Joint Resummation

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Some older work with Anna Kulesza, Eric Laenen and Werner Vogelsang

- 1. General comments about resummation and joint resummation

- 2. Color singlet production in hadron-hadron scattering
  review of Q_T resummation and the "nonperturbative window"

- 3. Soft gluon refactorization and joint "combinatoric" resummation

- 4. Regularization and inverse transforms

About resummation and joint resummation

- Resummation is just a systematic summation of some (usually logarithmic) corrections at all orders: $(\alpha_s \ \ln^2 x)^n$, $(\alpha_s \ln x)^n$ etc.

- Variable x is large or small. It may be:

  – An observable. Fit a rapidly changing curve.

  – An integration variable. Confirm the stability of the expansion in α_s and test for large corrections.

- When an inclusive or differential cross section is sensitive to the emission of soft gluons, encounter “Sudakov” double logarithms, and their resummation is generally an exponentiation in a transform space.

- In either case, but particularly for resummed observables, a necessity and a bonus is to infer a phenomenology of nonperturbative parameters.

- **The classic resummed observable:**
  Q_T resummation for electroweak singlet production (Drell Yan, Higgs . . .). Fourier transform.

- **Classic resummed integration variable:**
  Threshold resummation for $1 - z$, $z = Q/\hat{s}$. Mellin or Laplace transform.

- **Joint resummation combines the two classic cases. Originally, it was developed to treat transverse momentum enhancements in a semi-inclusive cross section (high-p_T photon)**
  Laenen, GS, Vogelsang (PRL (2000), PRD (2001))

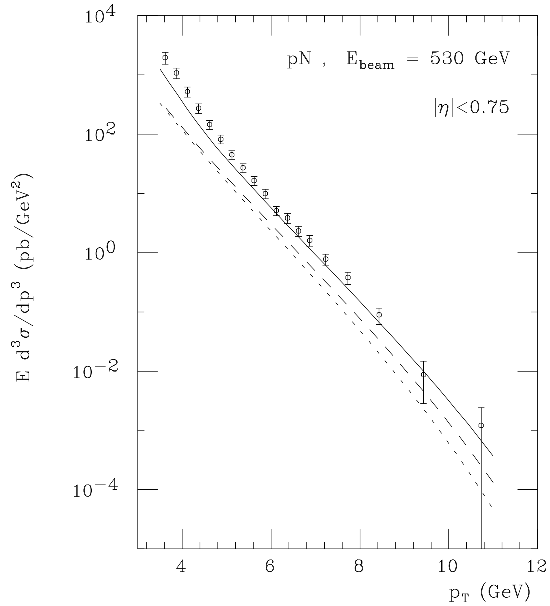


Figure 2. $Ed^3\sigma_{\mathrm{pN}\to\gamma X}/dp^3$ for pN collisions at $\sqrt{s} =$ 31.5 GeV. The dotted line represents the full NLO calculation, while the dashed and solid lines respectively incorporate pure threshold resummation [8] and the joint resummation described in this paper. Data have been taken from [9].

- **Was subsequently developed and extended to differential Drell-Yan and Higgs production, and a variety of inclusive BSM scenarios.**
  Kulesza, GS, Vogelsang (PRD (2002) NPB (2003))

- **Both Fourier and Mellin transforms. Developed only in hadron-hadron scattering (to my knowledge). Let's review the role for resummation in Drell-Yan cross sections.**

- **We will encounter an alternative, "combinatoric" resummation.**

2. Color singlet production in hadron-hadron scattering: to Q_T resummation and the nonperturbative "window"

"Collinear" factorization for Drell-Yan at large Q_T: isolating the hard scattering

- In terms of PDFs:

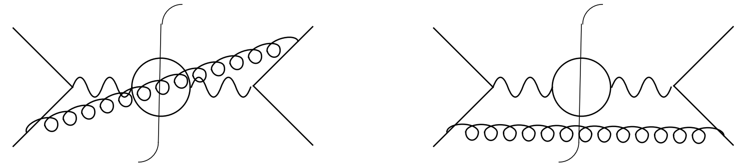
$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2 d^2\mathrm{Q}_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1 p_1,\xi_2 p_2,\mathrm{Q}_T)}{dQ^2 d^2\mathrm{Q}_T} \times \phi_{a/N}(\xi_1,\mu)\,\phi_{\bar{a}/N}(\xi_2,\mu)$$

- **Recall**: μ is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.

