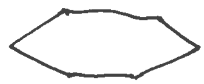
Outline

- Introduce TMD Factorization & evolution
- Re-examine in threshold Region
- WRITE TMD — threshold consistency condition
- Explain 3-Loop calculation (Briefly)

  $\hookrightarrow \gamma^{R} = \gamma^{thr} + \beta$ terms
- $\langle \int \rangle =$ 

- ARGUE special conformal ward ID.

  PROHIBITS SINGLE Logs in finite part of polygons.
- $\therefore\ \gamma^{R} = \gamma^{thr}$ in conformal theory

  β terms calculatable at $4-2\epsilon^{*}$ dim

  with $\epsilon^{*} = \beta(\alpha_s)$ in QCD

  (1608.04920)

Model Problem $pp \to \gamma^*/Z \to e^+e^-$

Measure the momentum spectrum of e^+e^- A

TMD Factorization

$$\frac{d\sigma}{dY\, dq_T^2\, dQ^2} = \sigma_0 \int \frac{d^4q}{(2\pi)^3}\, \delta^+(n\cdot q\, \bar{n}\cdot q - Q^2)\, \delta\left(y - \frac{1}{2}\ln\frac{n\cdot q}{\bar{n}\cdot q}\right) \delta^{(2)}(\vec{Q}_T - \vec{q}_\perp)$$

$$\int d^4b\, H(Q)\, B_{n, q/N_A}(0, n\cdot b, \vec{b}_\perp)\, B_{\bar{n}, \bar{q}/N_B}(\bar{n}\cdot b, 0, \vec{b}_\perp)$$

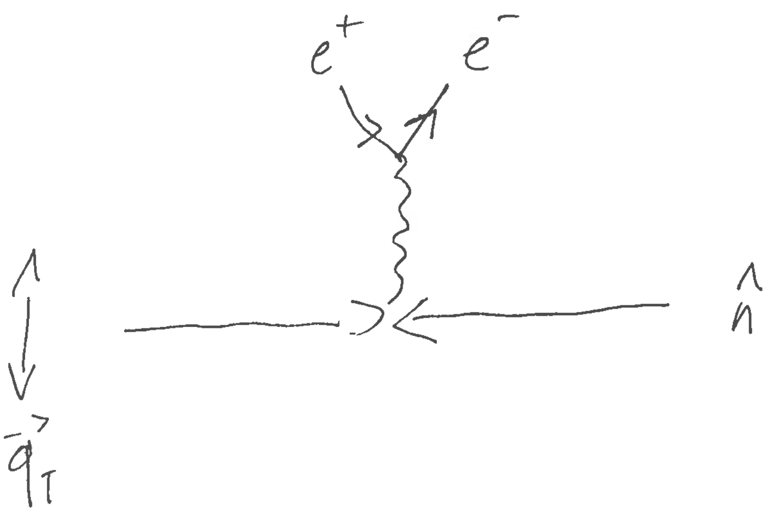
$$\mathrm{Exp}[iq\cdot b] \quad S_{n\bar{n}}(0, 0, \vec{b}_\perp) \quad + \quad q \leftrightarrow \bar{q}$$

THRESHOLD FACTORIZATION

$$\frac{d\sigma}{dY\, dq_\perp^2\, dQ^2} = \sigma_0 \int \frac{d^4q}{(2\pi)^3}\, \delta^+(n\cdot q\, \bar{n}\cdot q - Q^2)\, \delta\left(y - \frac{1}{2}\ln\frac{n\cdot q}{\bar{n}\cdot q}\right) \delta^{(2)}(\vec{Q}_T - \vec{q}_\perp)$$

$$\int d^4b\, H(Q)\, f^{thr}_{q/N_A}(0, n\cdot b, \vec{0})\, f_{\bar{q}/N_B}(\bar{n}\cdot b, 0, \vec{0})$$

$$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp)\, \mathrm{Exp}[iq\cdot b]$$



OPERATOR DEF.

