ $$= -\frac{2i\sin(2\pi\epsilon)s^{-2-2\epsilon}\Gamma(-1-2\epsilon)\Gamma(1+2\epsilon)\Gamma^3(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(-3\epsilon)}$$

# Explicitly Evaluate The Resulting Definite Integrals




$$= \frac{2^{1-4\epsilon} s^{-1+2\epsilon} i}{\Gamma(1-2\epsilon)} \int_{-s/2}^{0} \mathrm{d}s_1 \int_{0}^{-2s_1} \mathrm{d}s_2 \int_{s_1}^{0} \mathrm{d}s_3 \int_{s_4^-}^{s_4^+} \mathrm{d}s_4$$

$$\frac{1}{s_2 s_3 \left(s_2 + 2s_1\right)} \Big( -(s_2 s_3 - 2 s_1 s_4)^2/4 - s^2 (s_2 + 2s_3)^2/16$$

$$+ s(s_2 s_3 s_4 - (s_2 - 2s_3)(s_2 s_3 + 2 s_1 s_4)/4 - s_1 s_2 s_3) \Big)^{-1/2-\epsilon}$$

Mathematica $\Longrightarrow$ $$= -\frac{2i \sin(2\pi\epsilon) s^{-2-2\epsilon} \Gamma(-1-2\epsilon)\Gamma(1+2\epsilon)\Gamma^3(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(-3\epsilon)}$$

Needless to say, this agrees with the known result!

W. L. van Neerven, Nucl. Phys. **B268** (1986) 453

# Outlook

Several ideas for further research come to mind:

- Take a closer look at integrals cut in several channels sequentially. (*e.g.* The $s$-channel + $t$-channel cut of a one-loop box)

  S. Abreu *et. al.*, JHEP **1410** (2014) 125

- Look at iterated cuts in a single channel.
- Apply to the analytical calculation of a $pp \to t\bar{t}$ soft function.
- Look at more complicated examples numerically with sector decomposition using the program `pySecDec`.

  S. Borowka *et. al.*, arXiv:1703.09692