Inclusive jets and their substructure

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Outline

- Inclusive jets
- Subjets *Kang, FR, Vitev `16*
  - Inclusive subjets *Kang, FR, Waalewijn `17*
  - Centered around an axis
- Conclusions

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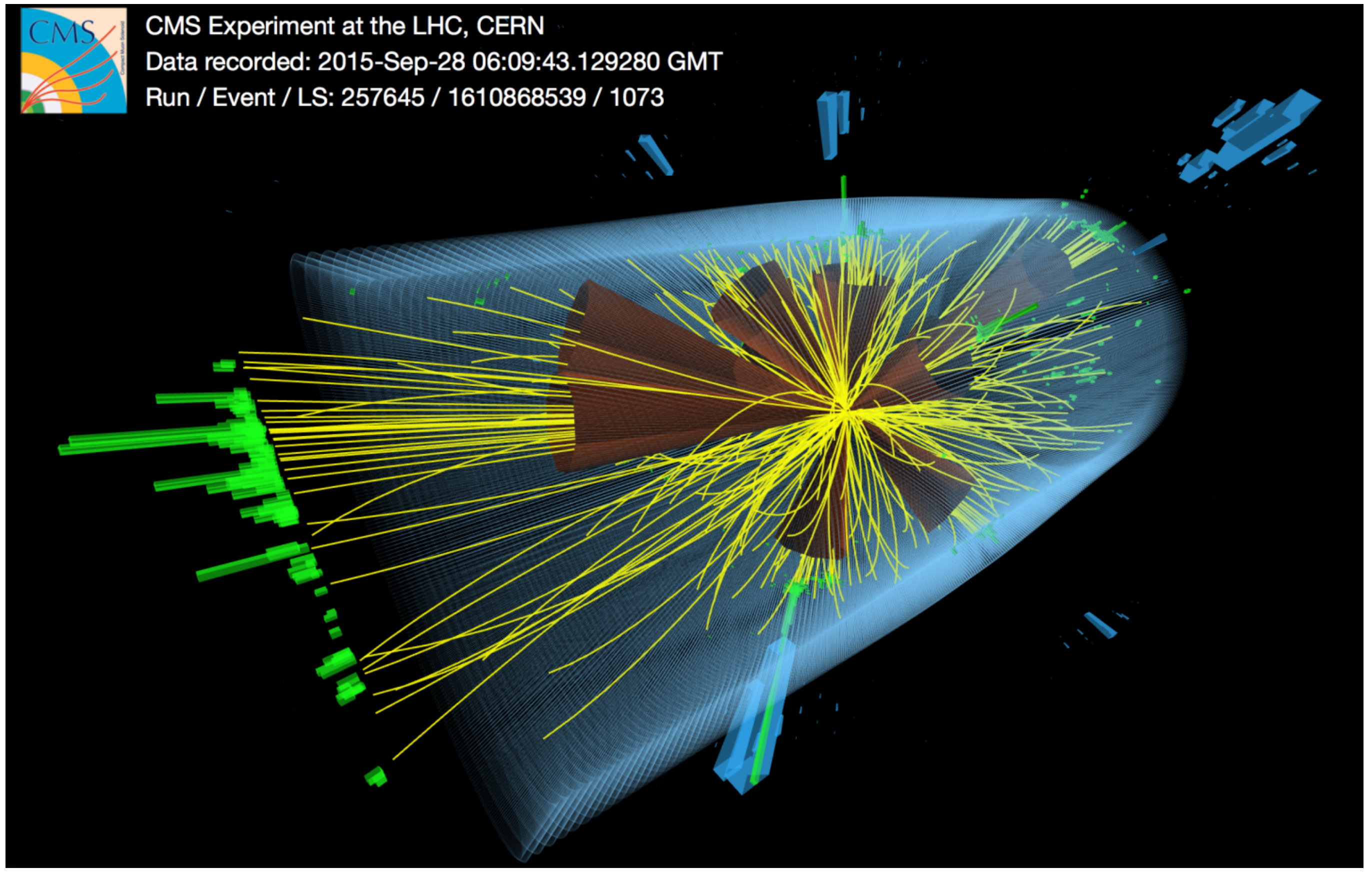
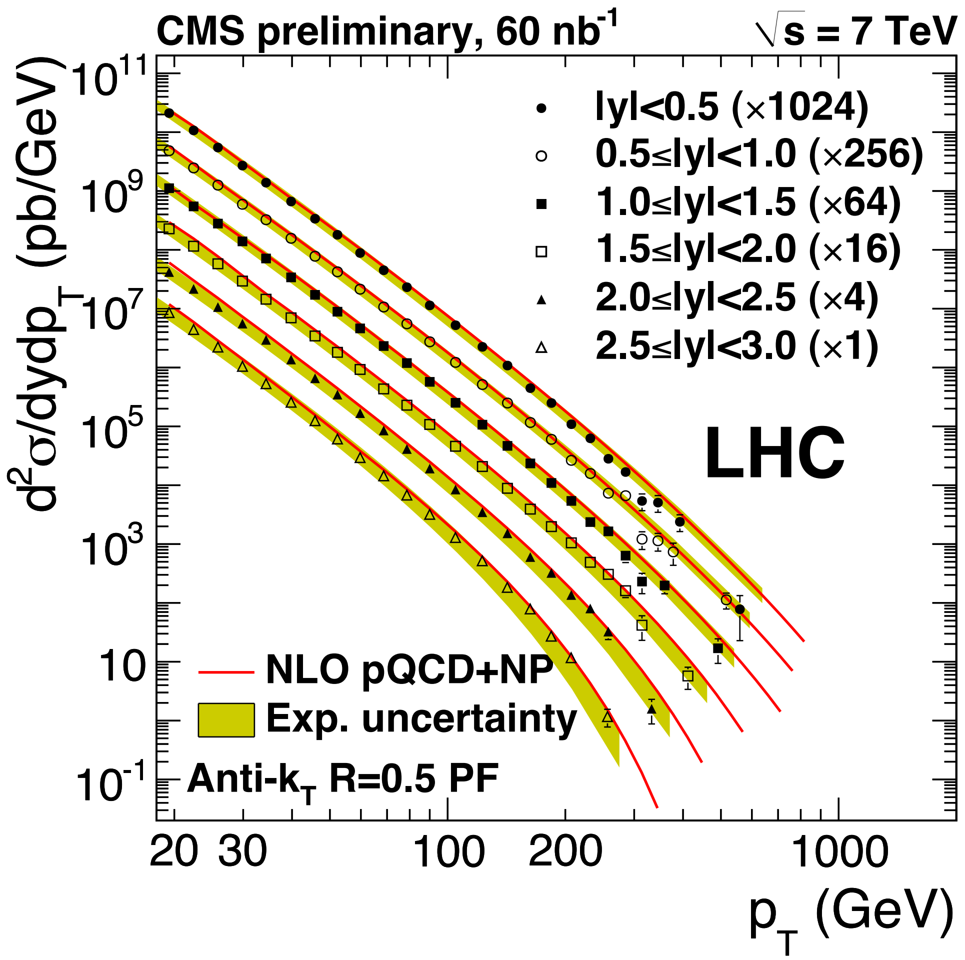
Inclusive Jet Production $pp \to \mathrm{jet}X$

- PDFs and α_s are constrained by collider jet data
- High p_T jets are a promising observable for the search of BSM physics at the LHC
- Baseline for jet quenching in heavy-ion collisions





Inclusive Jet Production $pp \to \mathrm{jet}X$

Factorization

Kang, FR, Vitev `16

$R \sim 1$

$$\frac{d\sigma^{pp\to\mathrm{jet}X}}{dp_T d\eta} = \sum_{a,b,c} f_a \otimes f_b \otimes H_{ab} \longrightarrow$$

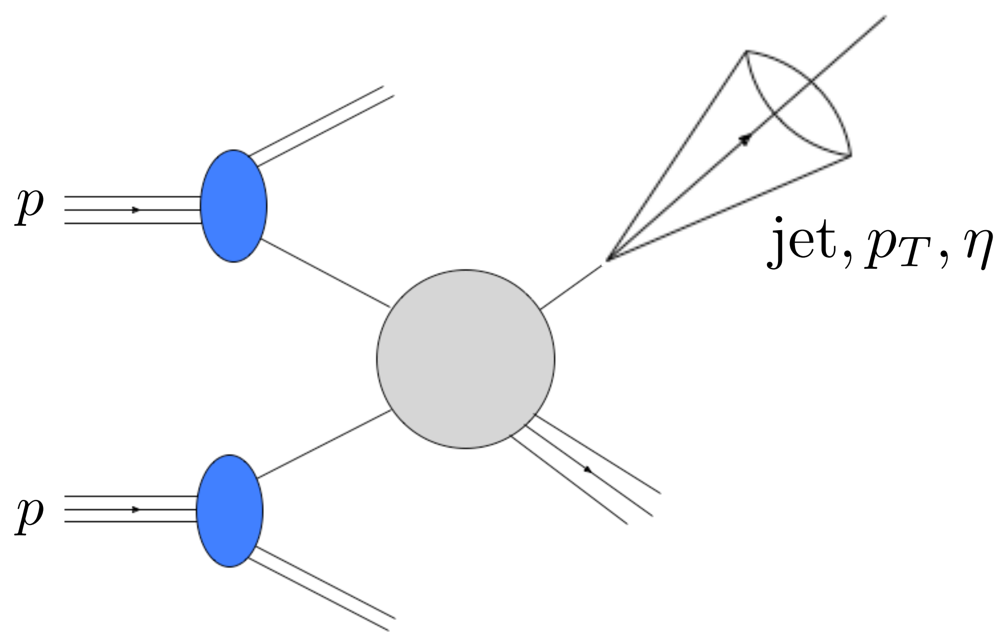
$R \ll 1$

$$\frac{d\sigma^{pp\to\mathrm{jet}X}}{dp_T d\eta} = \sum_{a,b,c} f_a \otimes f_b \otimes H_{ab}^c \otimes \boxed{J_c} + \mathcal{O}(R^2)$$

(e.g. <4% difference for R=0.7)