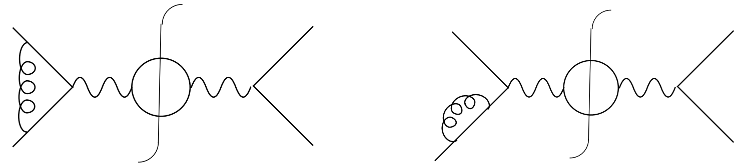
- μ appears in $\hat{\sigma}$ through $\alpha_s(\mu)$ and $\ln(\mu/Q)$ asymptotic freedom comes in here: choosing $\mu \sim Q$ can improve perturbative predictions (with important refinements available)

- Evolution: $\mu df(x,\mu)/d\mu = \int_x^1 P(x/\xi)\; f(\xi,\mu)$ makes energy extrapolations possible.

- **At order α_s Gluon emission contributes at $Q_T \neq 0$**



Virtual corrections contribute only at $Q_T = 0$



- **The result is finite for $\mathrm{Q}_T \neq 0$ . . .**

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathrm{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathrm{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2} \times \left[\frac{1}{\mathrm{Q}_T^2} \frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

Fine as long as $\mathrm{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

Q_T integral $\rightarrow \frac{\ln(1-z)}{1-z}$; z integral $\rightarrow \frac{\ln \mathrm{Q}_T^2}{\mathrm{Q}_T^2}$.

Both singularities cancel in the inclusive cross section.

Both inspire resummation of higher order corrections. Joint resummation joins the two.

The leading singularity in Q_T

- Because the gluon is massless: $1 - z \sim 2k_0/Q \geq 2|\mathrm{k}_T|/Q$

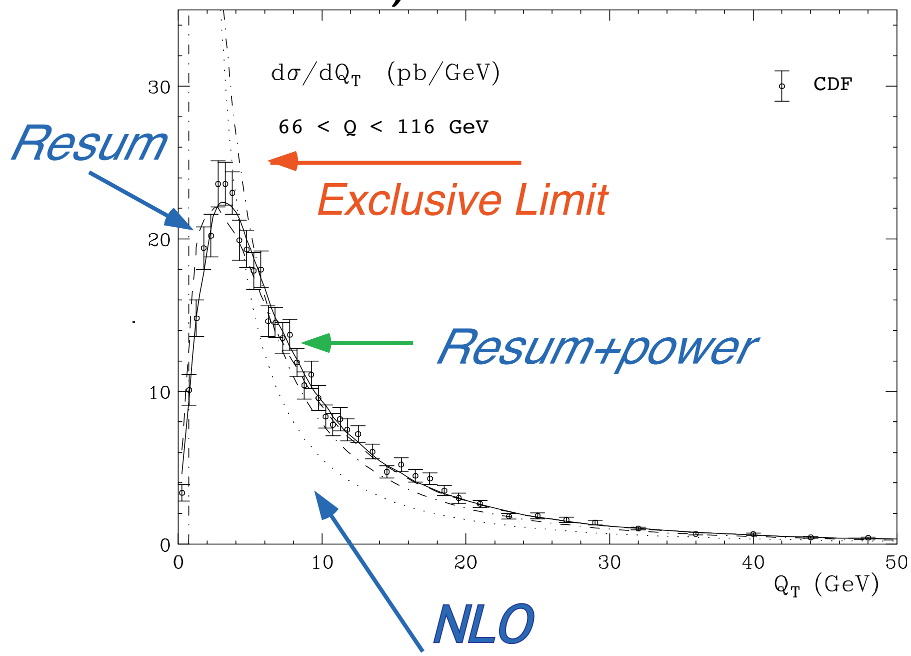
- z integral: If Q^2/S not too big, PDFs nearly constant:

$$\frac{1}{\mathrm{Q}_T^2} \int_{1-Q^2/S}^{1-\mathrm{Q}_T^2/Q^2} \frac{dz}{1-z} = \frac{1}{\mathrm{Q}_T^2} \ln\left[\frac{Q^2}{\mathrm{Q}_T^2}\right]$$