$$B_n(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp) = \mathrm{tr}\langle N(P)|\bar{\chi}_n(\bar{n}\cdot b, n\cdot b, b_\perp)\frac{\bar{n}\!\!\!/}{2}\chi_n(0)|N(P)\rangle \quad (8)$$

$$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp) = \frac{1}{d_a}\mathrm{tr}\langle 0|T\{S_{\bar{n}}^{+}(0)S_n(0)\}\bar{T}\{S_n^{+}(b)S_{\bar{n}}(b)\}|0\rangle$$

$$\chi_n(x) = W_n^{+}(-\infty, x)\,\psi_n(x)$$

$$W_n(x) = P\exp\left[ig\int_{-\infty}^{0} ds\, \bar{n}\cdot A(s\bar{n}+x)\right]$$

$$S_n(x) = P\exp\left[ig\int_{-\infty}^{0} ds\, n\cdot A(x+sn)\right]$$

QUESTIONS

- ~~What~~ How do we calculate efficiently the control quantities for TMD-Evolution?
- What is the relationship between threshold, TMD, & Collinear factorization?
- Ultimately, what is the relationship between quantities evolving in energy[a] versus quantities evolving in rapidity?

This last POINT IS PERHAPS THE MOST PROFOUND.

- BK small-x & Non-global Logs (essentially space-like to time-like PARTON SHOWERS)

- Threshold to transverse momentum Resummation [energy-ordered emissions versus Rapidity]

$$\boxed{\text{pQCD is a quasi-conformal theory}}$$

$pp \to e^+e^- + X$ thru γ^*/Z

what is ~~its~~ the momentum spectrum of the e^+e^- paiR

Natural "Jet axis" - Beam

DemAND: $\frac{q_T^2}{Q^2} \ll 1$ $\quad Q^2$ e^+e^- invariant mass

This Restricts All Radiation Recoiling against e^+e^- pair

$\hookrightarrow \underset{\uparrow \text{ small}}{\vec{q}_T} = \sum_{i \in QCD} \underset{\uparrow \text{ small}}{\vec{p}_{i\perp}}$ $\qquad$ call $\lambda \sim \frac{q_\perp}{Q}$

In a theory of quarks & Gluons, what is the allowed on-shell states in the sum?

- First Need particles with enough large momentum to cReate the e^+e^- pair
- transverse fluctuations order λ
- "on-shell" that is, propagators have homogeneous power counting & cutcosky rule would give non-zero Result

$n = (1, \hat{n})$

$\bar{n} = (1, -\hat{n})$

$\qquad\qquad \bar{n}\cdot p \quad , \; n\cdot p \; , \; p_\perp$

$p_n \sim Q(1, \lambda^2, \lambda)$

$\lambda \to 0$

$p_s \sim Q(\lambda, \lambda, \lambda)$

$$p_n^2 \simeq \underbrace{\bar{n}\cdot p_n}_{} \; n\cdot p_n - p_{n\perp}^2 \sim Q^2\lambda^2$$

$\mathcal{O}(Q^2\lambda^2)$

$\lambda \to 0 \rightsquigarrow$ scales each term to ~~exact~~ 0 homogeneously

$$p_s^2 = \bar{n}\cdot p_s \; n\cdot p_s - p_{s\perp}^2 \sim Q^2\lambda^2$$

$\mathcal{O}(Q^2\lambda^2)$

$\rightsquigarrow$ no obstruction to probing "Deep IR" of Loops

SCET: form EFT with soft + collinear modes → manifest power counting & gauge invariance at $\mathcal{L}$ & operator Level

↳ Construct Basis of operators that are gauge invariant with momentum transfer expanded according to P.C.

Sterman – Libby

↳ Momentum Region of QCD Diagrams that give rise to pinched integrals

↳ Find operators which reproduce expansion of diagrams in these momentum Regions

$$\frac{d\sigma}{dY\, d\vec{Q}_T^{\,2}\, dQ^2} = \sigma_0 \int \frac{d^4q}{(2\pi)^3}\, \delta^+(n\cdot q\, \bar{n}\cdot q - Q^2)\, \delta\left(y - \tfrac{1}{2}\ln\frac{n\cdot q}{\bar{n}\cdot q}\right) \delta^{(2)}(\vec{q}_T - \vec{Q}_T)$$

$$\int d^4b \;\; H(Q)\; B_n(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp)\; B_{\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp)$$

$$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp)$$

$$\mathrm{Exp}\left[\frac{i}{2}\left(n\cdot q\, \bar{n}\cdot b + \bar{n}\cdot q\, n\cdot b\right) - i\vec{q}_\perp\cdot\vec{b}_\perp\right]$$

