

Scale evolution equations for collinear twist-3 functions

Shinsuke Yoshida
(LANL)

based on collaboration with: L. Gamberg (Penn State Berks)

Z. B. Kang (UCLA)

M. Schlegel (Tübingen)

K. Tanaka (Juntendo)

H. Xing (ANL/Northwestern)

Workshop@UCLA
05/06/2017

Large single spin asymmetry

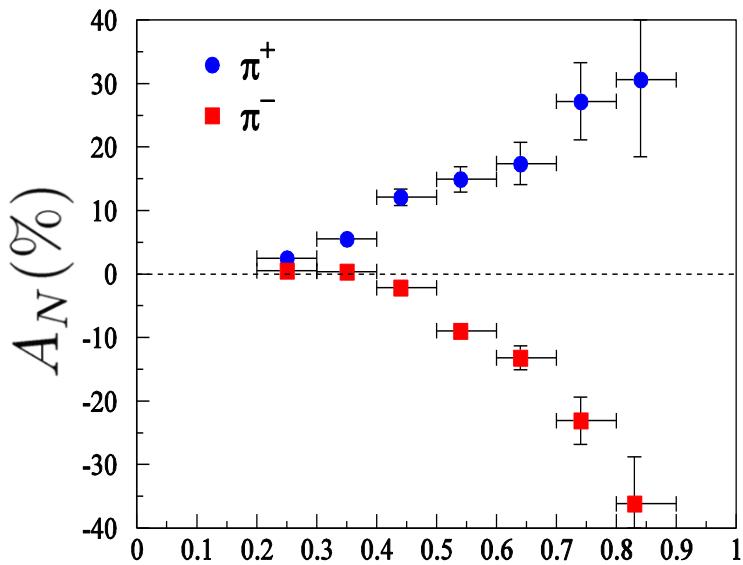
Single spin Asymmetry (SSA)

$$A_N = \frac{d\sigma^{\uparrow} - d\sigma^{\downarrow}}{d\sigma^{\uparrow} + d\sigma^{\downarrow}}$$

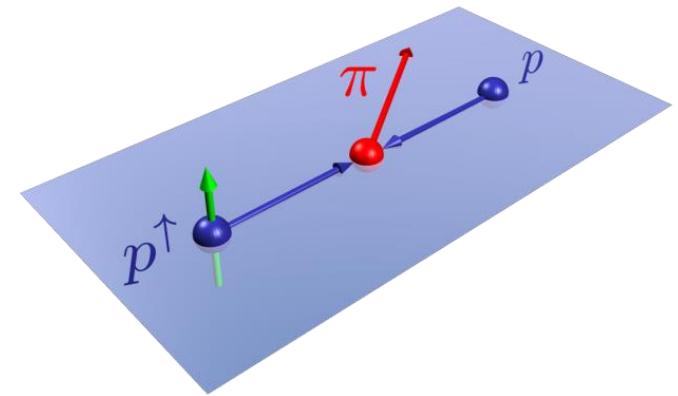
FNAL-E704 ($p^{\uparrow} p \rightarrow \pi X$)

$\sqrt{s} = 20$ GeV

P.L. B264 ('91) 462
P.L. B261 ('91) 201



$$x_F (= \frac{2p_l}{\sqrt{s}})$$



left-right asymmetry of pion distribution

SSA evaluated by the conventional pQCD

$A_N \sim \frac{\alpha_s m_q}{q_T}$: small! (Kane *et al.* ('78))

What is the origin of the large SSA ?

New pQCD frameworks

Transverse Momentum Dependent(TMD) factorization

- Applicable in small $P_T(Q \gg P_T \geq \Lambda_{QCD})$ region
- Nonperturbative functions depend on the transverse momentum of partons
 - c.f Sivers function $f_{1T}^\perp(x, \textcolor{red}{k}_\perp)$
- Some nonperturbative functions have process dependence

collinear factorization

- Applicable in large $P_T(P_T \gg \Lambda_{QCD})$ region
- Unique applicable framework for $pp^\uparrow \rightarrow \pi X, pp \rightarrow \Lambda^\uparrow$
- twist-3 functions don't have simple physical interpretation

New pQCD frameworks

Transverse Momentum Dependent(TMD) factorization

- Applicable in small $P_T(Q \gg P_T \geq \Lambda_{QCD})$ region
- Nonperturbative functions depend on the transverse momentum of partons
 - c.f Sivers function $f_{1T}^\perp(x, \textcolor{red}{k}_\perp)$
- Some nonperturbative functions have process dependence