$\Rightarrow$ Prediction for Q_T dependence:

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathrm{Q}_T)}{dQ^2 d^2\mathrm{Q}_T} = \frac{\alpha_s C_F}{\pi^2} \frac{1}{\mathrm{Q}_T^2} \ln\left[\frac{Q^2}{\mathrm{Q}_T^2}\right]$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2} \phi_{a/N}(\xi_1,\mu)\, \phi_{\bar{a}/N}(\xi_2,\mu)$$

- **Compare to: Z p_T (here old CDF data)**



- **$\ln Q_T/Q_T$ works pretty well for large Q_T**

- **At smaller Q_T reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)**

- **Most events are at "low" $Q_T \ll Q = m_Z$.**

Getting to $Q_T \ll Q$: Transverse momentum resummation

(Logs of Q_T)/Q_T to all orders

How? Further factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to the affect cross section,

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathrm{Q}_T)}{dQ^2 d^2\mathrm{Q}_T}\ .$$

Summarized by: Q_T-factorization :.

(Collins & Soper (1983) Collins, Soper, GS (1985))

$$\frac{d\sigma_{NN\to QX}}{dQ^2 d^2Q_T} = \int d\xi_1 d\xi_2\; H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$

$$\times \int d^2\mathrm{k}_{1T} d^2\mathrm{k}_{2T} d^2\mathrm{k}_{sT}\, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$

$$\times\; \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})\; U_{a\bar{a}}(k_{sT}, n)$$

New factorization variables: n^μ apportions gluons k (schematically):

$$p_i\cdot k < n\cdot k \;\Rightarrow\; k\; \in \mathcal{P}_i$$

$$p_1\cdot k, p_2\cdot k \;>\; n\cdot k \;\Rightarrow\; k \in S$$

Convolution in $Q_T \Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

- The functions $\mathcal{P}$ (and U, which can be absorbed into the $\mathcal{P}$'s) defined at fixed gluon transverse momentum. TMDs / a special form of beam functions, can be related to collinear PDFs. Note this factorization also applies to partonic cross sections, $d\sigma_{ab\to QX}$ (and that's how we compute the short-distance functions).
- Aside: threshold resummation is similarly based on a convolution in gluon emission energy near threshold.
GS NPB (1987).

Transform: cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q^2,b)}{dQ^2} = \int \frac{d^2Q_T}{(2\pi)^2}\, \mathrm{e}^{i\mathrm{b}\cdot\mathrm{Q}_T}\, \frac{d\sigma_{NN\to QX}}{dQ^2 d^2Q_T}$$

$$= \int d\xi_1 d\xi_2\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$

$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can resum by separating variables.
(Contopanagos, Laenen, GS (NPB 1997))

the LHS is independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \qquad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

- Equivalently, use of RG formalism as in original CS analysis and SCET

• Solve and transform back to Q_T: all the (Logs of Q_T)/Q_T:

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)\,\right]$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\ \phi_{a/N}(\xi_1, 1/b)\, \phi_{\bar{a}/N}(\xi_2, 1/b)$$

"Sudakov" exponent suppresses large $b \leftrightarrow$ small Q_T:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T)) \right]$$

Lower limit: $1/b$ (NLL)

• Comments:

The functions $A_i(\alpha_s)$ and $B_i(\alpha_s)$ are anomalous dimensions,

and can be calculated by comparison to low orders.

In particular, $A_q(\alpha_s) = \frac{\alpha_s}{\pi} C_q \left(1 + \frac{\alpha_s}{\pi} K + \ldots\right)$, $K = C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5n_F}{9}$ is the numerator of the $1/(1-x)$ term in splitting function $P_{ii}(x)$.