$n\cdot q$ & $\bar{n}\cdot q$ is ORDER Q^2 [central rapidities]

---

send $B_n(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp) \to B_n(0, n\cdot b, \vec{b}_\perp)$

$B_{\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp) \to B_{\bar{n}}(\bar{n}\cdot b, 0, \vec{b}_\perp)$

$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp) \to S_{n\bar{n}}(0, 0, \vec{b}_\perp)$

This is to prevent any large momentum irragating the small l.c. momentum components of $n/\bar{n}/s$ sectors

↳ would lead to propagators that are not "on-shell"

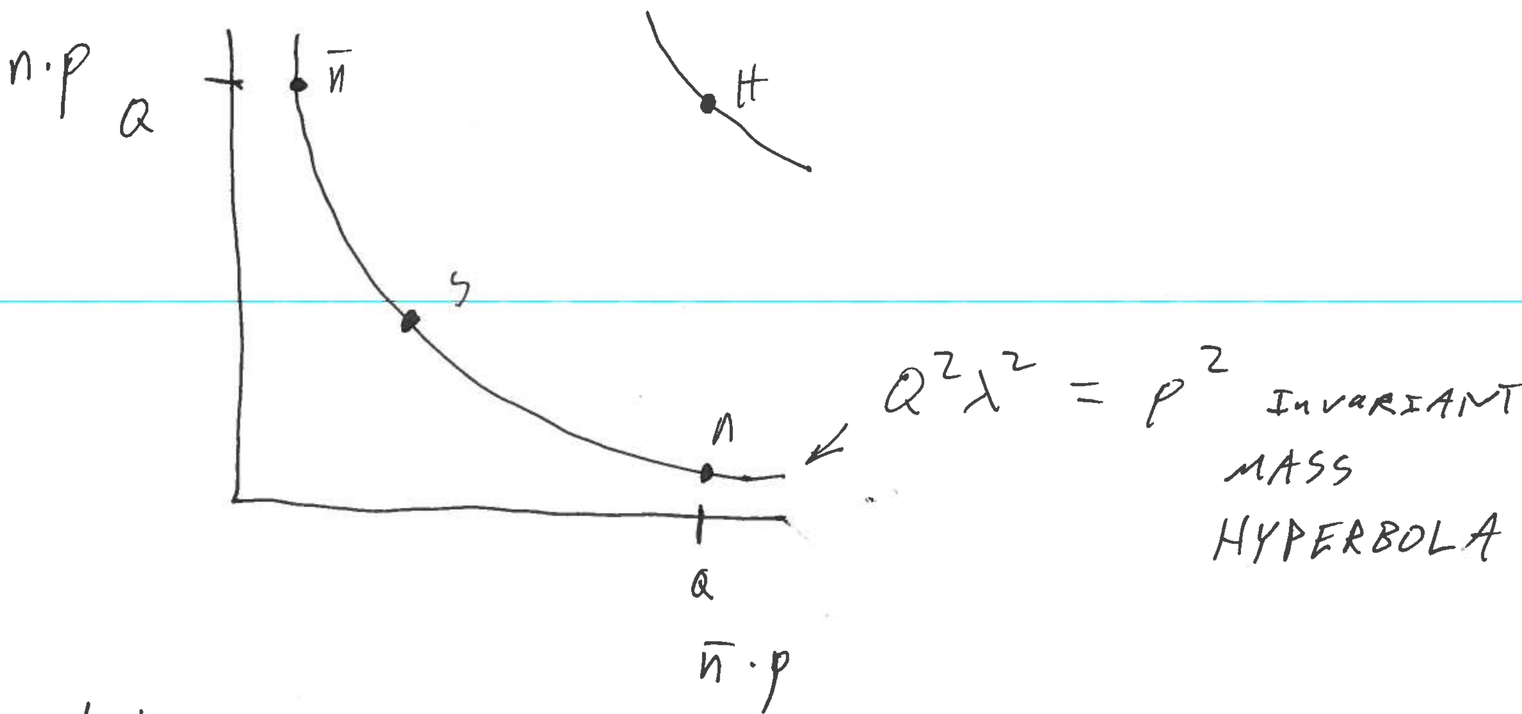
↳ multipole expansion

↳ NECESSARY TO FIND THE CORRECT DIV. AND THEREFORE CORRECT RESUMMATION

PROBLEM:

In this case we have two species of divergences: UV & RAPIDITY

why?