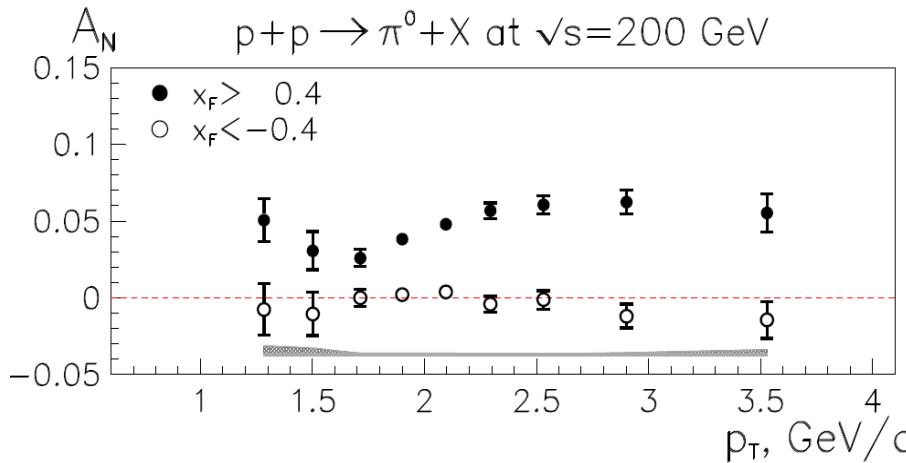
collinear factorization

- Applicable in large $P_T(P_T \gg \Lambda_{QCD})$ region
- Unique applicable framework for $pp^\uparrow \rightarrow \pi X, pp \rightarrow \Lambda^\uparrow$
- twist-3 functions don't have simple physical interpretation

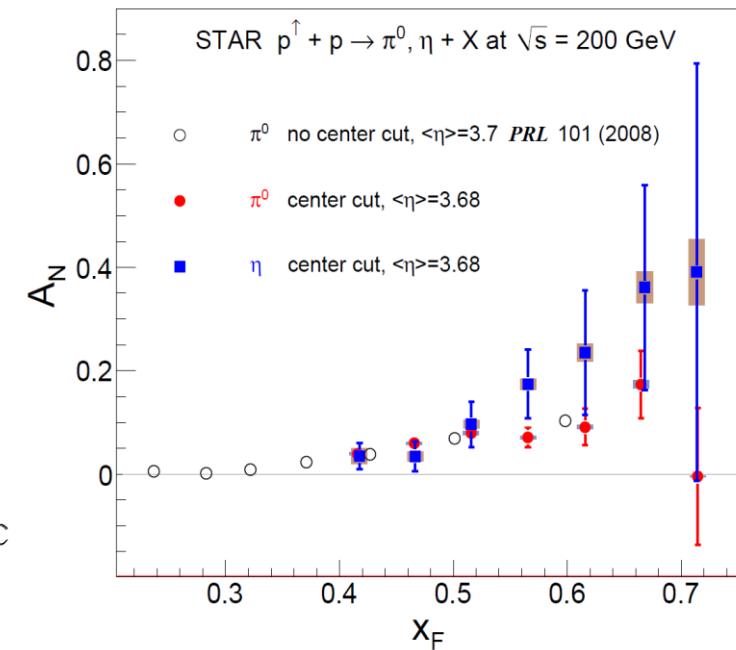
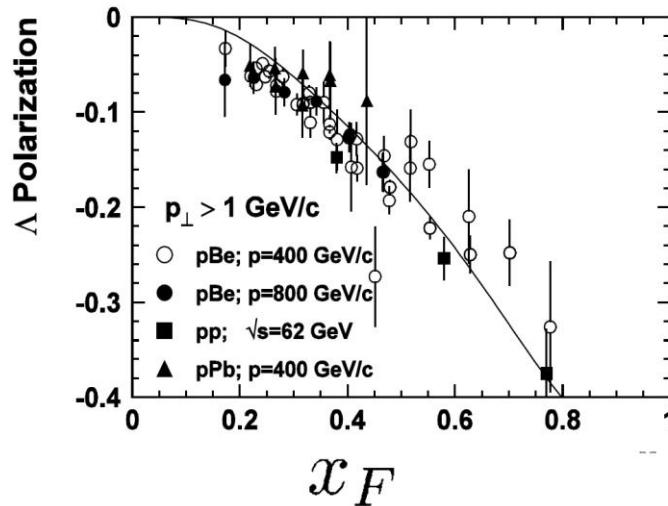
Large single spin asymmetries in high- P_T

Large SSAs in high energy scatterings were first observed in 1970s.

$p^\uparrow p \rightarrow \pi X$



$pp \rightarrow \Lambda^\uparrow X$ (in 80's and 90's)



Leading order calculations

$p^\uparrow p \rightarrow \pi X$

- Twist-3 distribution effect of p^\uparrow (**Sivers type**)

Qiu, Sterman(1998) Kouvaris, Qiu, Vogelsang, Yuan(2006) Koike, Tomita(2009)
Beppu, Kanazawa, Koike, Yoshida(2014)

- Twist-3 distribution effect of p (**Boer-Mulders type**)

Kanazawa, Koike(2000)

- Twist-3 fragmentation effect of π (**Collins type**)

Metz, Pitonyak(2013)

Completed !