- The pattern:

$$E_{a\bar{a}} \sim 2C_a \frac{\alpha_s(Q)}{\pi} \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[\left\{ 1 + \left(\frac{\alpha_s(Q)}{\pi} \right) \left(K - \frac{\beta_0}{4\pi} \right) \right\} \ln \left(\frac{Q^2}{k_T^2} \right) + 2 \frac{\alpha_s(Q)}{\pi} \right]$$

$$\sim \alpha_s \ln^2(bQ)(1 + \alpha_s \ln(bQ) + \ldots)$$

$$+\alpha_s \ln(bQ)(1 + \alpha_s \ln(bQ) + \ldots) + \ldots$$

- These are LL($A^{(1)}$), NLL ($B^{(1)}$, $A^{(2)}$), and so on
- Evaluating a resummed cross sections: re-enter NPQCD. We start with:

$$E_{a\bar{a}}^{\rm PT} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_a(\alpha_s(k_T)) \ln \left(\frac{Q^2}{k_T^2} \right) + B_a(\alpha_s(k_T)) \right]$$

With running coupling:

$$\alpha_s(k_T) = \frac{\alpha_s(Q)}{1 + \frac{\alpha_s(Q)}{4\pi} \beta_0 \ln \left(\frac{k_T^2}{Q^2} \right)} = \frac{4\pi}{\beta_0 \ln \left(\frac{k_T^2}{\Lambda_{\rm QCD}^2} \right)}$$

Singularity in integral at $b^2 = Q^2 \exp[-4\pi/\beta_0 \alpha_s(Q)] \sim \frac{1}{\Lambda^2}$.

A nonperturbative window– perturbation theory provides hints on its is “completion” in the infrared.

- How to do the inverse transform with the running coupling when $k_T^{\rm min} \sim 1/b$ gets small?

- At least four approaches:

1) Work in Q_T-space directly to some approximation
(Dokshitzer, Diakanov & Troyan (PL 1978) ... then Ellis & Veseli (NPB 1998) Kulesza & Stirling (NPB 1999) who re-derived it from b-space.)

Consistent with the EFT approach– never follow the RG to the Landau pole.

2) Insert a "soft landing" on the k_T integral by replacing

$$1/b \to \sqrt{1/b^2 + 1/b_*^2}$$

for some fixed b_*. (CS, CSS "b_*" prescription, ResBos)

3) Extrapolation of $E^{\rm PT}$ into NP region (Qiu, Zhang (PRD 2000)).

4) Minimal: avoid the singularity at $1/b = \Lambda_{\rm QCD}$ by monkeying with the b-space contour integral. (This technique introduced in threshold resummation; then adapted by Laenen, GS and Vogelsang (PRD 2000), and Bozzi, Catani, de Florian and Grazzini (NPB 2006). See below.)

Any of these "define" PT. All will fit the data qualitatively, and with a little work quantitatively.

But all require new parameters for quantitative fit. This is not all bad ... perturbative resummation demands forms of nonperturbative completion. Viz. "resurgence".

3. Soft gluon refactorization and joint "combinatoric" resummation

A factorization that reflects the exact phase space for soft gluon emission near threshold. A convolution in transverse momentum (as Q_T resummation) and energy fraction (threshold resummation). To derive parton hard scattering functions, factorize partonic cross section.

$$\frac{d\sigma_{ab\to V}}{dQ^2 d^2\mathrm{Q}_T} = \frac{1}{S}\,\sigma^{(0)}_{ab\to V}(Q^2)\; h_{ab}(\alpha_s(Q))$$
$$\times \int dx_a \int dx_b \int dw_s\; \delta(1 - Q^2/S - (1-x_a) - (1-x_b) - w_s)$$
$$\times \int d^2\mathrm{k}_a d^2\mathrm{k}_b\; d^2\mathrm{k}_s\; \delta^2\left(\mathrm{Q}_T + \mathrm{k}_a + \mathrm{k}_b + \mathrm{k}_s\right)$$
$$\times\; \mathcal{R}_{a/a}(x_a, \mathrm{k}_a, Q)\;, \mathcal{R}_{b/b}(x_b, \mathrm{k}_b, Q) U_{ab}(w_s, Q, \mathrm{k}_s)$$

For this analysis, introduce a generalized parton-in-parton distribution at measured energy and k_T:

$$\mathcal{R}_{f/f}(x, \mathrm{k}, 2p_0, \epsilon) = \frac{1}{2N_C}\frac{p_0}{2p\cdot u}\int \frac{d\lambda}{2\pi}\,\frac{d^2\mathrm{b}}{(2\pi)^2}\;\mathrm{e}^{-i\lambda x p_0 + i\mathrm{b}\cdot\mathrm{k}}$$
$$\times \langle f(p)|\bar{q}_f(\lambda\hat{n} + \mathrm{b})\gamma\cdot u q_f(0)|f(p)\rangle\,,$$

here in $A^0 = 0$ gauge.