$n/\bar{n}/s$ distinguished in invariant mass → connected via dilatations

→ Dim-Reg breaks dilatation

→ keeps Momentum INTEGRALS In either Hard or IR Regions

HOWEVER: $n/\bar{n}/s$ Related by boosts, which dim reg preserves.

- dim reg cannot keep $n/\bar{n}/s$ to their assigned momentum Region

NEED ADDITIONAL REGULATOR

↳ tilt Wilson lines of Light-cone

↳ "δ-regulator" ⇝ mass for eikonal lines

↳ "Analytic Regulators" → Becher & Bell & Neubert
P.N. et al

can be made gauge invariant ~~pe~~ perturbatively

$$\int \frac{d^d p}{(2\pi)^{d-1}} \frac{\delta(p^2)\, e^{-ib_\perp \cdot p_\perp}}{n\cdot p\ \bar{n}\cdot p} \to \frac{1}{2\pi}\int_0^\infty \frac{dn\cdot p}{n\cdot p} \times \int \frac{dp_\perp^{2-2\epsilon}}{(2\pi)^{2-2\epsilon}} \frac{e^{ip_\perp\cdot b_\perp}}{p_\perp^2}$$

N.B. Regulator dictates soft/collinear overlap

↳ each regulator will need a distinct subtraction

- Once REGULATED & ~~u~~ SUBTRACTED
  Rapidity divergences cancel between collinear & soft functions, but a residual scale from extracting the rapidity div. will remain in each function

so

$$\mathcal{B}_n(x_A, \vec{b}_\perp) = \int \frac{d\, n\cdot b}{2\pi} e^{\frac{i}{2} x_A \bar{n}\cdot P_A\, n\cdot b}\, \mathcal{B}_n(0, n\cdot b, \vec{b}_\perp)$$

$$\mathcal{B}_n(x_A, \vec{b}_\perp) \xrightarrow{\text{renorm}} \mathcal{B}_n(x_A, \vec{b}_\perp; \mu, \nu)$$

$$S_{n\bar{n}}(\vec{b}_\perp) \longrightarrow S_{n\bar{n}}(b_\perp; \mu, \nu)$$

C.S. EQN

$$\nu^2 \frac{d}{d\nu^2} \ln \mathcal{B}_n(x_A, \vec{b}_\perp; \mu, \nu) = -\frac{1}{2}\gamma_R\left(\mu|\vec{b}_\perp|; \alpha_s(\mu)\right)$$

$$\nu^2 \frac{d}{d\nu^2} \ln S_{n\bar{n}}(b_\perp; \mu, \nu) = \gamma_R\left(\mu|\vec{b}_\perp|; \alpha_s(\mu)\right)$$

$$= \int_{b^{-2}}^{\mu^2} \frac{d\mu'^2}{\mu'^2} 2\Gamma(\alpha_s(\mu')) + \gamma^R(\alpha_s(b^{-2}))$$

(Note: $\mu^2 \frac{d}{d\mu^2}$ gives UV Anom, which depends on ν)

& $\nu^2 \frac{d}{d\nu^2} d\sigma = \mu^2 \frac{d}{d\mu^2} d\sigma = 0$

EACH REGULATOR Has its ups & downs

	Expo	Loops	Non perturbative	subtractions
Wilson Lines:	Yes	1	Yes	non-zero
δ:	Yes as $\delta \to 0$	2	Yes-ish	non-zero
Analytic:	Yes	2	No	zero

THEY ALL SUCK AT 3 LOOPS

$\hookrightarrow$ REQUIRED FOR % Level perturbative uncertainty

To Get to 3 Loops, we need new regulator.

- How to get there: Look back at "Derivation" of TMD - Factorized FORMULA.