Ellis, Kunszt, Soper `90, Aversa, Chiappetta, Greco, Guillet `90, Jäger, Stratmann, Vogelsang `04, Currie, Glover, Pires `16, …

see also:
Kaufmann, Mukherjee, Vogelsang `15
Dai, Kim, Leibovich `16



Inclusive Jet Production $pp \to \mathrm{jet}X$

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Factorization

$R \sim 1$

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$\longrightarrow$

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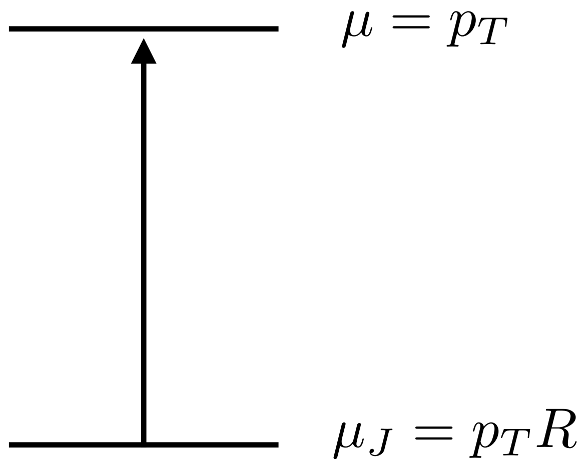
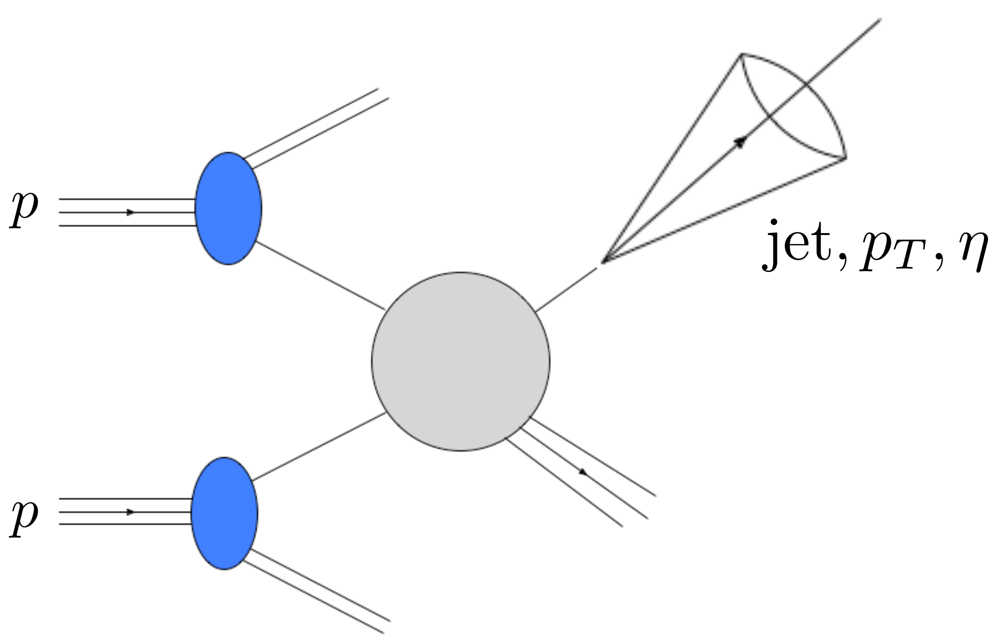
timelike DGLAP for <u>semi-inclusive jet function</u>

$$\boxed{\mu \frac{d}{d\mu} J_i = \sum_j P_{ji} \otimes J_j}$$



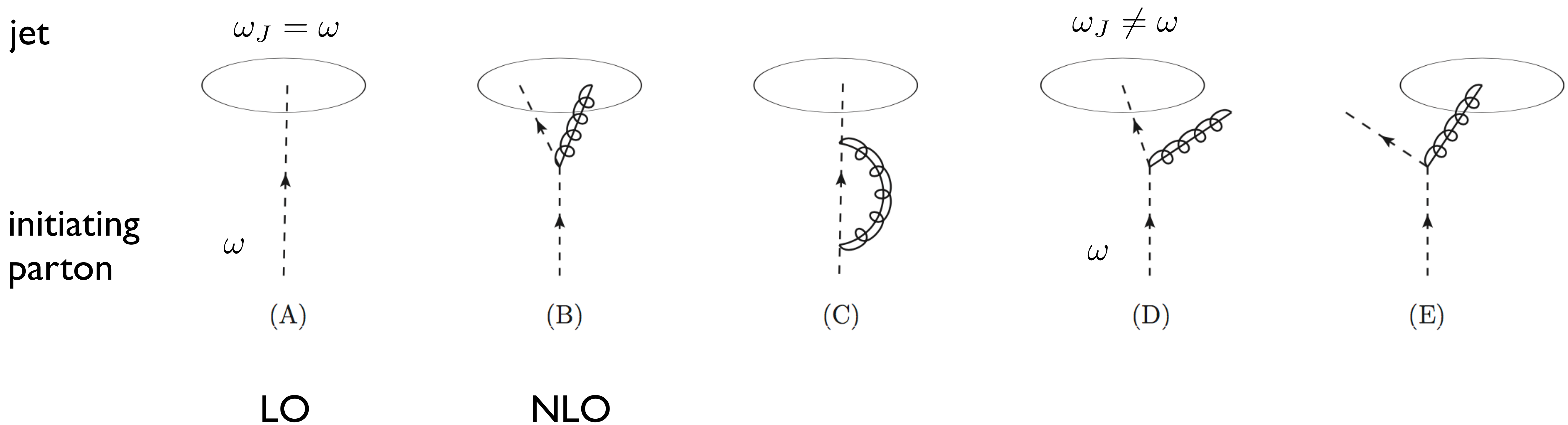


resummation of $\alpha_s^n \ln^n R$

see also: *Dasgupta, Dreyer, Salam, Soyez `15, `16*

Semi-inclusive jet function in SCET up to NLO

- The siJFs describe how a parton is transformed into a jet with radius R and energy fraction z



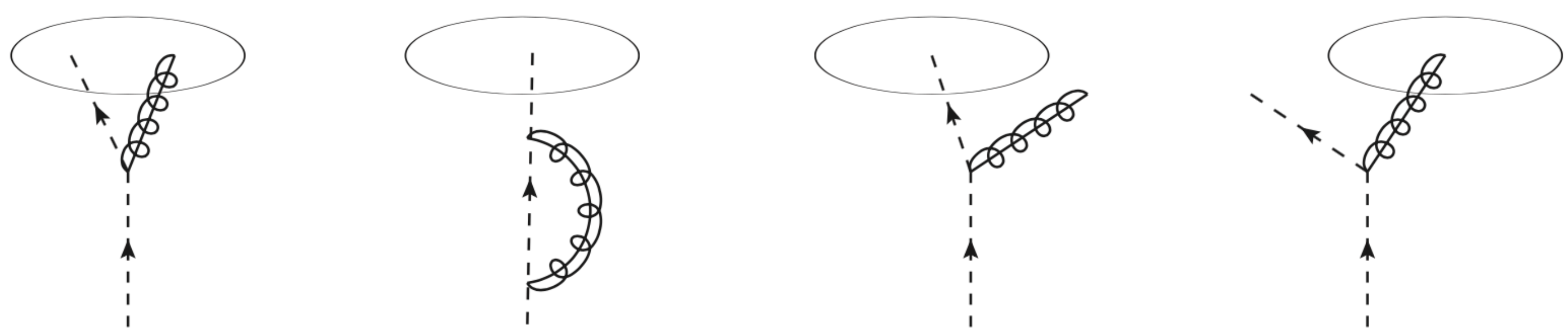
where $z = \omega_J/\omega$ momentum sum rule: $\int_0^1 dz\, z J_i(z, \omega R, \mu) = 1$

Semi-inclusive jet function in SCET at NLO

$\overline{\text{MS}}$ scheme, anti-k_T

$$
\begin{aligned}
J_q^{(1)}(z,\omega_J) =& J_{q\to qg}(z,\omega_J) + J_{q\to q(g)}(z,\omega_J) + J_{q\to (q)g}(z,\omega_J) \\
=& \frac{\alpha_s}{2\pi}\left[\overset{\text{UV}}{\frac{1}{\epsilon}} + \ln\left(\frac{\mu^2}{\omega_J^2\tan^2(R/2)}\right)\right]\left[P_{qq}(z)+P_{gq}(z)\right] \\
& - \frac{\alpha_s}{2\pi}\left\{C_F\left[2\left(1+z^2\right)\left(\frac{\ln(1-z)}{1-z}\right)_+ + (1-z)\right] - \delta(1-z)d_J^{q,\text{alg}} \right. \\
& \left. + P_{gq}(z)2\ln(1-z) + C_F z\right\},
\end{aligned}
$$

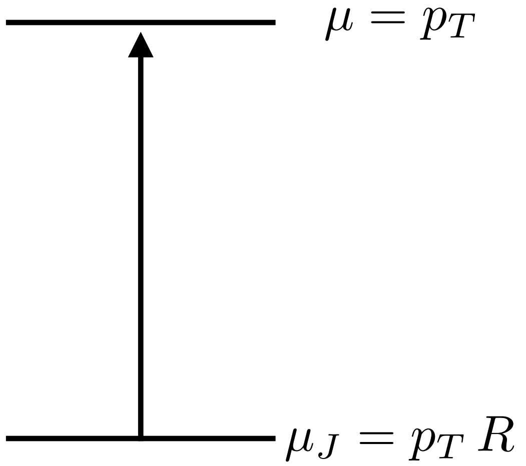
$$
d_J^{q,\text{anti-k}_\text{T}} = C_F\left(\frac{13}{2} - \frac{2\pi^2}{3}\right)
$$



(B) (C) (D) (E)

Renormalization and RG evolution

Bare - renormalized semi-inclusive jet function



$$J_{i,\text{bare}}(z,\omega_J) = \sum_j \int_z^1 \frac{dz'}{z'} Z_{ij}\left(\frac{z}{z'},\mu\right) J_j(z',\omega_J,\mu)$$

