First Two-Loop Amplitudes with the Numerical Unitarity Method

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Based on work with S. Abreu, F. Febres-Cordero, H. Ita, M. Jaquier and M. Zeng.

[arXiv:1703.05273] and [arXiv:1703.05255]

A Bottleneck: Two-loop Amplitude Calculations

Feynman diagrams

Tensor reduction
[Tarasov 96; Anastasiou, Glover, Oleari 99]
IBPs
[Tkachov, Chetyrkin 81]
↓

Sum of master integrals

$$A = \sum_{\Gamma \in \Delta} \sum_{i \in M_\Gamma} c_{\Gamma,i}(D) I_{\Gamma,i}.$$

Differential equations
[Gehrmann, Remiddi 01]
↓

Integrated form

- Standard procedure.
- Process specific.
- Large intermediate expressions.
- Can we repeat the leap made at NLO? [OPP '07]

The Idea of Numerical Unitarity

- Numerical method for reduction to master integrals.
- Start with an ansatz for loop-amplitude integrand

$$\mathcal{A}(\ell_l) = \sum_{\Gamma\in\Delta} \sum_{i\,\in\, M_\Gamma\cup S_\Gamma} \frac{c_{\Gamma,i}(D)\, m_{\Gamma,i}(\ell_l, D)}{\prod_{j\in P_\Gamma} \rho_j}.$$

- Determine coefficients $c_{\Gamma,i}(D)$ on-shell from cut equations

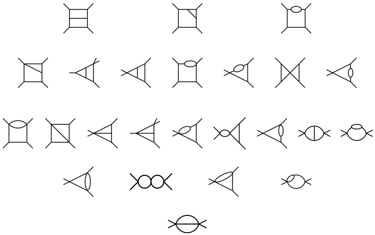
$$\sum_{\text{states}} \prod_{i\in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma'\geq\Gamma,\\ i\,\in\, M_{\Gamma'}\cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(D)\, m_{\Gamma',i}(\ell_l^\Gamma, D)}{\prod_{j\in(P_{\Gamma'}\setminus P_\Gamma)} \rho_j(\ell_l^\Gamma)}.$$

[BDDK '94, '95]

Ingredients for Numerical Unitarity @ 2-Loops

$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma, \\ i \in M_{\Gamma'} \cup S_{\Gamma'}}} \frac{c_{\Gamma',i}(D)\, m_{\Gamma',i}(\ell_l^\Gamma, D)}{\prod_{j \in (P_{\Gamma'} \setminus P_\Gamma)} \rho_j(\ell_l^\Gamma)} .$$

- Integrand parameterization, $m_{\Gamma,i}$.
- Sub-leading poles.
- Products of trees, T_Γ.
- Colour decomposition.
- Coefficient determination, $c_{\Gamma,i}$.



4-gluon Amplitude Hierarchy

Integrand Parameterization

[Abreu, Febres-Cordero, Ita, Jaquier, B.P., Zeng '17], [Ita '15]

- Surface terms, S_Γ, for a topology Γ, from total derivatives.

$$0 = \int \prod_{l=1,2} d^D\ell_l \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k\in P_\Gamma} \rho_k} \right],$$

- Write loop momenta in natural coordinates.

$$\ell_l = \sum_{j\in B_l^p} v_l^j r^{lj} + \sum_{j\in B_l^t} v_l^j \alpha^{lj} + \sum_{i\in B^{ct}} n^i \alpha^{li} + \sum_{i\in B^\epsilon} n^i \mu_l^i$$

- Obtain either IBPs,

$$m_{\Gamma,u}(\ell_l) = \left[-(\nu_i - 1) f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha^j} + \left(D - \frac{n_\alpha + 1}{2} \right) (f_1^1 + f_2^2) \right],$$

- Or D-dimensional transverse terms,

$$\alpha^{lj}\alpha^{l'j} - b_1 \frac{\mu_{ll'}}{\epsilon}, \quad l, l' \in \{1,2\},$$

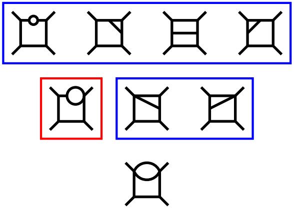
$$(\alpha^{lj})^2(\alpha^{l'j})^2 - b_1 \frac{\mu_{ll}}{\epsilon}(\alpha^{l'j})^2 - b_2 \frac{\mu_{ll'}}{\epsilon}\alpha^{lj}\alpha^{l'j}, \; l \neq l'.$$

Master integrands, M_Γ, fill remaining space.