$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, b_\perp) \to S_{n\bar{n}}(0, 0, \vec{b}_\perp^{\,2})$ [to keep out large momentum]

If we keep $\bar{n}\cdot b, n\cdot b$

$$\int \frac{d^d p}{(2\pi)^{d-1}} \frac{\delta(p^2)\, e^{ib\cdot p}}{n\cdot p\, \bar{n}\cdot p} \to \frac{1}{2\pi} \int_0^\infty \frac{d n\cdot p}{n\cdot p}\, e^{\frac{i n\cdot p\, \bar{n}\cdot b}{2} + i\underbrace{\frac{n\cdot b\, |p_\perp|^2}{n\cdot p}}_{\text{regulated!}}}$$

But what is $S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, \vec{b}_\perp)$?

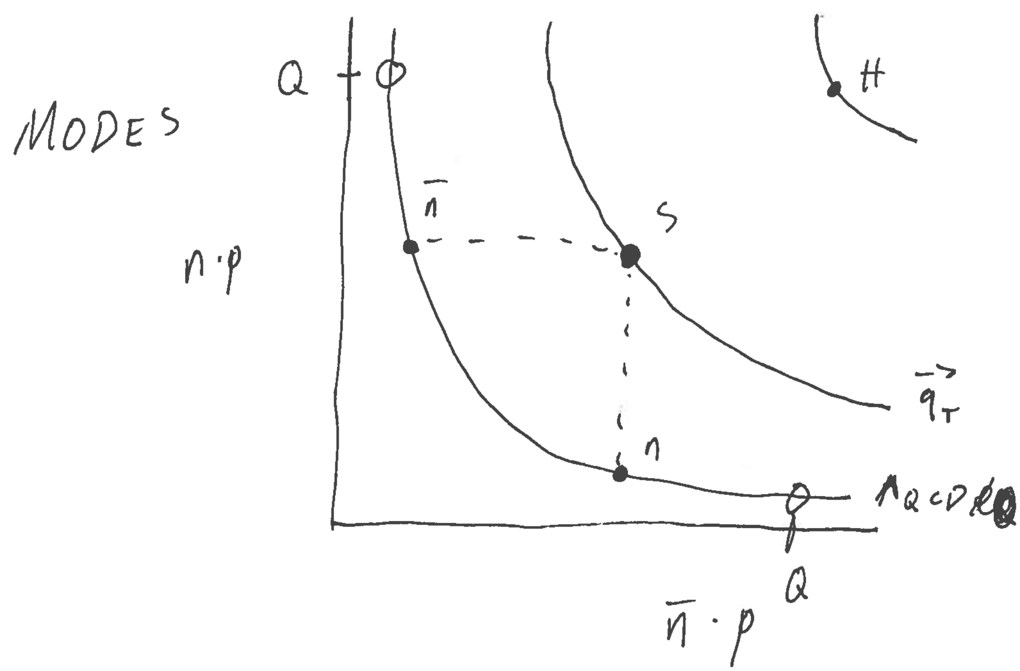
That is, what kind of ~~function~~ factorization ?

- If we require very little ~~red~~ energy in final state:

$$\frac{d\sigma}{dY\,dq_T^2\,dQ^2} = \sigma_0 \int \frac{d^4q}{(2\pi)^3}\,\delta(Q^2 - n\cdot q\,\bar{n}\cdot q)\,\delta\left(y - \tfrac{1}{2}\ln\frac{n\cdot q}{\bar{n}\cdot q}\right)\delta^{(2)}(\vec{q}_T - \vec{q}_\perp)$$

$$\int d^4b\, e^{iq\cdot b}\, H(Q^2)\, f_n^{thr}(0, n\cdot b, 0)\, f_{\bar{n}}^{thr}(\bar{n}\cdot b, 0, 0)\, S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, b_\perp)$$