$pp \rightarrow \Lambda^\uparrow X$

- Twist-3 distribution effect of p (**Boer-Mulders type**)

Kanazawa, Koike(2001) Zhou, Yuan, Liang(2008) Koike, Yabe, Yoshida(2015)

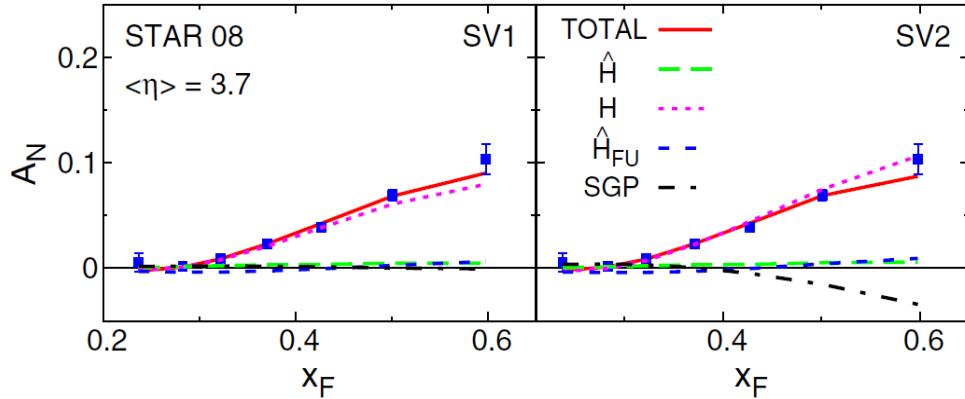
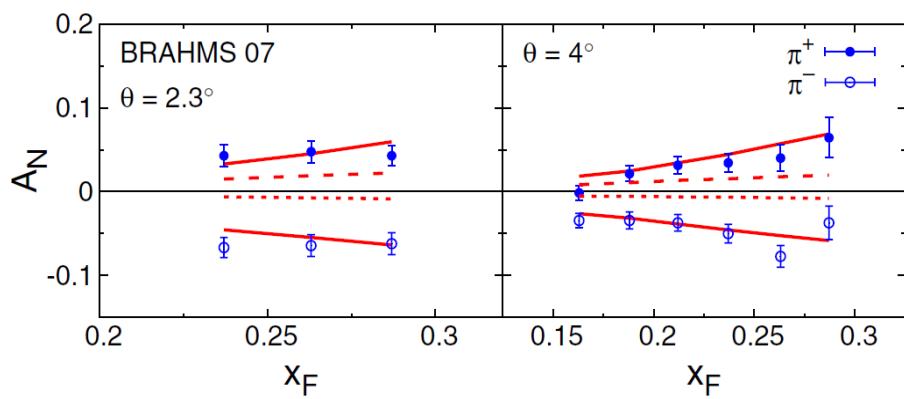
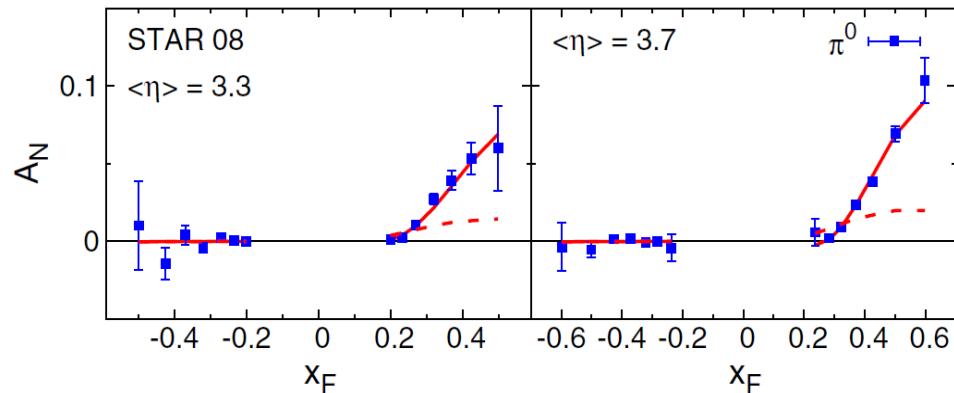
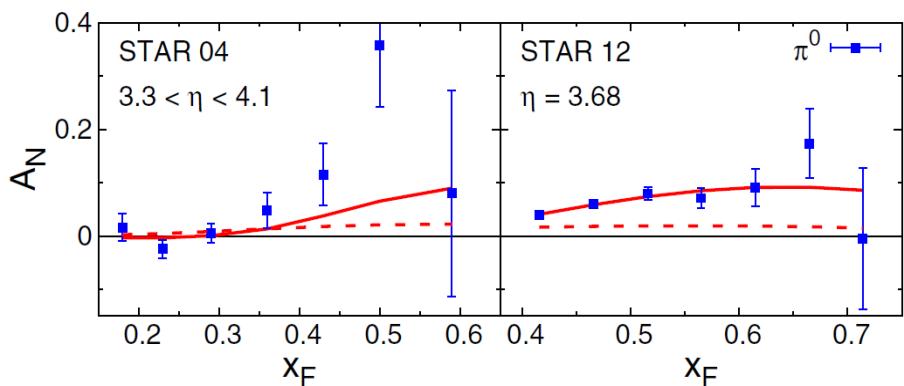
- Twist-3 fragmentaion effect of Λ^\uparrow (**Sivers-fragmentation type**)

Koike, Metz, Pitonyak, Yabe, Yoshida(2017)

Last piece: 3-gluon fragmentation

Numerical Simulation

K. Kanazawa, Y. Koike, A. Metz and D. Pitonyak, Phys. Rev. D 89(2014)



- TMD associated functions $G_F(x, x)$ and $H_1^{\perp(1)}$ give tiny contribution
- Twist-3 fragmentation effect can be dominant source of the large SSA

Evolution equation for the twist-3 distribution function $G_F(x, x)$

QCD factorization theorem

Cross section formula can be decomposed into the nonperturbative function and the hard cross section.

$$d\sigma = f(x, \mu_F) \otimes d\hat{\sigma}(x, Q^2, \mu_F)$$

- Cancellation of the infrared divergence (collinear singularity) has to be proven order by order.
- Scale evolution equation for the factorization scale μ_F is necessary for quantitative calculation.