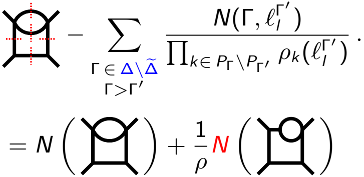
Sub-leading Poles

[Abreu, Febres-Cordero, Ita, Jaquier, B.P. '17]

- Beyond one loop multiple poles can be associated to a given factorization limit.
- Only leading poles given by product of trees.



- Some numerators ($\widetilde{\Delta}$) lack associated cut equation.
- Numerators determined from descendant cut equations.

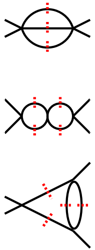


A Proof of Principle - Gluon Gluon Scattering

- Aim: To test the Numerical Unitarity approach @ 2 loops for phenomenologically relevant amplitudes.
- We recompute the known 4-gluon scattering amplitude in an arbitrary helicity configuration [Anastasiou et al'01] [Bern et al '02].
- A gym with all necessary ingredients and an analytic target.

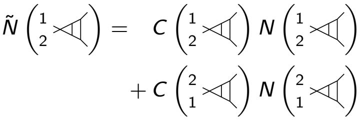
Tree Amplitudes and Colour

- Off-shell recursion. [Berends, Giele '87]
- Flexible spin DoF - "D_s".
- Extensive caching across different helicities and orderings.



- Leading colour produced by projecting colour decomposition of product of trees to integrand.

[B.P., Ochirov '16]



Solving for Master Integral Coefficients

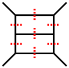
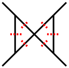
$$\sum_{\substack{\Gamma \in \widetilde{\Delta} \\ \Gamma \geq \Gamma' \\ i \in M_\Gamma}} \frac{c_{\Gamma,i}(D)\, m_{\Gamma,i}(\ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})} = \sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) - \sum_{\substack{\Gamma \in \Delta \setminus \widetilde{\Delta} \\ \Gamma > \Gamma'}} \frac{N(\Gamma, \ell_l^{\Gamma'})}{\prod_{k \in P_\Gamma \setminus P_{\Gamma'}} \rho_k(\ell_l^{\Gamma'})}\,.$$

- Sample randomly on-shell phase space ℓ_l^Γ to constrain $c_{\Gamma,i}$.
- PLU factorization for $n \times n$ systems.
- QR factorization for $n \times m$ overdetermined systems.
- LAPACK for double precision. [Anderson et al '99]
- MPACK for high precision. [Nakata '10]

Can determine master integral coefficients for fixed D and D_s.

Ds-Dependence of Coefficients

- Amplitude depends on $(D_s - 2)$, the number of spin states.
- D_s dependence of amplitude is at most quadratic.
- One power of D_s from each separable component e.g.

 is linear in D_s  is quadratic in D_s

- Interpolate from products of trees with different D_s. [Giele et al '08]
- Then use t'Hooft Veltman scheme, $D_s = D$.

D Dependence of Coefficients

- Coefficients inherit D-dependence from surface terms recall

$$m_{\Gamma,u}(\ell_l) = \left[-(\nu_i - 1)f_i + \rho_i \frac{\partial f_i}{\partial \rho_i} + \frac{\partial u_j}{\partial \alpha^j} + \left(D - \frac{n_\alpha + 1}{2}\right)(f_1^1 + f_2^2)\right].$$

- Resulting coefficients are rational functions of D, i.e.

$$c(D) = \frac{P(D)}{Q(D)} = \frac{p_0 + p_1 D + ... + p_i D^i}{q_0 + q_1 D + ... + q_{j-1} D^{j-1} + D^j}.$$

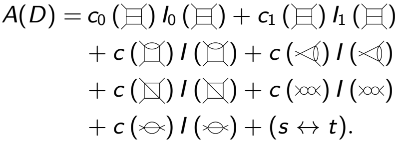
- Can reconstruct $c(D)$ by sampling various D.

related [Peraro '16], [Schabinger, von Manteuffel '14]

- Only $P(D)$ is kinematically dependent.
- We fix $Q(D)$ once in high precision.