That is, we want to eat up most of the energy in creating the e^+e^- pair, so that



momentum of parton fed into hard interaction

$n\cdot Q - n\cdot p_c \sim \vec{q}_T$

$\bar{n}\cdot Q - \bar{n}\cdot p_c \sim \vec{q}_T$

residual momentum in PDF

f^{thr} is the "threshold PDF", i.e.,

$$\mu^2 \frac{d}{d\mu^2} f^{thr}(x) = \int_x^1 \frac{dz}{z} P\left(\frac{x}{z}\right) f(z)$$

$$P(z) = \gamma\, \delta(1-z) + \Gamma_{cusp} \left[\frac{1}{1-z}\right]_+$$

Note that if we integrate over $q_\perp^2$, we send

$$S_{n\bar n}(\bar n \cdot b,\, n\cdot b,\, b_\perp) \longrightarrow S_{n\bar n}(\bar n\cdot b,\, n\cdot b,\, 0_\perp)$$

This is the standard threshold soft function, known to 3 Loops.

Indeed $S_{n\bar n}(\bar n\cdot b, n\cdot b, b_\perp)$ has Lots of nice properties:

- Calculable in pure dim. Reg.
- $b_\perp \to 0$ smooth
- RG structure is the same as traditional thr. factorization softfunction [guaranteed by form of factORization theorem & "Sum Rule"]
- Invariant under $\bar{n}$ & n rescaling $\bar{n} \to \alpha \bar{n}$ & $n \to \alpha n$

Therefore

we can write

$$S_{n\bar{n}}(\bar{n}\cdot b, n\cdot b, b_\perp; \mu) = S\left(\frac{\bar{n}\cdot b\, n\cdot b}{\mu^2}; \frac{b_\perp^2}{\bar{n}\cdot b\, n\cdot b}; \alpha_s(\mu)\right)$$

set $\bar{n}\cdot b = n\cdot b = it$

$$\ln S\left(\frac{t^2}{\mu^2}, \frac{b_\perp^2}{t^2}, \alpha_s(\mu)\right) = \int_{1/t^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2}\left\{2\Gamma(\alpha_s(\mu'))\ln(\mu'^2 t^2) + \gamma^{thr}(\alpha_s(\mu'))\right\}$$

$$+ \ln S\left(1, \frac{b_\perp}{t^2}; \alpha_s(t^{-1})\right)$$

How do we recover TMD-evolution?

By treating t as a regulator!

$t \to 0$ limit is TMD, but is now logarithimically divergent

$$\lim_{\tau \to 0} \tau^2 \frac{d}{d\tau^2} \ln S\left(\frac{t^2}{\mu^2}, \frac{b^2}{t^2}; \alpha_s(\mu)\right) = \int_{1/t^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left\{ 2\Gamma\left(\alpha_s(\mu')\right) \right\}$$

$$- \gamma^{thr}\left(\alpha_s\left(1/t^2\right)\right)$$

$$+ \lim_{\tau \to 0} \frac{\tau^2 d}{dt^2} \ln S\left(1, \frac{b_\perp^2}{t^2}, \alpha_s\left(\frac{1}{t^2}\right)\right)$$

$$= -\gamma_R\left(\mu |b_\perp| ; \alpha_s(\mu)\right)$$

$$= -\int_{b^{-2}}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \Gamma\left(\alpha_s(\mu')\right) - \gamma^R\left(\alpha_s(b^{-2})\right)$$

This can be understood as a refactorization of TMD in threshold region $\to$ Joint Resummation

Why this fully differential soft function?

1 We can calculate at 3-loops the taylor series about $b_\perp = 0$ -> threshold soft integrals & IBP

~~2 Basis of functions~~

However, this is not enough. Need the $t \to 0$ limit, i.e., $b_\perp \to \infty$ [∞ # of terms in taylor expansion, assuming ∞ radius of convergence]

2 HOWEVER, based on experience with $N=4$ SYM try a bootstrap approach

↳ write down a finite basis of functions (HPL)

↳ solve for coefficients by matching taylor series

↳ now we know $\ln S\left(1, \frac{b_\perp^2}{t^2}, \alpha_s(1/t^2)\right)$ everywhere.

↳ check solution by predicting higher order terms.