- All singularities in $1/(1-z)$ generated from an eikonal cross section. Includes all $1/Q_T$ that gives $1/(1-z)$ singularities.

$$\sigma_{a\bar{a}}^{(\mathrm{eik})}(w_s, Q, \mathrm{k}, \mu, \epsilon) \;=\; Q \int \frac{d\lambda}{2\pi}\, \frac{d^2\mathrm{b}}{(2\pi)^2}\, \mathrm{e}^{-i\lambda w_s Q + i\mathrm{b}\cdot\mathrm{k}} \times \mathrm{Tr}\, \langle 0|\, \bar{\mathrm{T}}\left[\mathcal{W}_{a\bar{a}}^{\dagger}(0)\right]\, \mathrm{T}\left[\mathcal{W}_{a\bar{a}}(\lambda\hat{n}+\mathrm{b})\right]\, |0\rangle\,,$$

with $\mathcal{W}$ a product of Wilson lines.

- This gives for the refactorized electroweak cross section:

$$\frac{d\sigma_{AB\to V}}{dQ^2\, dQ_T^2} \;=\; \sum_{ab}\, \hat{\sigma}_{ab\to V}^{(\mathrm{H})}(Q^2)\, \int_{\mathcal{C}} \frac{dN}{2\pi i}\, \exp\left\{\int_{\mu_F^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \sum_{i=a,b} \gamma_{ii}(N, \alpha_s(\mu'))\right\} \times \tilde{\phi}_{a/A}(N+1, \mu_F)\tilde{\phi}_{b/B}(N+1, \mu_F)\, \tau^{-N} \times \int \frac{d^2\mathrm{b}}{(2\pi)^2} \mathrm{e}^{i\mathrm{b}\cdot\mathrm{Q}_T}\, \exp\left[E_{ab}^{(\mathrm{eik})}(N, b, Q, Q)\right] \exp\left[D_{ab}^{(\mathrm{eik})}(N, b)\right]\,,$$

- E and D start at LL and NNLL, respectively.
- A little more on where this comes from: the exponentiation of the eikonal cross section is combinatoric.
- Its logarithm has an exact diagrammatic interpretation in terms of so-called "web diagrams" . . .

- **The eikonal cross section as an exponential of the "web function"**

$$\bar{\sigma}_{ab}^{(\mathrm{eik})}(N, \mathrm{b}Q, \epsilon) \;=\; \exp\left\{ 2 \int \frac{d^{4-2\epsilon}k}{\Omega_{1-2\epsilon}}\, \theta\left(\frac{Q}{\sqrt{2}} - k^+\right)\, \theta\left(\frac{Q}{\sqrt{2}} - k^-\right) \right.$$
$$\left. \times\, w_{ab}\left(k^2, \frac{k\cdot\beta k\cdot\beta'}{\beta\cdot\beta'}, \mu^2, \alpha_s(\mu), \epsilon\right) \left(\mathrm{e}^{-N(k\cdot\hat{n}/Q) - i\mathrm{b}\cdot\mathrm{k}} - 1\right) \right\},$$

- **The web function for $k \neq 0$ starts with single gluon contribution a**

$$w_{a\bar{a}}^{(1)(\mathrm{real})}(k) = \frac{2C_a\alpha_s}{\pi} \left(4\pi\mu^2 \mathrm{e}^{-\gamma_E}\right)^{\epsilon} \frac{1}{k_T^2}\, \delta_+(k^2)\,.$$

- **It is RG invariant and boost invariant**

$$\mu \frac{d}{d\mu}\, w_{ab}\left(k^2, \frac{k\cdot\beta k\cdot\beta'}{\beta\cdot\beta'}, \mu^2, \alpha_s(\mu), \epsilon\right) = 0$$

- **Web diagrams are the all-orders generalization of single-gluon exchange.**
  Erdogan, GS PRD (2015)