Dynamical twist-3 functions

$$\begin{aligned} & \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{i\lambda x_1} e^{i\mu(x_2 - x_1)} \langle pS_\perp | \bar{\psi}_j(0) gF^{\alpha n}(\mu n) \psi_i(\lambda n) | pS_\perp \rangle \\ &= \frac{M_N}{4} \epsilon^{\alpha p n S_\perp} (\not{p})_{ij} G_F(x_1, x_2) + i \frac{M_N}{4} S_\perp^\alpha (\gamma_5 \not{p})_{ij} \tilde{G}_F(x_1, x_2) \cdots \end{aligned}$$

We discuss the scale evolution equation for $G_F(x, x)$.

History of evolution equation for $G_F(x, x)$

- One loop calculation for F-type operator

Z. B. Kang and J. W. Qiu, Phys. Rev. D79, 016003 (2009)

J. Zhou, F. Yuan and Z. T. Liang, Phys. Rev. D79, 114022 (2009)

V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)

A. Schafer and J. Zhou, Phys. Rev. D85, 117501 (2012)

J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)

Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

$$d\sigma = f(x, \mu_F) \otimes d\hat{\sigma}(x, Q^2, \mu_F)$$

- NLO transverse-momentum-weighted single spin asymmetry

W. Vogelsang and F. Yuan, Phys. Rev. D **79**, 094010 (2009)

Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D **87**, 034024 (2013)

SY, Phys. Rev. D **93**, 054048 (2016)

History of evolution equation for $G_F(x, x)$

- One loop calculation for F-type operator

Z. B. Kang and J. W. Qiu, Phys. Rev. D79, 016003 (2009)

J. Zhou, F. Yuan and Z. T. Liang, Phys. Rev. D79, 114022 (2009)

V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)

A. Schafer and J. Zhou, Phys. Rev. D85, 117501 (2012)

J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)

Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

$$d\sigma = \boxed{f(x, \mu_F)} \otimes d\hat{\sigma}(x, Q^2, \mu_F)$$

- NLO transverse-momentum-weighted single spin asymmetry

W. Vogelsang and F. Yuan, Phys. Rev. D **79**, 094010 (2009)

Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D **87**, 034024 (2013)

SY, Phys. Rev. D **93**, 054048 (2016)

History of evolution equation for $G_F(x, x)$

- One loop calculation for F-type operator

Z. B. Kang and J. W. Qiu, Phys. Rev. D79, 016003 (2009)

J. Zhou, F. Yuan and Z. T. Liang, Phys. Rev. D79, 114022 (2009)

V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)

A. Schafer and J. Zhou, Phys. Rev. D85, 117501 (2012)

J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)

Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

$$d\sigma = \boxed{f(x, \mu_F)} \otimes d\hat{\sigma}(x, Q^2, \mu_F)$$

- NLO transverse-momentum-weighted single spin asymmetry

W. Vogelsang and F. Yuan, Phys. Rev. D **79**, 094010 (2009)

Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D **87**, 034024 (2013)

SY, Phys. Rev. D **93**, 054048 (2016)

History of evolution equation for $G_F(x, x)$

- One loop calculation for F-type operator

Z. B. Kang and J. W. Qiu, Phys. Rev. D79, 016003 (2009)

J. Zhou, F. Yuan and Z. T. Liang, Phys. Rev. D79, 114022 (2009)

V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)

A. Schafer and J. Zhou, Phys. Rev. D85, 117501 (2012)

J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)

Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

$$\boxed{d\sigma} = f(x, \mu_F) \otimes d\hat{\sigma}(x, Q^2, \mu_F)$$

- NLO transverse-momentum-weighted single spin asymmetry

W. Vogelsang and F. Yuan, Phys. Rev. D **79**, 094010 (2009)

Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D **87**, 034024 (2013)

SY, Phys. Rev. D **93**, 054048 (2016)

P_h -weighted cross section in SIDIS

We discuss the semi-inclusive deep inelastic scattering,

$$e(\ell) + p(p, S_\perp) \rightarrow e(\ell') + h(P_h) + X.$$

The cross section formula can be expressed by the Lorentz invariant variables,

$$S_{ep} = (p + \ell)^2, \quad Q^2 = -(\ell - \ell')^2 = -q^2, \quad x_B = \frac{Q^2}{2p \cdot q}, \quad z_h = \frac{p \cdot P_h}{p \cdot q}.$$