Result:

$$\gamma^R(\alpha_s) = \sum_{i=1}^{\infty} \gamma^R_{i-1} \left(\frac{\alpha_s}{4\pi}\right)^i$$

$$\gamma^{thr}(\alpha_s) = \sum_{i=1}^{\infty} \gamma^{thr}_{i-1} \left(\frac{\alpha_s}{4\pi}\right)^i$$

$$\gamma^R_0 = \gamma^{thr}_0 \qquad \text{1-loop}$$

$$\gamma^R_1 = \gamma^{thr}_1 + \#\,\beta_0 \qquad \text{2-Loop}$$

$$\gamma^R_2 = \gamma^{thr}_2 + \#\,\beta_1 + \#\,\beta_0 \qquad \text{3-Loop}$$

↳ It appears in a conformal theory

$$\gamma^R_i = \gamma^{thr}_i \qquad \text{to all orders}$$

We had

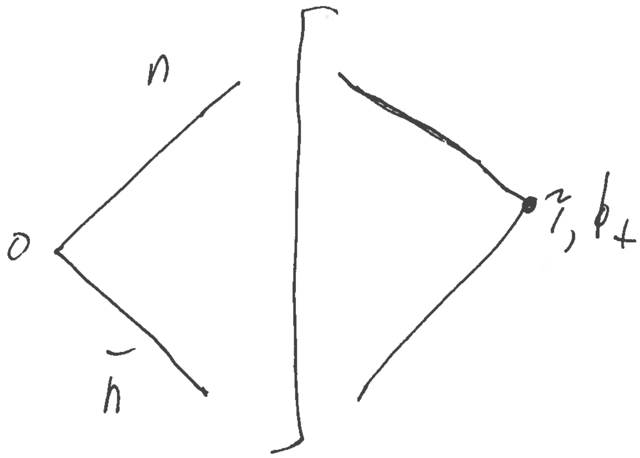
$$\lim_{\tau \to 0} \tau^2 \frac{d}{d\tau^2} \ln S\left(\frac{t}{\mu}, \frac{b_\perp}{t}; \alpha_s\right) = -2\Gamma(\alpha_s) \ln(\mu^2 b^2) - \gamma^R(\alpha_s)$$

$$= 2\Gamma(\alpha_s) \ln(\mu^2 \tau^2) - \gamma^{thr}(\alpha_s)$$

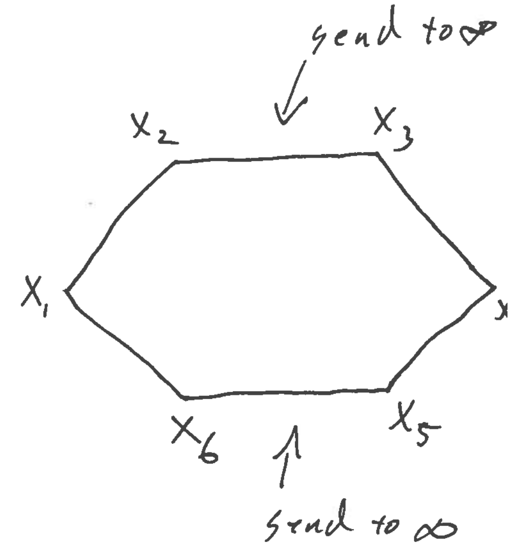
$$+ \lim_{\tau \to 0} \tau^2 \frac{d}{d\tau^2} \ln S\left(1, \frac{b_\perp^2}{t^2}; \alpha_s\right)$$

This function can only have double logarithmic asymptotics.

Why? Physics of Polygonal Wilson Loops



= limit of



$$X_1 = 0$$

$$x_{ij}^2 = (x_i - x_j)^2 \qquad X_4 = (\tau, b_\perp, 0)$$

i & $i+1$ Light-like seperated

ARXIV 0712.1223 → Conformal WARD ID, PLANAR THEORIES

W_6 is hexagon

Finite

$$\ln W_6 = \ln Z_6 + \ln F_6$$

Divergences

$$\sum_{i=1}^{6} \left(2x_i^{\nu}\, x_i \cdot \partial_i - x_i^2\, \partial_i^{\nu} \right) \ln F_6 = \frac{1}{2} \Gamma(a) \sum_{i=1}^{6} x_{i,i+1}^{\nu} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}$$

Special conformal ward identity

Single logarithmic terms

$$\ln F_n \sim \sum_{i=1}^{n} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}$$