- **The virtual corrections cross section exponentiate identically into webs by a combinatoric argument, and the transform of the eikonal cross section is the exponential of the transform of the web function, after collinear subtraction: at $\mu_F = Q$.**
  (Gatheral (PL 1983) Frenkel & Taylor (NPB 1984) Mitov, Sung & GS (PRD 2010))

- **Accurate to exponential corrections:**

$$\bar{\sigma}_{a\bar{a}}^{(\mathrm{eik})}(N, \mathrm{b}Q, \epsilon) \;=\; \exp\Big[\,2\int \frac{d^{2-2\epsilon}k_T}{\Omega_{1-2\epsilon}} \int_0^{Q^2-k_T^2} dk^2\; w_{a\bar{a}}\left(k^2, k_T^2+k^2, \mu^2, \alpha_s(\mu), \epsilon\right) \times \left(\int_{(k_T^2+k^2)/\sqrt{2}Q}^{Q/\sqrt{2}} \frac{dk^+}{2k^+}\, \mathrm{e}^{-N\sqrt{2}\left(\frac{k^+}{2Q}+\frac{k_T^2+k^2}{2Qk^+}\right)-i\mathrm{b}\cdot\mathrm{k}_T} - \ln\sqrt{\frac{Q^2}{k_T^2+k^2}}\right)\Big]$$

- **Some future analysis may use the k^2 integral to organize an infrared extension.**

- For convenience in PT the exponent can be organized into a leading part that is close to k_T resummation, plus a part that NNLL and beyond,

$$\hat{\sigma}_{ab}^{(\mathrm{eik})}(N, \mathrm{b}, Q, \mu) \;=\; \exp\left[E_{ab}^{(\mathrm{eik})}(N, b, Q, \mu)\right]\; \exp\left[D_{ab}^{(\mathrm{eik})}(N, b)\right]$$

- with the leading term as an integral of k_T only, is a functional of the anomalous dimension $A(\alpha_s)$,

$$\begin{aligned} E_{ab}^{(\mathrm{eik})}&(N, b, Q, \mu_F) \\ &= \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \left\{ \sum_{i=a,b} A_i\left(\alpha_s(k_T)\right) \left[J_0\left(bk_T\right)\; K_0\left(\frac{2Nk_T}{Q}\right) + \ln\left(\frac{\bar{N}k_T}{Q}\right)\right]\right\} \\ &\quad - \ln \bar{N} \int_{\mu_F^2}^{Q^2} \frac{dk_T^2}{k_T^2} \sum_{i=a,b} A_i\left(\alpha_s(k_T)\right)\;. \end{aligned}$$

- The difference between a pure k_T integral and the full phase space starts at NNLL

$$\begin{aligned} D_{ab}^{(\mathrm{eik})}(N, b) \;=&\; \int_0^{Q^2} dk_T^2\; \mathcal{A}_{ab}\left(\alpha_s(k_T), k_T, Q\right) \left[\mathrm{e}^{-i\mathrm{b}\cdot\mathrm{k}_T}\; K_0\left(\frac{2Nk_T}{Q}\right) + \ln\left(\frac{k_T}{Q}\right)\right] \\ &+ \int_0^{Q^2} dk_T^2 \int_0^{Q^2-k_T^2} dk^2\; w_{ab}\left(k^2, k_T^2+k^2, \mu^2, \alpha_s(\mu), \epsilon\right) \\ &\times \left[\mathrm{e}^{-i\mathrm{b}\cdot\mathrm{k}_T} \left\{ K_0\left(2N\sqrt{\frac{k_T^2+k^2}{Q^2}}\right) - K_0\left(\frac{2Nk_T}{Q}\right)\right\} + \ln\left(\frac{\sqrt{k_T^2+k^2}}{k_T}\right)\right]\;. \end{aligned}$$