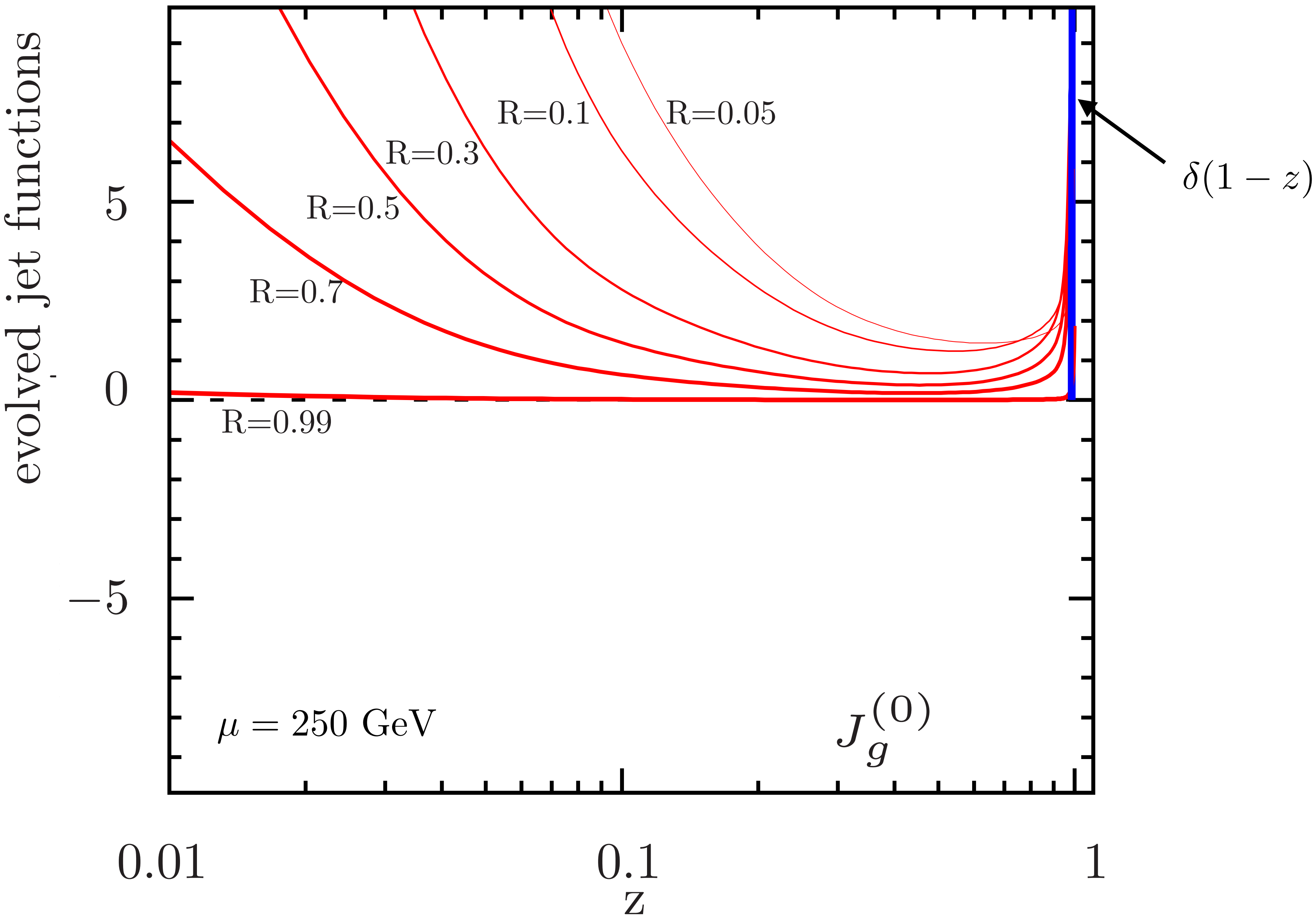
RG equation:

$$\boxed{\mu\frac{d}{d\mu}J_i(z,\omega_J,\mu) = \frac{\alpha_s(\mu)}{\pi}\sum_j \int_z^1 \frac{dz'}{z'} P_{ji}\left(\frac{z}{z'},\mu\right) J_j(z',\omega_J,\mu)}$$

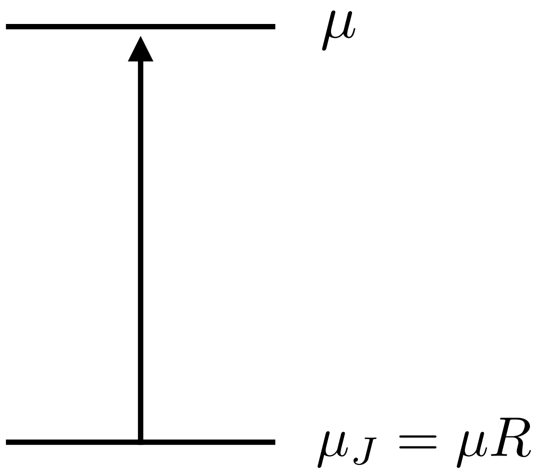
DGLAP evolution equation like for FFs. Resums single $\ln R$: $\mathrm{NLO{+}NLL}_R$

$\longrightarrow$ solve in Mellin moment space

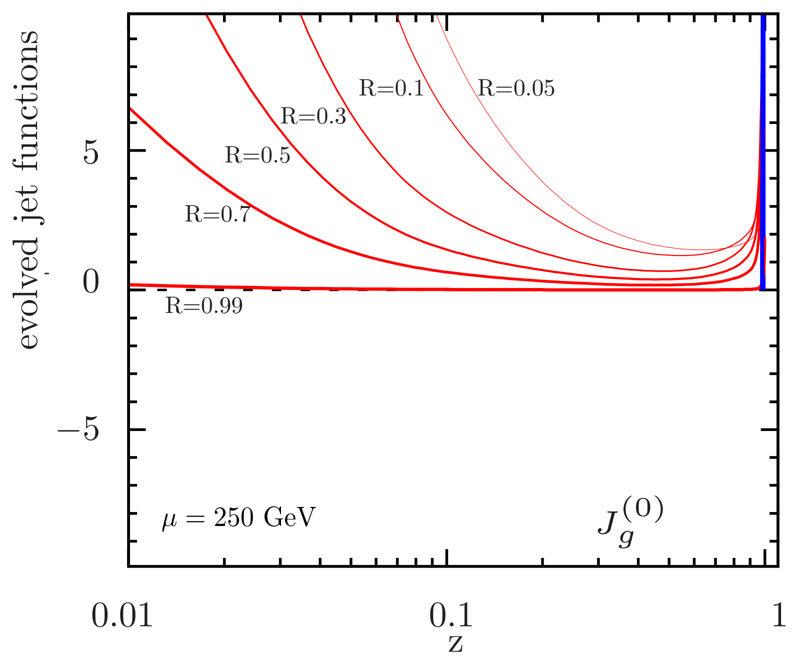
see also
Dasgupta, Dreyer, Salam, Soyez `15, `16



NLL_R DGLAP evolution



see
Vogt `04 (Pegasus), Anderle, FR, Stratmann `15

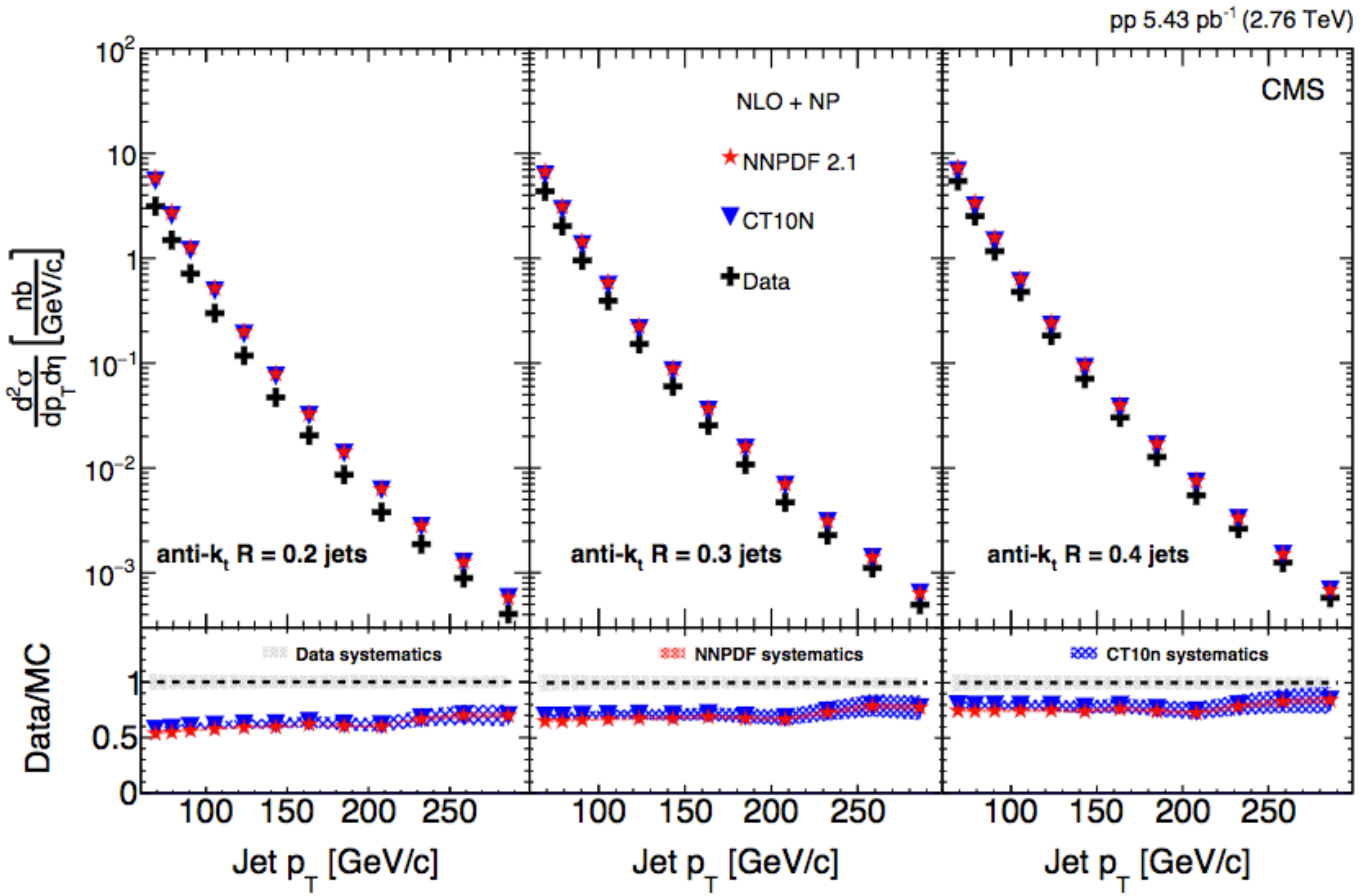


$$\frac{d\sigma^{pp\to \mathrm{jet}X}}{dp_T d\eta} = \sum_{a,b,c} f_a \otimes f_b \otimes H_{ab}^c \otimes J_c$$