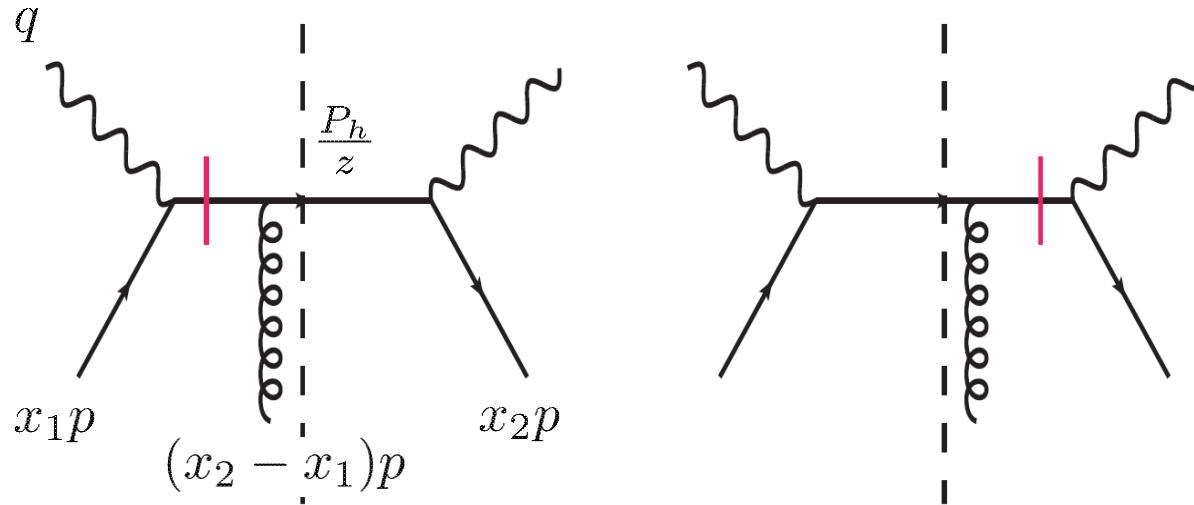
NLO transverse-momentum-weighted single spin asymmetry is defined as

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle}{dx_B dQ^2 dz_h d\phi} \equiv \int d^2 P_{h\perp} \epsilon^{\alpha\beta-+} S_{\perp\alpha} P_{h\perp\beta} \left(\frac{d^6 \Delta\sigma}{dx_B dQ^2 dz_h dP_{h\perp}^2 d\phi d\chi} \right)$$

LO cross section

The SSA, naively T-odd observable, requires a complex phase.

This can be given by the pole contribution in collinear factorization approach.



$$\frac{1}{(\frac{P_h}{z} - (x_2 - x_1)p)^2 + i\epsilon} = P \frac{1}{(\frac{P_h}{z} - (x_2 - x_1)p)^2} - i\pi\delta\left((\frac{P_h}{z} - (x_2 - x_1)p)^2\right)$$

The **red barred** propagator gives the pole contribution $i\pi\delta(x_1 - x_2)$.

$$\frac{d^4 \langle P_{h\perp} \Delta \sigma \rangle^{\text{LO}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G_F(x_B, x_B) D(z_h)$$

$$G_F(x_B, x_B) = \frac{1}{\pi M_N^2} \int d^2 k_\perp k_\perp^2 f_{1T}^{\perp \text{ (SIDIS)}}(x_B, k_\perp)$$

Real emission diagrams in NLO cross section

Pole contributions can be classified into four types.

H. Eguchi Y. Koike and K. Tanaka, Nucl. Phys. B763 (2007) 198

Y. Koike and K. Tanaka, arXiv:0907.2797

1. soft gluon pole(SGP)

$$x_1 = x_2$$

$$G_F(x, x)$$

~~$$\tilde{G}_F(x, x)$$~~

2. soft fermion pole(SFP)

$$x_1 = 0 \quad \text{or} \quad x_2 = 0$$

$$G_F(x, 0)$$

$$\tilde{G}_F(x, 0)$$

3. hard pole(HP)