Cyclic $i \to i+1$
only adjecents contribute

$n=6$

$$\frac{x_{13}^2 \; x_{24}^2 \; x_{35}^2 \; x_{46}^2 \; x_{51}^2 \; x_{62}^2}{x_{62}^2 \; x_{13}^2 \; x_{24}^2 \; x_{35}^2 \; x_{46}^2 \; x_{51}^2} = 1$$

Only adjacents contribute

$$D \ln \langle W_n \rangle = -\frac{2i\varepsilon}{g^2 \mu^{2\varepsilon}} \int d^d x \, \langle \mathcal{L}(x) W_n \rangle / \langle W_n \rangle$$

$$K^\nu \ln \langle W_n \rangle = -\frac{4i\varepsilon}{g^2 \mu^{2\varepsilon}} \int d^d x \, x^\nu \langle \mathcal{L}(x) W_n \rangle / \langle W_n \rangle$$

If these are nonzero as $\varepsilon \to 0$, anomalous ward ID.

$$D = \sum_{i=1}^{n} (x_i \cdot \partial_i) \qquad K^\nu = \sum_{i=1}^{n} x_i^\nu \, x_i \cdot \partial_i - x_i^2 \partial_i^\nu$$

In conformal theory, this is true due to the explicit breaking of invariance in $4-2\varepsilon$ dimensions

$$\frac{2i\varepsilon}{g^2\mu^{2\varepsilon}} \frac{\langle \mathcal{L}(x) W_n \rangle}{\langle W_n \rangle}$$

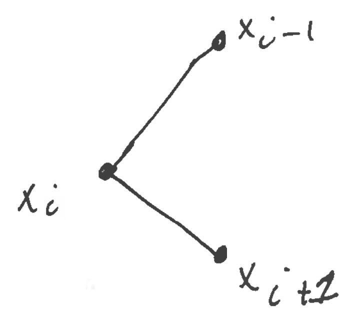
$$= \sum_{\ell \geq 1} a^\ell \sum_{i=1}^{n} \left(-x_{i-1,i+1}^2 \mu^2\right)^{\ell\varepsilon} \left\{ \frac{1}{2}\left(\frac{\Gamma_{cusp}^{(\ell)}}{\ell \varepsilon} + \Gamma_{col}^{(\ell)} \right) \delta^{(D)}(x-x_i) + \Upsilon^{(\ell)}(x\,;\, x_{i-1}, x_i, x_{i+1}) \right\}$$

terms proportional to $\delta^{(D)}(x-x_i)$

Follow from

$$\left(\mu\frac{\partial}{\partial\mu} - D\right)\langle W_n \rangle = 0$$

and the fact cusp divergences only depend on adjacent points



Cyclicity must follow from renormalization properties of wilson loops Brandt et. al.

$$\sum_{k,l,m} \overset{a_{klm}}{\downarrow} \sum_i \ln \frac{x_{i,i+k}^2}{x_{i+l,i+m}^2}$$

Cyclicity in i implies

$$\check{\partial}_i \ln x_{i,i+k}^2 = \frac{\check{x}_i}{x_{i,i+k}^2}$$

$$\sum_{k,l} \phi \sum_{i=1}^{n} a_{kl} \ln\left(\frac{x_{i,i+k}^2}{x_{i+l,i+m}^2}\right) \quad \sum_{i=1}^{n} 2\check{x}_i \, x_i \cdot \partial_i - x_i^2 \check{\partial}_i$$

Proof:
Fix an l, relabel sum on i in Bottom terms as $i \to i-l$ after

$$\sum_i \ln\left(\frac{x_{i,i+k}^2}{x_{i+l,i+m}^2}\right)$$

$$\sum_{i=1} \frac{2\check{x}_i \, x_i^2 - x_i^2 \check{x}_i}{x_{i,i+k}^2} = \sum_{i=1} \frac{\check{x}_i \, x_i^2}{x_{i,i+k}^2}$$

relabel here

$$\hookrightarrow \sum_i \ln(x_{i,i+k}^2) - \sum_i \ln(x_{i+l,i+m}^2)$$

Then $l-m=k$ to get term to annihilate under action of $\check{K}$

i.e. term vanishes!