- Resummation of $\alpha_s^n \ln^n R$
- Adopt a prescription used for quarkonium fragmentation functions

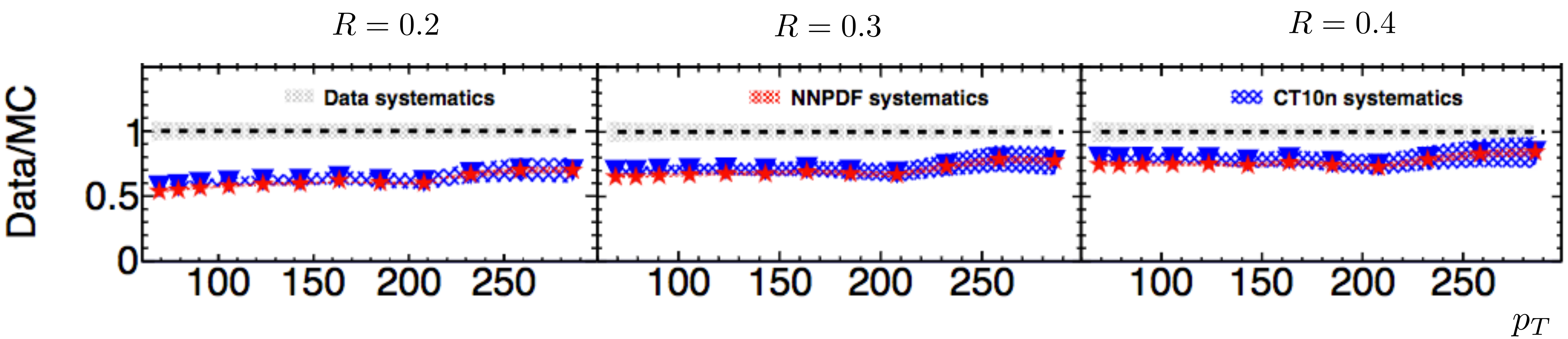
  Bodwin, Chao, Chung, Kim, Lee, Ma `16

Inclusive Jet Production in SCET $pp \to \mathrm{jet}X$



CMS data arXiv:1609.05383

Inclusive Jet Production in SCET $pp \to \mathrm{jet}X$



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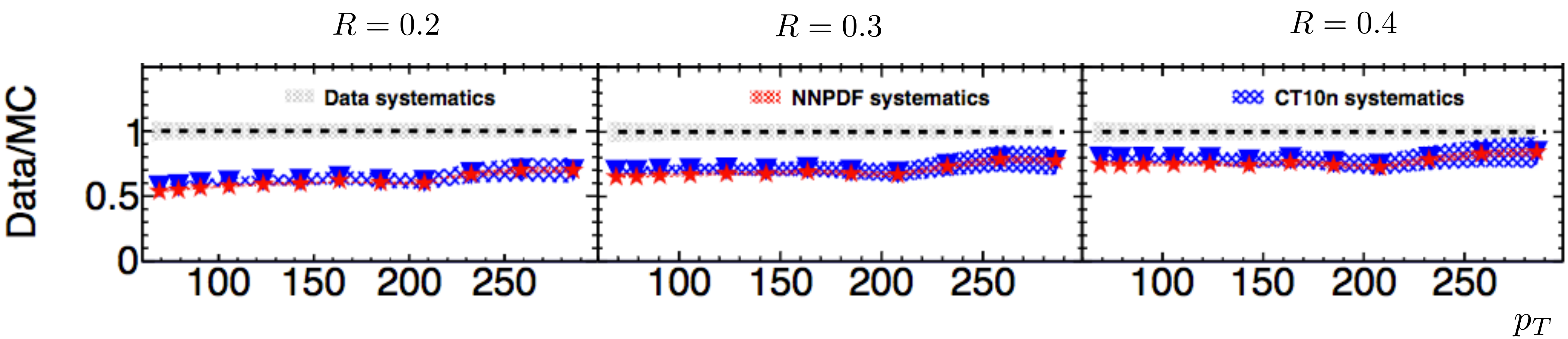
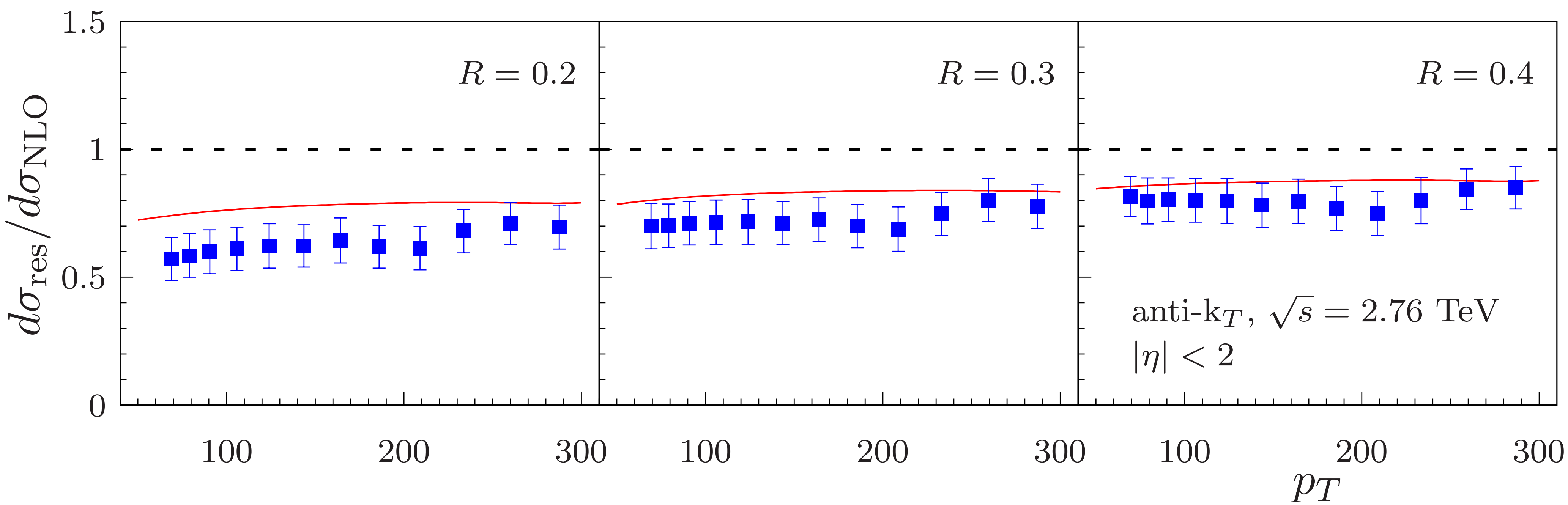
Inclusive Jet Production in SCET $pp \to \mathrm{jet}X$





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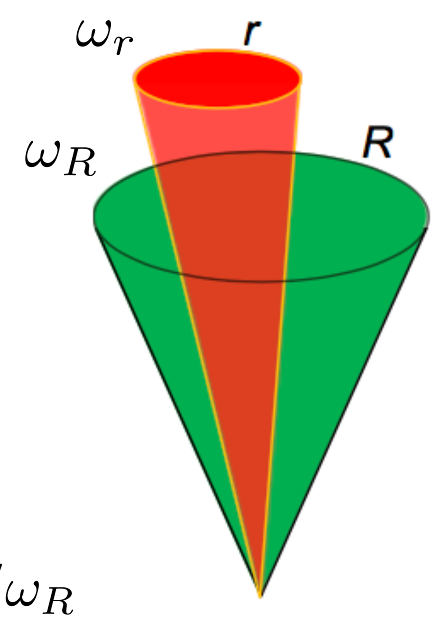
Inclusive subjets

- Recluster particles of identified jet with a smaller jet parameter $r < R$
- Longitudinal and transverse energy profile of jets
- More differential than traditional jet shapes
- Requires resummation of $\ln R,\ \ln(r/R)$

$$pp \to (\text{jet}\, j_r) + X$$

$$F(z_r, r; \eta, p_T, R) = \frac{\mathrm{d}\sigma}{\mathrm{d}\eta\, \mathrm{d}p_T\, \mathrm{d}z_r} \Big/ \frac{\mathrm{d}\sigma}{\mathrm{d}\eta\, \mathrm{d}p_T}$$



$$z_r = \omega_r / \omega_R$$

- Applications for tagging or heavy-ion physics

see also: *Kaufmann, Vogelsang, Mukherjee `16;*
Kang, FR, Vitev `16; Dai, Kim, Leibovich `16;
Neill, Scimemi, Waalewijn `16;
Elder, Produra, Thaler, Waalewijn, Zhou `17;
Kang, Liu, FR, Xing `17