$$x_1 = x_B \quad \text{or} \quad x_2 = x_B$$

$$G_F(x, x_B)$$

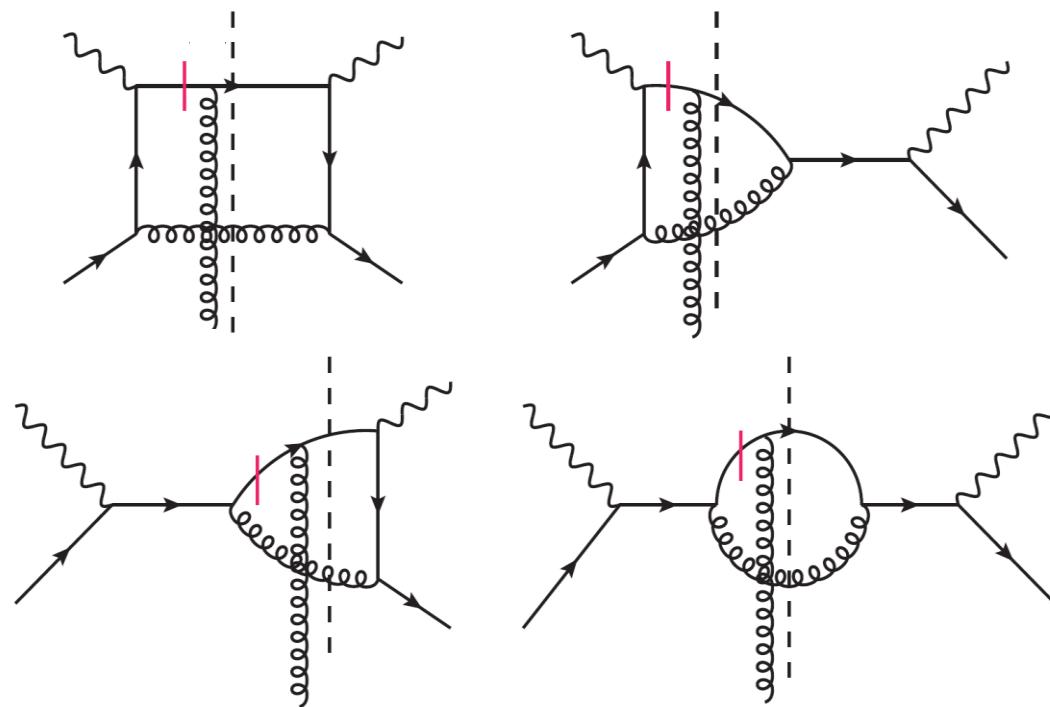
$$\tilde{G}_F(x, x_B)$$

4. new hard pole(HP2)

$$x_1 = x_B - x, \quad x_2 = x_B \quad \text{or} \quad x_1 = x_B, \quad x_2 = x_B - x$$

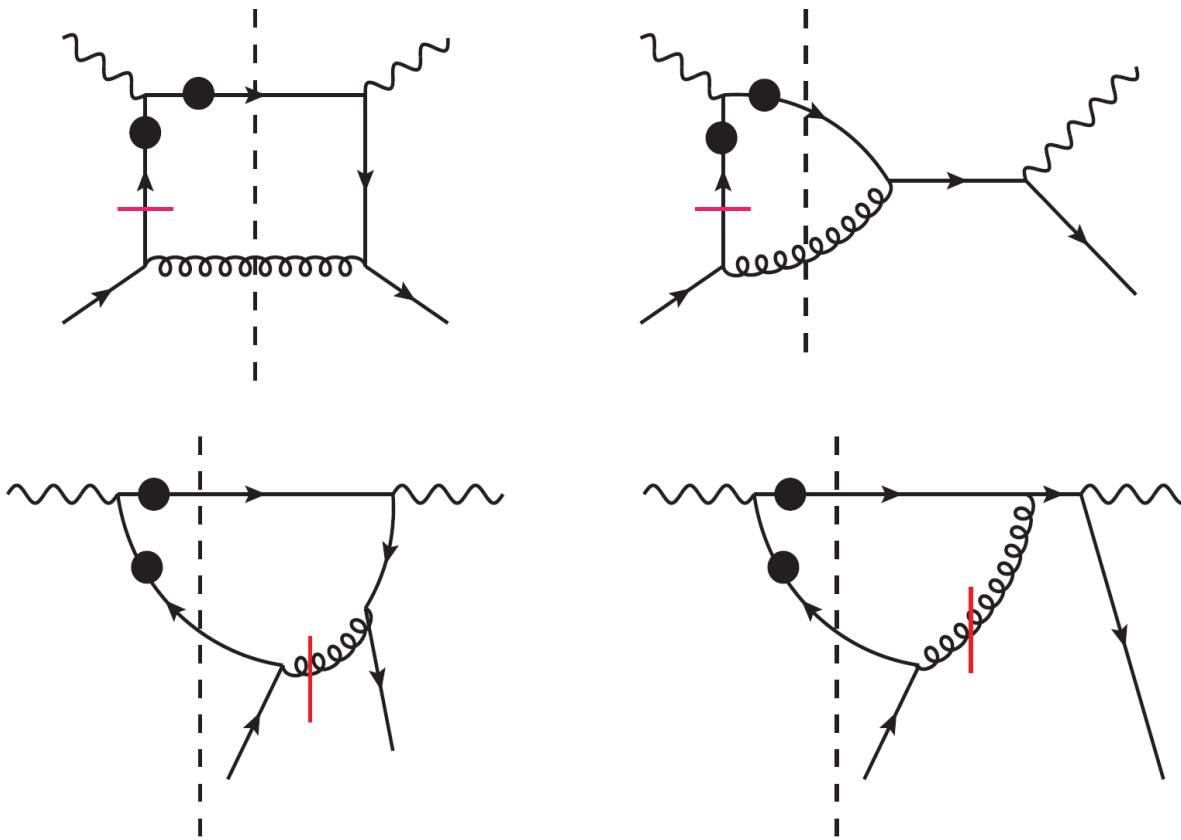
$$G_F(x_B - x, x_B) \quad \tilde{G}_F(x_B - x, x_B)$$