Gives a manageable expression for the resummed cross section in transform space,

$$\hat{\sigma}_{ab}^{(\mathrm{eik})}(N, \mathrm{b}, Q, \mu) = \exp\left[D_{ab}^{(\mathrm{eik})}(N, b)\right] \; \exp\left[E_{ab}^{(\mathrm{eik})}(N, b, Q, \mu)\right] \, ,$$

where the leading N and b-dependence (LL and NLL) is entirely contained in the exponent

$$\begin{aligned} E_{a\bar{a}}^{\mathrm{eik}}(N, b, Q, \mu, \mu_F) \; &= \; 2\int_0^{Q^2} \frac{dk_T^2}{k_T^2} \, A_a(\alpha_s(k_T)) \left[J_0(bk_T) \, K_0\left(\frac{2Nk_T}{Q}\right) + \ln\left(\frac{\bar{N}k_T}{Q}\right)\right] \\ &\quad - 2\ln\left(\bar{N}\right) \int_{\mu_F^2}^{Q^2} \frac{dk_T^2}{k_T^2} A_a(\alpha_s(k_T)) \; . \end{aligned}$$

The second term on the right accounts for the difference between the physical scale Q and the factorization scale μ_F.

- Collinear finite but the usual Landau pole.

- For comparison with Q_T resummation, k_T of the web is chosen as a variable – accurate to NLL but more physical for low scales would have been k^2.

  Here, J_0 and K_0 are the usual Bessel functions, and $\bar{N} = N \mathrm{e}^{\gamma_\mathrm{E}}$

  Dependence on the renormalization scale is implicit through the expansion of $\alpha_s(k_T)$ in powers of $\alpha_s(\mu)$.

- The expansion of the Bessel functions suggest even powers of b and N/Q.

4. Regularization and inverse transforms

Regulating the exponenet

In Kulesza, GS, Vogelsang (PRL 2000) we approximated the exponent by a "cutoff" form that is accurate to next-to-leading logarithm in both transform variables:

$$E^{\mathrm{eik}}_{a\bar{a}}(N,b,Q,\mu,\mu_F) \;=\; 2\int_{Q^2/\chi^2}^{Q^2}\frac{dk_T^2}{k_T^2}\,A_a(\alpha_s(k_T))\,\ln\left(\frac{\bar{N}k_T}{Q}\right) - 2\ln\left(\bar{N}\right)\int_{\mu_F^2}^{Q^2}\frac{dk_T^2}{k_T^2}A_a(\alpha_s(k_T))$$

Here, the function $\chi(\bar{N},\bar{b})$ organizes the logarithms of N and b in joint resummation,

$$\chi(\bar{N},\bar{b}) = \bar{b} + \frac{\bar{N}}{1+\eta\,\bar{b}/\bar{N}}$$

η and $\bar{b}\equiv bQ\mathrm{e}^{\gamma_E}/2$. This form introduces only power-suppressed dependence in both Q_T and $1-z$ at fixed order, but of course still encounters a singularity from the running coupoling for b or N large enough. "Landau pole".

- The full inverse transform takes the form

$$\frac{d\sigma^{\rm res}_{AB}}{dQ^2\,dQ_T^2} = \sum_a \sigma_a^{(0)} \int_{C_N} \frac{dN}{2\pi i}\,\tau^{-N} \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \times\ \mathcal{C}_{a/A}(Q,b,N,\mu,\mu_F)\ \exp\left[\, E^{\rm PT}_{a\bar{a}}(N,b,Q,\mu)\right]\ \mathcal{C}_{\bar{a}/B}(Q,b,N,\mu,\mu_F)$$

- where the $\mathcal{C}$'s include the PDFs.

- Inverting the transforms

  DY cross section at $\mu_F = Q$, using azimuthal symmetry

$$\frac{d\sigma_{AB\to V}}{dQ^2\,dQ_T^2} = \sum_{ab} \sigma^{(\rm H)}_{ab\to V}(Q^2) \times \int_{\mathcal{C}} \frac{dN}{2\pi i}\, \tilde{\phi}_{a/A}(N+1,\mu_F)\tilde{\phi}_{b/B}(N+1,\mu_F)\ \tau^{-N} \times \int db\, b\ J_0(Q_T)\ \hat{\sigma}^{(\rm eik)}_{c\bar{d}}(N,{\rm b},Q,\mu)\,,$$

- Follow Catani Mangano Nason (NPB 1996)) in a “minimal” Mellin inversion (contour $\mathcal{C}$ to right of all perturbative singularities, but to the left of the Landau pole from the k_T integral.

- "Minimal" inverse Fourier transform (LSV). Nontrivial because $J_0(x)$ grows exponentially in $\mathrm{Im}x$ in both half-planes.