Inclusive subjets

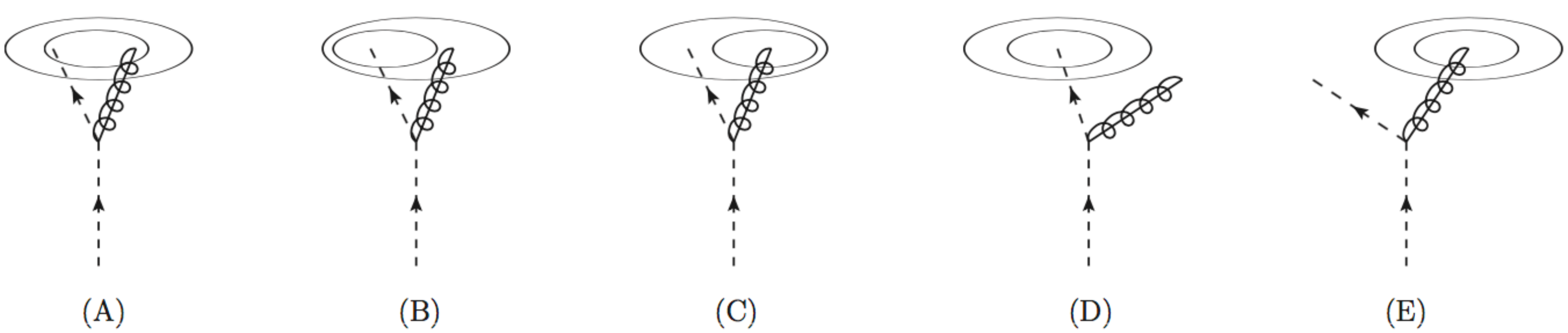
- Factorization for $R \ll 1$

$$\frac{d\sigma}{d\eta\, dp_T\, dz_r} = \sum_{a,b,c} f_a(x_a,\mu) \otimes f_b(x_b,\mu) \otimes \mathcal{H}_{ab}^c\,(x_a, x_b, \eta, p_T/z, \mu) \otimes \mathcal{G}_c^{\mathrm{jet}}(z, z_r, \omega_R, \mu)$$

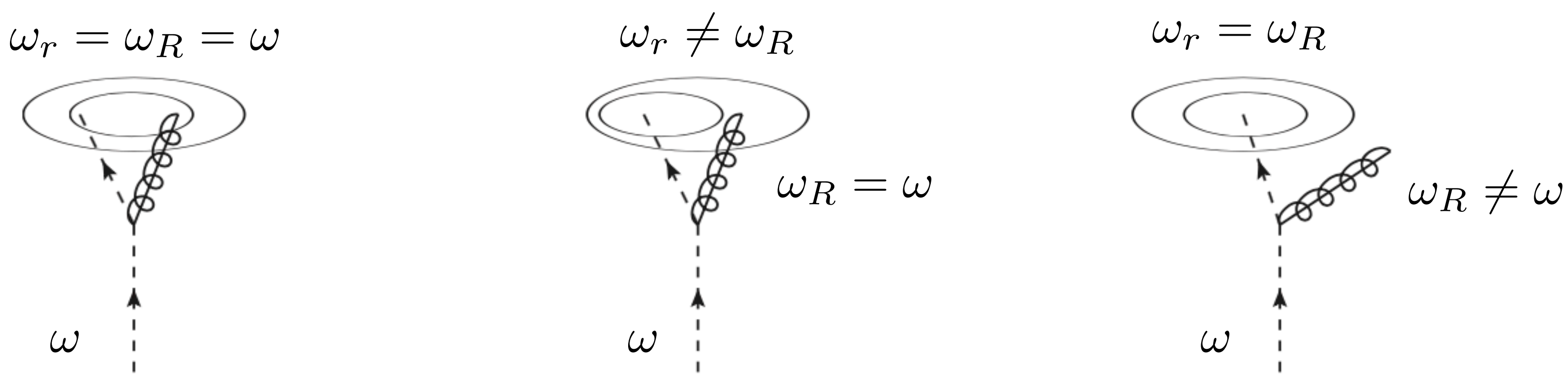
same hard functions as before

semi-inclusive subjet function (siSJF)

- The quark siSJF at NLO



Inclusive subjets



(A) (B) (D)

$$z = \frac{\omega_R}{\omega}\,, \qquad z_r = \frac{\omega_r}{\omega_R}$$

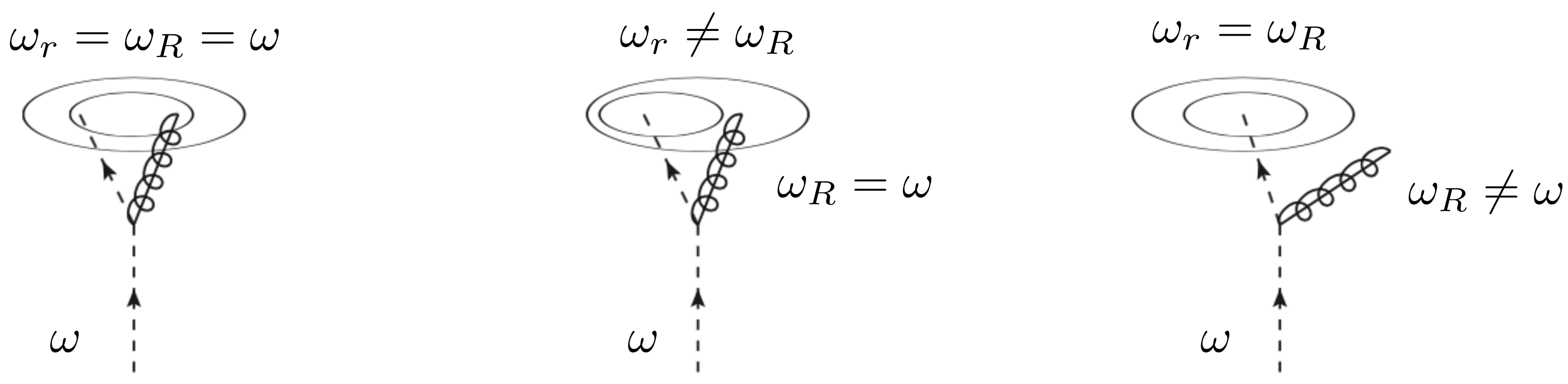
$$\begin{aligned}\mathcal{G}_{q,\mathrm{bare}}^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = {} & \delta(1-z)\delta(1-z_r) + \frac{\alpha_s}{2\pi}\bigg\{\delta(1-z_r)\Big(\frac{1}{\epsilon} + L_R\Big)\left[P_{qq}(z) + P_{gq}(z)\right] \\ & + \delta(1-z) L_{r/R}\left[P_{qq}(z_r) + P_{gq}(z_r)\right] + C_F\,\delta(1-z_r)\bigg[\delta(1-z)\Big(\frac{13}{2} - \frac{2\pi^2}{3}\Big) \\ & - 2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\ln(1-z)\,\frac{1+(1-z)^2}{z} - 1\bigg]\bigg\}\end{aligned}$$

where $L_{r/R} = L_r - L_R\,, \qquad L_R = \ln\left(\frac{4\mu^2}{\omega_R^2 R^2}\right)\,, \quad L_r = \ln\left(\frac{4\mu^2}{\omega_R^2 r^2}\right)$

$r \sim R \ll 1$

anti-k$_T$-in-anti-k$_T$

Inclusive subjets



(A) (B) (D)

$$z = \frac{\omega_R}{\omega}\,, \qquad z_r = \frac{\omega_r}{\omega_R}$$

UV

IR

$$\begin{aligned}\mathcal{G}^{\rm jet}_{q,\rm bare}(z, z_r, \omega_R, \mu) = \;& \delta(1-z)\delta(1-z_r) + \frac{\alpha_s}{2\pi}\Big\{\delta(1-z_r)\left(\frac{1}{\epsilon} + L_R\right)\left[P_{qq}(z) + P_{gq}(z)\right] \\ & + \delta(1-z) L_{r/R}\left[P_{qq}(z_r) + P_{gq}(z_r)\right] + C_F\,\delta(1-z_r)\Big[\delta(1-z)\left(\frac{13}{2} - \frac{2\pi^2}{3}\right) \\ & - 2(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\ln(1-z)\,\frac{1+(1-z)^2}{z} - 1\Big]\Big\}\end{aligned}$$

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$r \sim R \ll 1$

anti-k$_T$-in-anti-k$_T$

Inclusive subjets

- Renormalization

$$\mathcal{G}_{i,\mathrm{bare}}^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} Z_{ik}\left(\frac{z}{z'}, \mu\right) \mathcal{G}_k^{\mathrm{jet}}(z', z_r, \omega_R, \mu)$$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{G}_i^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} \gamma_{ik}\left(\frac{z}{z'}, \mu\right) \mathcal{G}_k^{\mathrm{jet}}(z', z_r, \omega_R, \mu)$$

anomalous dimensions: AP splitting functions

Inclusive subjets

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$$\mathcal{G}_{i,\mathrm{bare}}^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} Z_{ik}\left(\frac{z}{z'}, \mu\right) \mathcal{G}_k^{\mathrm{jet}}(z', z_r, \omega_R, \mu)$$

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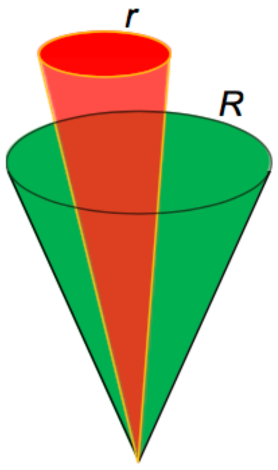
anomalous dimensions: AP splitting functions