The Finished Product

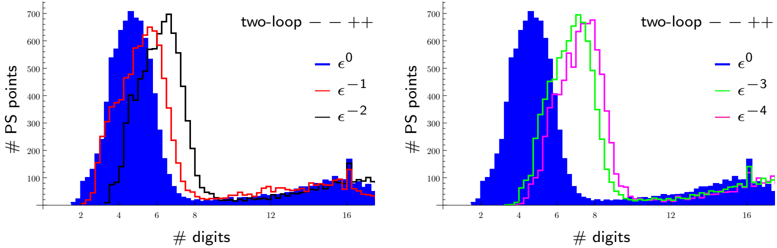
- We thus determine $A(D)$ for a given numerical (s, t).
- $A(D)$ is exact in D and we verify it against analytics.
- We insert the integrals and expand around $D = 4$.



With, $g_s = 1$, $\mu = 1$, $s = -\frac{1}{4}$ and $t = -\frac{3}{4}$ we find:

$\mathcal{A}/(\mathcal{A}_0 N_c^2)(4\pi)^4$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_g^-, 2_g^+, 3_g^-, 4_g^+)$	8.00000	55.6527	176.009	332.296	486.502
$(1_g^-, 2_g^-, 3_g^+, 4_g^+)$	8.00000	55.6527	164.642	222.327	-8.39044

Numerical Stability of MMPP Amplitude

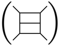


- Comparison to analytics over 10000 phase space points.
- Rescue system based on accuracy of universal $\frac{1}{\epsilon}$ pole.
- Coefficient precision often good to $\sim$ 8 digits.
- Large precision loss when inserting integrals.

Encore - Full Amplitude Reconstruction

- Rescaled integral coefficients are rational functions of $x = \frac{t}{s}$.
- Reconstruct using same techniques as for the regulator.

  related [Peraro '16], [Schabinger, von Manteuffel '14]
- Exact analytic results from numerics, e.g (mmpp):

c 

$$c\left(\,\cdot\,\right) = \frac{9x + \frac{\epsilon\left(-x^3 - \frac{32x^2}{11} - \frac{97x}{44} - \frac{5}{22}\right)}{\frac{x^2}{33} + \frac{2x}{33} + \frac{1}{33}} + \frac{\epsilon^2\left(-x^3 - \frac{385x^2}{51} - \frac{937x}{102} - \frac{77}{34}\right)}{-\frac{2x^2}{51} - \frac{4x}{51} - \frac{2}{51}} + \cdots}{-9 + 66\epsilon - 184\epsilon^2 + 240\epsilon^3 - 144\epsilon^4 + 32\epsilon^5}$$

- Requires 15 evaluations of the amplitude in high precision.

Conclusions

- We set up a numerical algorithm for two-loop amplitudes.
- A proof of principle calculation of the 4-gluon amplitude.
- Analytic results can be reconstructed from numerical samples.
- The method shows promise for phenomenological calculations.

Univariate Rational Function Reconstruction

- Thiele's formula gives $c(D)$ as a continued fraction [Peraro '16]

$$c(D) = a_0 + \cfrac{D - D_0}{a_1 + \cfrac{D - D_1}{a_2 + \cfrac{D - D_2}{\cdots + \cfrac{D - D_{N-1}}{a_N}}}}.$$

- Can fix coefficients a from values $c(D_i)$.
- Easy to convert back to canonical $\frac{P(D)}{Q(D)}$.
- Only $P(D)$ is kinematically dependent.
- We fix the rational coefficients of $Q(D)$ once in high precision, recovering the exact form with continued fractions.

Numerical Stability for MPMP