- How to "get off the real axis": an alternate representation of J_0:

$$2\pi \int_0^\infty db\, b\; J_0(bQ_T)\, f(b) = \pi \int_0^\infty db\, b\; [\, h_1(bQ_T, v) + h_2(bQ_T, v)\,]\; f(b)\,,$$

- $h_{1,2}(z, v)$, related to Hankel functions by $v \to \infty$ in

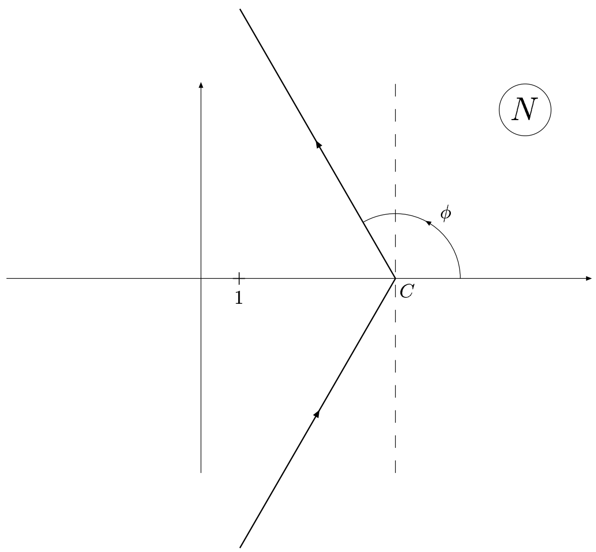
$$h_1(z, v) \;\equiv\; -\frac{1}{\pi} \int_{-iv\pi}^{-\pi+iv\pi} d\theta\, \mathrm{e}^{-iz\,\sin\theta}\,,$$
$$h_2(z, v) \;\equiv\; -\frac{1}{\pi} \int_{\pi+iv\pi}^{-iv\pi} d\theta\, \mathrm{e}^{-iz\,\sin\theta}\,.$$

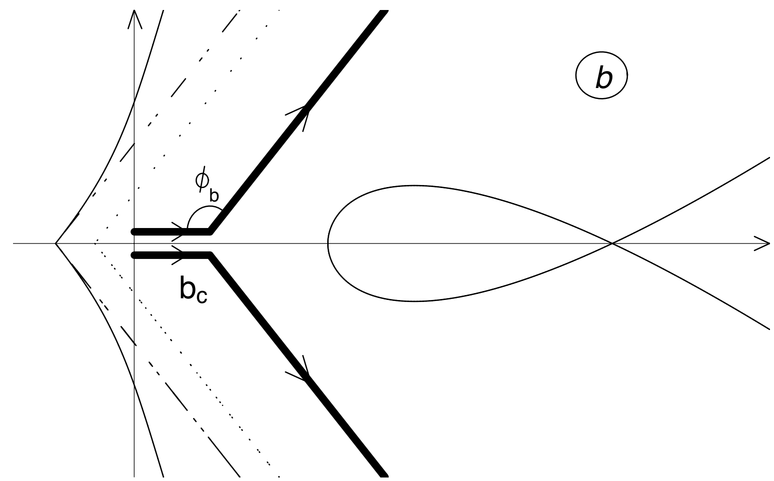
The $h_{1,2}$ become the usual Hankel functions $H_{1,2}(z)$ in the limit $v \to \infty$ (very rapidly).

- Positive and negative phases are separated by the h's, making it possible to treat the b integral as the sum of the two contours associated with h_1 (h_2) corresponding to closing the contour in the upper (lower) half plane.

- These contours avoid the rather complicated Landau poles, and are exactly equivalent to PT order-by-order.

- **The minimal contours in N and b space:**





- **All integrals converge exponentially.**

- **Joint resummation was developed to relate transverse momentum and threshold resummation.**

- **It has been implemented primarily at high energies, but potentially offers a method for incorporating threshold resummation for lower energy Drell-Yan, DIS and single-particle inclusive cross sections.**

- **The combinatoric resummation through web functions is "exact" for eikonal cross sections.**

- **It may offer as-yet undeveloped avenues for the incorporation of nonperturbative effects.**