- Matching for $r \ll R \ll 1$

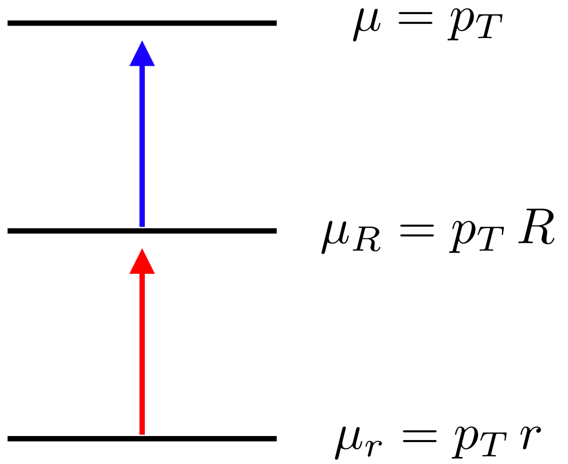


$$\mathcal{G}_i^{\mathrm{jet}}(z, z_r, \omega_R, r, R, \mu) = \sum_j \int_{z_r}^1 \frac{\mathrm{d}z_r'}{z_r'} \mathcal{J}_{ij}(z, z_r', \omega_R, R, \mu)\, J_j\left(\frac{z_r}{z_r'}, \omega_r, r, \mu\right)\left[1 + \mathcal{O}\left(\frac{r^2}{R^2}\right)\right]$$

matching coefficients
same as for hadron-in-jet

siJF for subjet of size r

Inclusive subjets



2x DGLAP

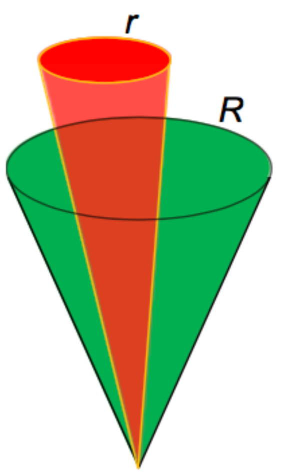
- Renormalization

$$\mathcal{G}_{i,\mathrm{bare}}^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} Z_{ik}\left(\frac{z}{z'}, \mu\right) \mathcal{G}_k^{\mathrm{jet}}(z', z_r, \omega_R, \mu)$$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{G}_i^{\mathrm{jet}}(z, z_r, \omega_R, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z'}{z'} \gamma_{ik}\left(\frac{z}{z'}, \mu\right) \mathcal{G}_k^{\mathrm{jet}}(z', z_r, \omega_R, \mu)$$

anomalous dimensions: AP splitting functions

- Matching for $r \ll R \ll 1$



$$\mathcal{G}_i^{\mathrm{jet}}(z, z_r, \omega_R, r, R, \mu) = \sum_j \int_{z_r}^1 \frac{\mathrm{d}z_r'}{z_r'} \mathcal{J}_{ij}(z, z_r', \omega_R, R, \mu)\, J_j\left(\frac{z_r}{z_r'}, \omega_r, r, \mu\right) \left[1 + \mathcal{O}\left(\frac{r^2}{R^2}\right)\right]$$

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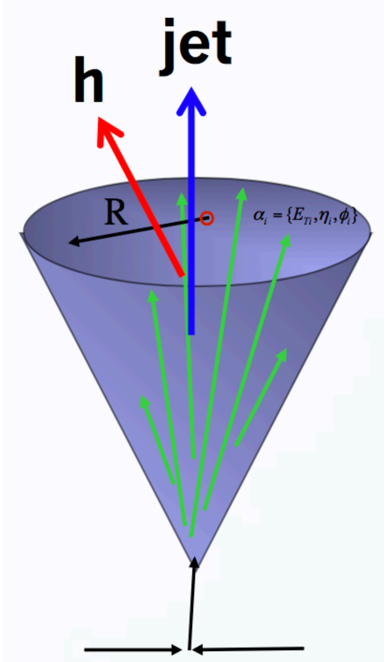


The jet fragmentation function

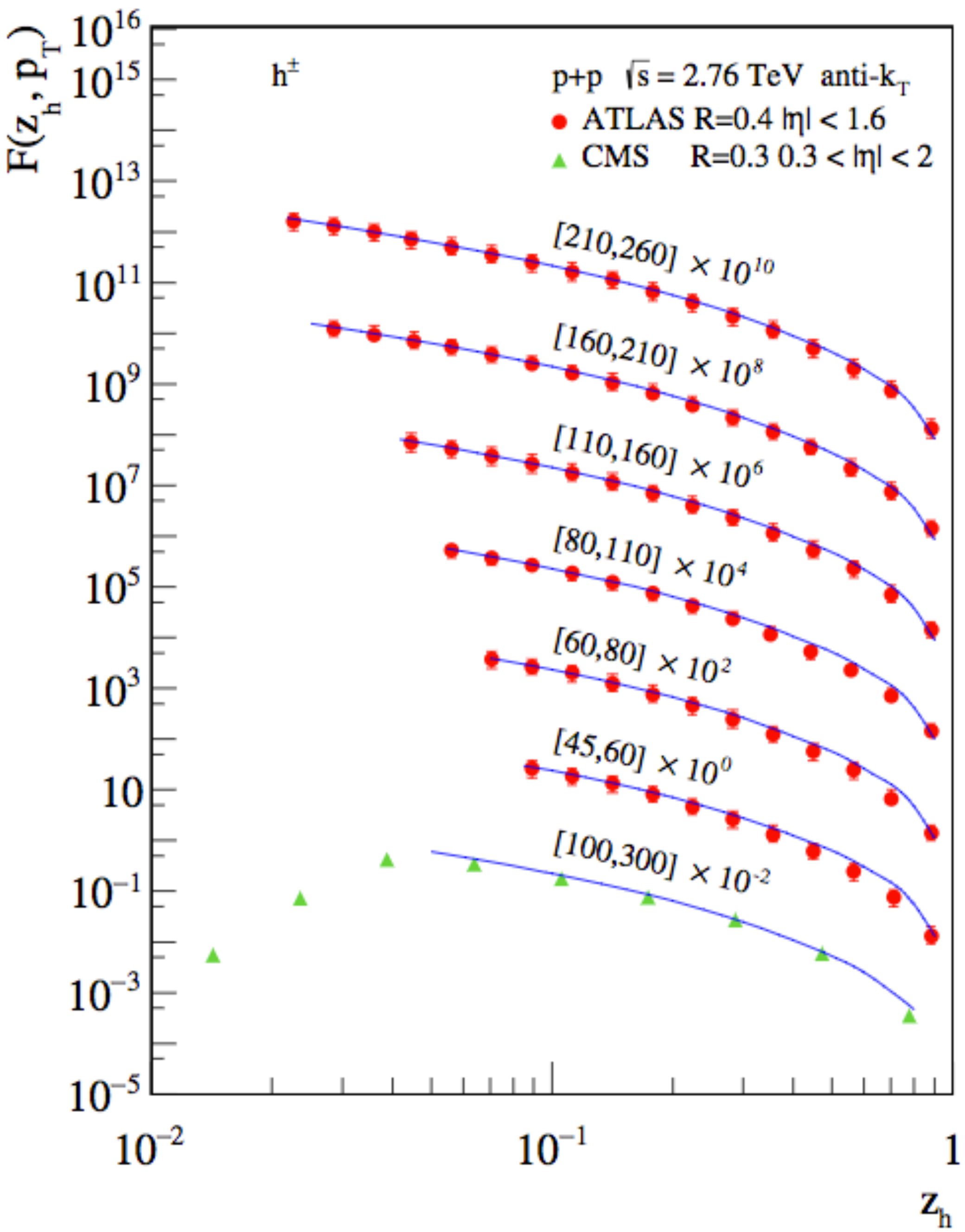
Hadron-in-jet momentum fraction

$$z_h = \omega_h/\omega_J$$

$$\frac{d\sigma^{pp\to(\text{jet }h)X}}{dp_T d\eta dz_h} = \sum_{a,b,c} f_a(x_a,\mu)\otimes f_b(x_b,\mu) \otimes \mathcal{H}_{ab}^c(x_a,x_b,\eta,p_T/z,\mu)\otimes \mathcal{G}_c^h(z,z_h,\omega_J,\mu)$$

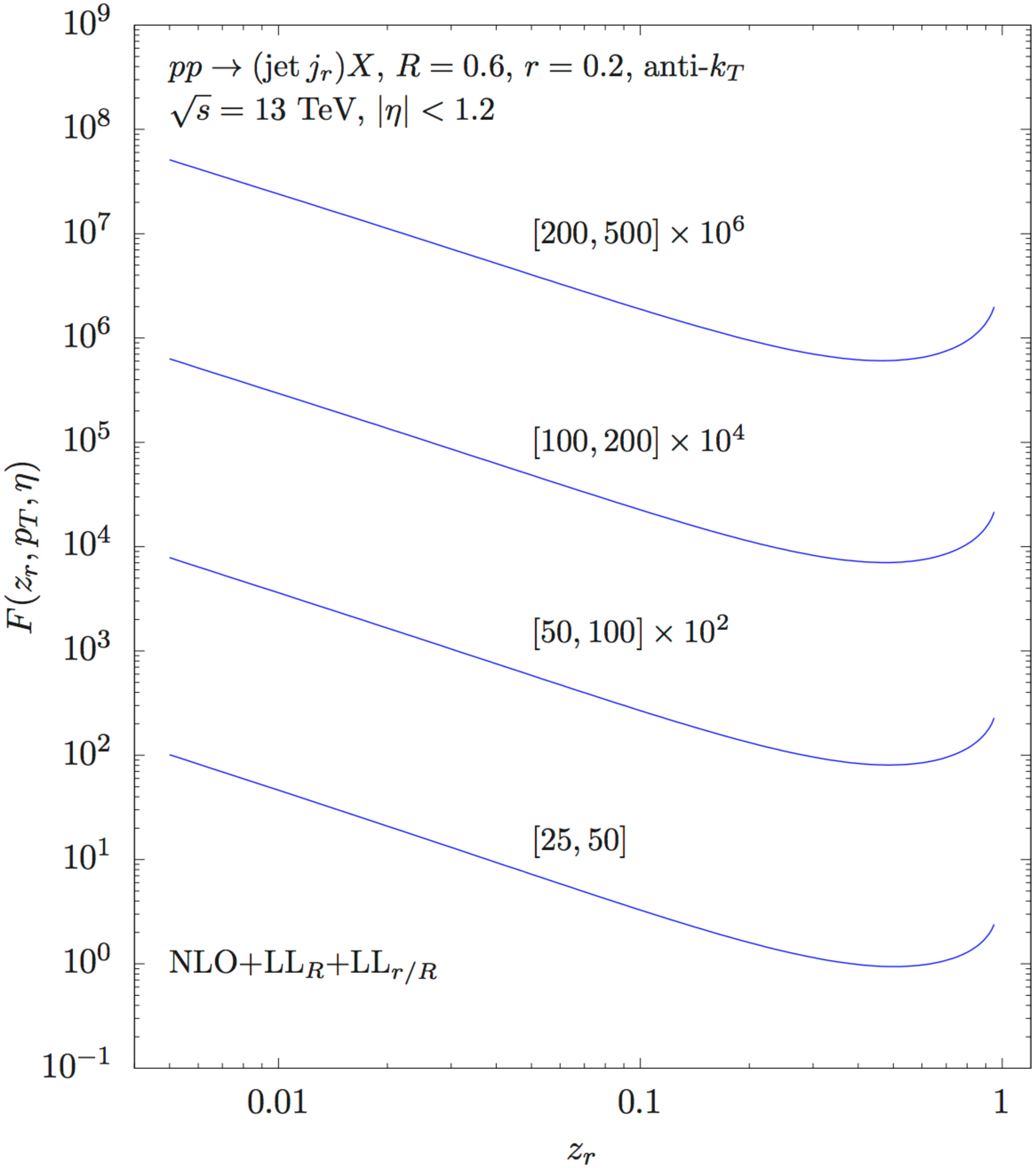
Semi-inclusive fragmenting jet function

$$\mathcal{G}_i^h(z,z_h,\omega_J,\mu) = \sum_j \int_{z_h}^1 \frac{dz_h}{z_h'}\mathcal{J}_{ij}(z,z_h',\omega_J,\mu)D_j^h\left(\frac{z_h}{z_h'},\mu\right)$$



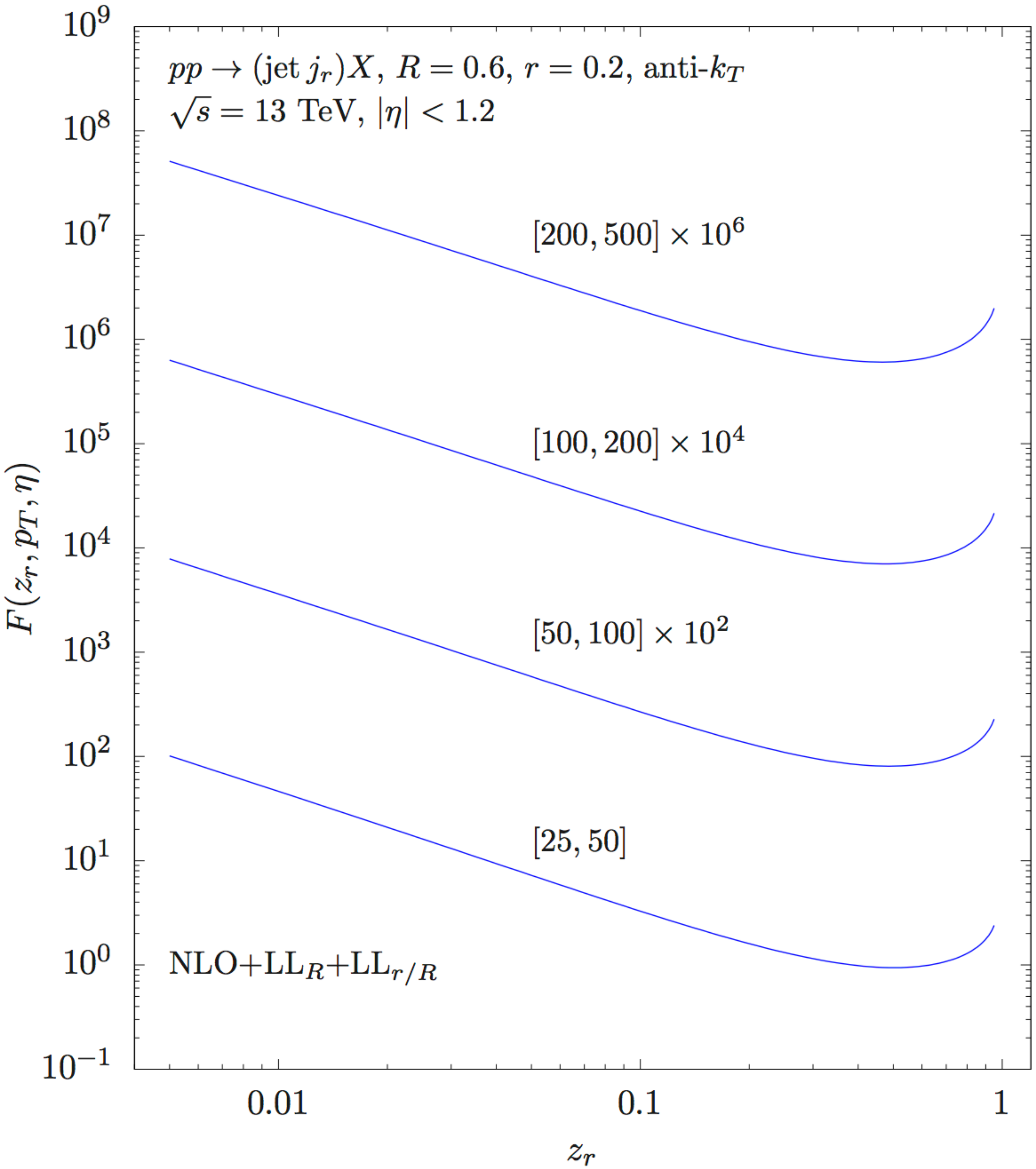
Procura, Stewart `10; Liu `11; Jain, Procura, Waalewijn `11, 12; Arleo, Fontannaz, Guillet, Nguyen `14; Kaufmann, Mukherjee, Vogelsang `15; Dai, Kim, Leibovich `16; Bain, Makris, Mehen `16; Kang, FR, Vitev `16 …

Inclusive subjets



Joint resummation of $\ln R,\ \ln(r/R)$

Inclusive subjets



Joint resummation of $\ln R,\ \ln(r/R)$

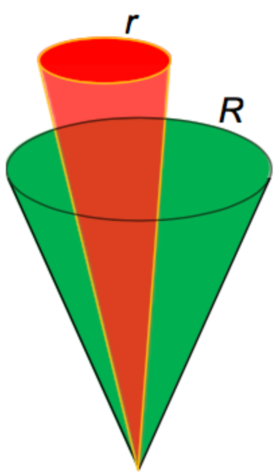
Fixed order result:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta\,\mathrm{d}p_T\,\mathrm{d}z_r} \sim \mathcal{G}_c^{\text{jet}}(z, z_r, \omega_R, \mu)$$

$$\boxed{\begin{aligned}&\mathcal{G}_q^{\text{jet}}(z, z_r < 1, \omega_R, \mu) = \\ &\frac{\alpha_s}{2\pi}\,\delta(1-z)L_{r/R}\left[P_{qq}(z_r) + P_{gq}(z_r)\right]\end{aligned}}$$

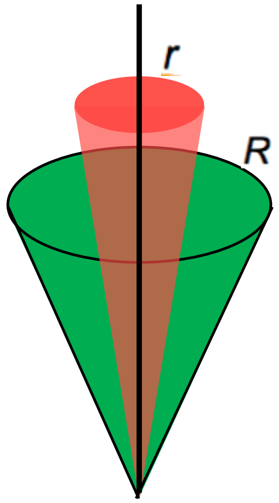
Subjets centered around a predetermined axis

Inclusive subjets



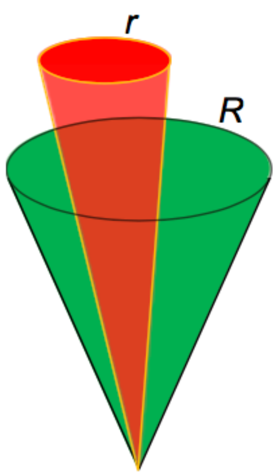
Centered around an axis

- recoil-free axis
  winner-take-all
- standard jet axis



Subjets centered around a predetermined axis

Inclusive subjets



Centered around an axis

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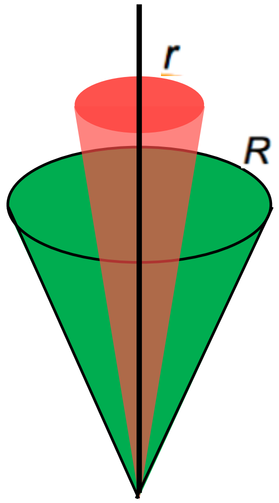


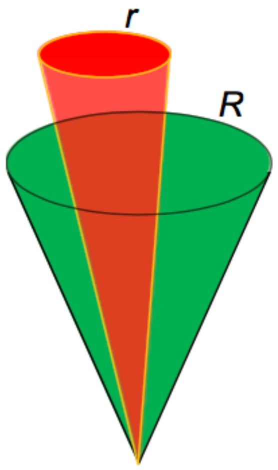
- recoil-free axis

  NLO:

$$\tilde{\mathcal{G}}_q^{\rm jet}(z, z_r, \omega_R, \mu) = \theta\Big(z_r - \frac{1}{2}\Big)\, \mathcal{G}_q^{\rm jet}(z, z_r, \omega_R, \mu)$$

Subjets centered around a predetermined axis

Inclusive subjets



Centered around an axis

- recoil-free axis
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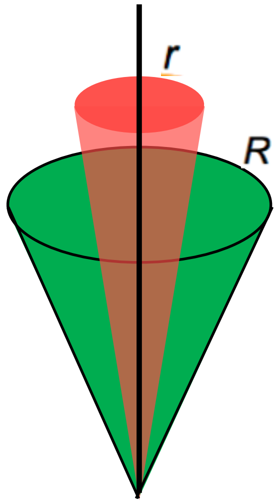
- standard jet axis



- recoil-free axis

  NLO:

$$\tilde{\mathcal{G}}_q^{\rm jet}(z, z_r, \omega_R, \mu) = \theta\Big(z_r - \frac{1}{2}\Big)\, \mathcal{G}_q^{\rm jet}(z, z_r, \omega_R, \mu)$$

Refactorization for $r \ll R$ and $\ln(r/R)$ resummation:

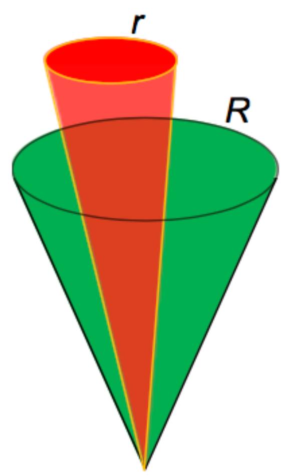
$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\, \tilde{J}_i^{\rm jet}(z_r, \omega_r, \mu) = \sum_k \int_z^1 \frac{\mathrm{d}z_r'}{z_r'}\, \tilde{\gamma}_{ik}\Big(\frac{z_r}{z_r'}, \mu\Big)\, \tilde{J}_k^{\rm jet}(z_r', \omega_r, \mu)$$

$$\tilde{\gamma}_{ij}^{(1)}(z_r, \mu) = \theta\Big(z_r > \frac{1}{2}\Big)\, \frac{\alpha_s}{\pi} P_{ji}(z_r)$$

modified DGLAP

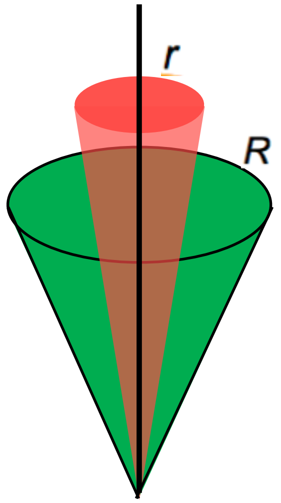
Subjets centered around a predetermined axis

Inclusive subjets



Centered around an axis

- recoil-free axis
  winner-take-all
- standard jet axis



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$$\tilde{\mathcal{G}}_q^{\rm jet}(z, z_r, \omega_R, \mu) = \theta\left(z_r - \frac{1}{2}\right) \mathcal{G}_q^{\rm jet}(z, z_r, \omega_R, \mu)$$

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$$\tilde{\gamma}_{ij}^{(1)}(z_r, \mu) = \theta\left(z_r > \frac{1}{2}\right)\frac{\alpha_s}{\pi}P_{ji}(z_r)$$

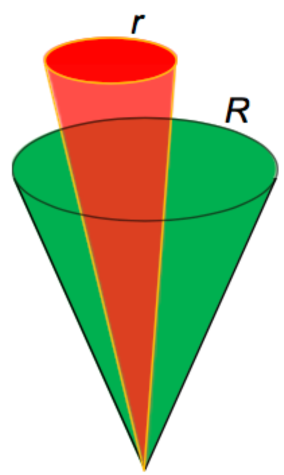
modified DGLAP

Jet shape, z_r average:

$$\int_0^1 \mathrm{d}z_r\, z_r\, \tilde{\mathcal{G}}_q^{\rm jet}(z, z_r, \omega_R, \mu) = J_q(z, \omega_R, \mu) + \delta(1-z)\,\frac{\alpha_s C_F}{2\pi} L_{r/R}\left(\frac{3}{8} - 2\ln 2\right)$$

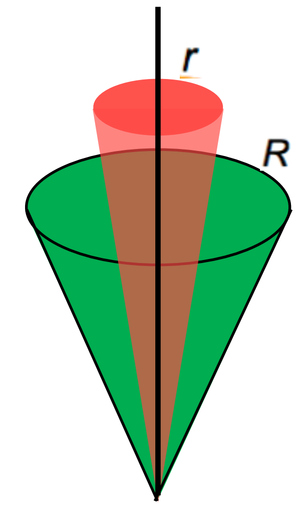
Subjets centered around a predetermined axis

Inclusive subjets



Centered around an axis

- recoil-free axis
  winner-take-all
- standard jet axis



- standard jet axis

Refactorization for $r \ll R$ and
$\ln(r/R)$ resummation:

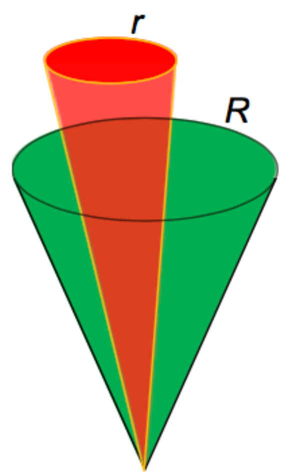
$$\hat{\mathcal{G}}_i^{\mathrm{jet}}(z, z_r, \omega_R, r, R, \mu) = H_{ij}(z, \omega_R R, \mu) \int \mathrm{d}^2 k_\perp \, C_j(z_r, \omega_r r, k_\perp, \mu, \nu) \, S_j(k_\perp, R, \mu, \nu)$$

hard collinear soft

RGs and rapidity RGs + non-global logarithms

Subjets centered around a predetermined axis

Inclusive subjets



Centered around an axis

- recoil-free axis
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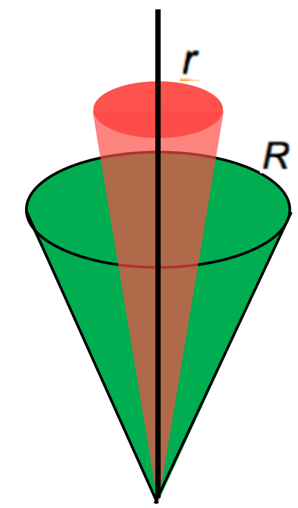


- standard jet axis

  Refactorization for $r \ll R$ and $\ln(r/R)$ resummation:

$$\hat{\mathcal{G}}_i^{\mathrm{jet}}(z, z_r, \omega_R, r, R, \mu) = H_{ij}(z, \omega_R R, \mu) \int \mathrm{d}^2 k_\perp \, C_j(z_r, \omega_r r, k_\perp, \mu, \nu) \, S_j(k_\perp, R, \mu, \nu)$$

hard collinear soft

RGs and rapidity RGs + non-global logarithms

Jet shape, z_r average:

$$\int_0^1 \mathrm{d}z_r \, z_r \, \hat{\mathcal{G}}_q^{\mathrm{anti\text{-}k}_T}(z, z_r, \omega_R, \mu) = J_q^{\mathrm{anti\text{-}k}_T}(z, \omega_R, \mu) + \delta(1-z) \frac{\alpha_s C_F}{2\pi} \left[-\frac{1}{2} L_{r/R}^2 + \frac{3}{2} L_{r/R} - \frac{9}{2} + \frac{6r}{R} - \frac{3r^2}{2R^2} \right]$$

see also: *Seymour `98; Li, Li, Yuan 11; Chien, Vitev `14 ...*

Outline

- Inclusive jets
- Subjets *Kang, FR, Vitev `16*
  - Inclusive subjets *Kang, FR, Waalewijn `17*
  - Centered around an axis
- Conclusions

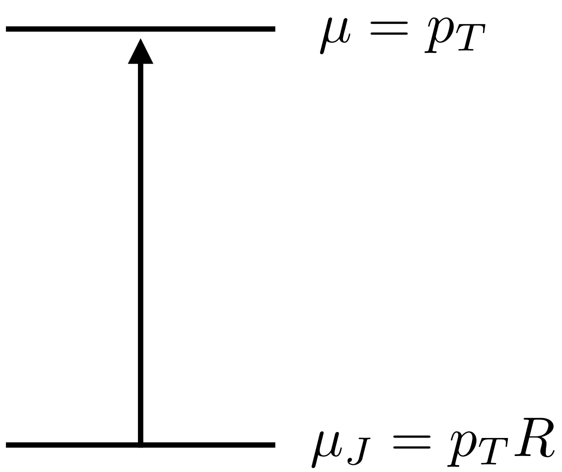
Conclusions

- Inclusive small-R jets and their substructure within SCET
- Inclusive and central subjets
- Standard jet shape

- Non-global logarithms
- Numerical results available soon for central subjets

Jet function evolution

$$\frac{d}{d\log\mu^2}\begin{pmatrix} J_S(z,\omega_J,\mu) \\ J_g(z,\omega_J,\mu)\end{pmatrix} = \frac{\alpha_s(\mu)}{2\pi}\begin{pmatrix} P_{qq}(z) & 2N_f P_{gq}(z) \\ P_{qg}(z) & P_{gg}(z)\end{pmatrix} \otimes \begin{pmatrix} J_S(z,\omega_J,\mu) \\ J_g(z,\omega_J,\mu)\end{pmatrix}$$



initial condition contains distributions in $1-z$

where $\quad J_S(z,\omega_J,\mu) = \sum_{q,\bar q} J_q(z,\omega_J,\mu) = 2N_f J_q(z,\omega_J,\mu)$ (singlet jet function)

solve in Mellin space: $\quad f(N) = \int_0^1 dz\, z^{N-1} f(z)$

$$(f\otimes g)(N) = f(N)\, g(N)$$

Jet function evolution

Vogt `04 (Pegasus), Anderle, FR, Stratmann `15

- solve in Mellin space to LL_R

$$\begin{pmatrix} J_S(N,\omega_J,\mu) \\ J_g(N,\omega_J,\mu) \end{pmatrix} = L\left(N,\alpha_s(\mu),\alpha_s(\mu_J)\right) \begin{pmatrix} J_S(N,\omega_J,\mu_J) \\ J_g(N,\omega_J,\mu_J) \end{pmatrix}$$

$$= \left[e_+(N) \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_J)}\right)^{-r_-(N)} + e_-(N) \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_J)}\right)^{-r_+(N)} \right] \begin{pmatrix} J_S(N,\omega_J,\mu_J) \\ J_g(N,\omega_J,\mu_J) \end{pmatrix}$$

- NLL_R :

$$\begin{pmatrix} J_S(N,\omega_J,\mu) \\ J_g(N,\omega_J,\mu) \end{pmatrix} = \left[1+\sum_{k=1}^{\infty} \alpha_s^k(\mu)\, U_k(N)\right] L\left(N,\alpha_s(\mu),\alpha_s(\mu_J)\right) \left[1+\sum_{k=1}^{\infty} \alpha_s^k(\mu_J)\, U_k(N)\right]^{-1} \begin{pmatrix} J_S(N,\omega_J,\mu_J) \\ J_g(N,\omega_J,\mu_J) \end{pmatrix}$$

- Mellin inverse

$$J_{S,g}(z,\omega_J,\mu) = \frac{1}{2\pi i} \int_{\mathcal{C}_N} dN\, z^{-N} J_{S,g}(N,\omega_J,\mu)$$