SGP contribution



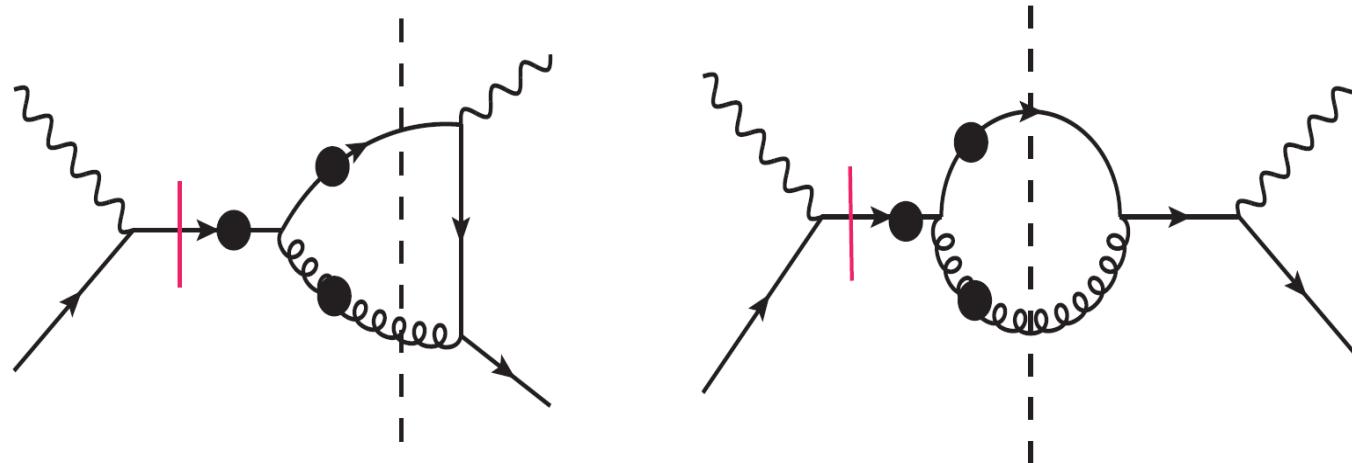
$$\begin{aligned} \frac{d^4 \langle P_{h\perp} \Delta \sigma \rangle^{\text{SGP}}}{dx_B dQ^2 dz_h d\phi} &= -\frac{\pi M_N \alpha_{em}^2 \alpha_s}{4x_B^2 S_{ep}^2 Q^2} \frac{1}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \int dz D(z) \int \frac{dx}{x} G_F(x, x) \\ &\quad \times \frac{1}{2N} \left[-\frac{2}{\epsilon^2} \delta(1-\hat{x}) \delta(1-\hat{z}) + \frac{1}{\epsilon} \frac{1+\hat{z}^2}{(1-\hat{z})_+} \delta(1-\hat{x}) - \frac{1}{\epsilon} \frac{2\hat{x}^3 - 3\hat{x}^2 - 1}{(1-\hat{x})_+} \delta(1-\hat{z}) \right] + \dots \end{aligned}$$

SFP contribution



Upper diagrams and lower diagrams cancel each other.

HP contribution



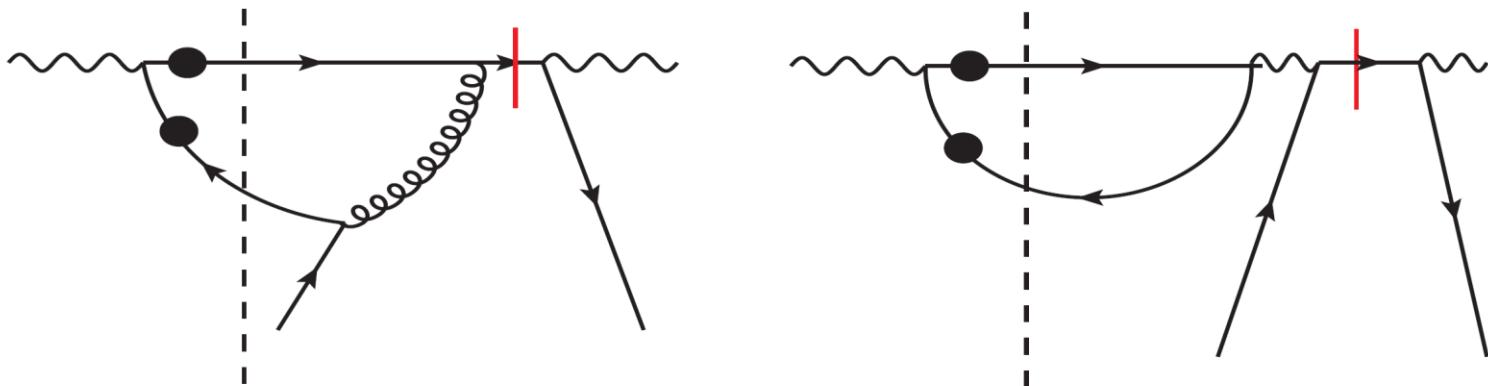
$$\begin{aligned}
 \frac{d^4 \langle P_{h\perp} \Delta \sigma \rangle^{\text{HP}}}{dx_B dQ^2 dz_h d\phi} = & -\frac{\pi M_N \alpha_{em}^2 \alpha_s}{4x_B^2 S_{ep}^2 Q^2} \frac{1}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \int dz D(z) \int \frac{dx}{x} \left(\hat{z} C_F + \frac{1}{2N} \right) \\
 & \times \left\{ G_F(x, x_B) \left[\frac{2}{\epsilon^2} \delta(1-\hat{x}) \delta(1-\hat{z}) + \frac{1}{\epsilon} \left(2\delta(1-\hat{x}) \delta(1-\hat{z}) - \frac{1+\hat{z}^2}{(1-\hat{z})_+} \delta(1-\hat{x}) \right. \right. \right. \\
 & \left. \left. \left. - \frac{1+\hat{x}}{(1-\hat{x})_+} \delta(1-\hat{z}) \right) \right] + \tilde{G}_F(x, x_B) \frac{1}{\epsilon} \delta(1-\hat{z}) \right\} + \dots
 \end{aligned}$$

Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D **87**, 034024 (2013)

SY, Phys. Rev. D **93**, 054048 (2016)

HP2 contribution

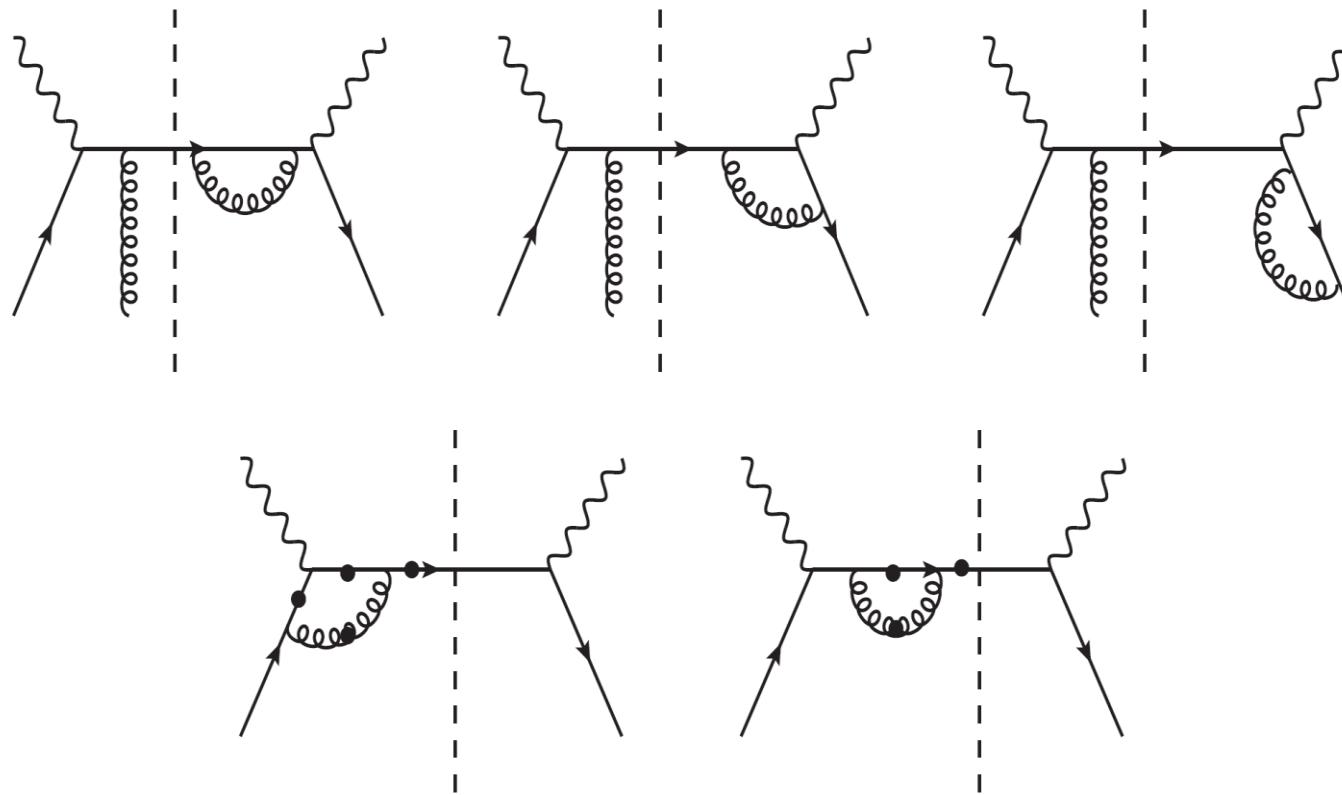
SY, Phys. Rev. D **93**, 054048 (2016)



$$\begin{aligned} \frac{d^4 \langle P_{h\perp} \Delta \sigma \rangle^{\text{HP2}}}{dx_B dQ^2 dz_h d\phi} &= -\frac{\pi M_N \alpha_{em}^2 \alpha_s}{4x_B^2 S_{ep}^2 Q^2} \frac{1}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \int dz D(z) \int \frac{dx}{x} \\ &\times \frac{1}{2N} \left\{ -G_F(x_B, x_B - x) \frac{1}{\epsilon} (1 - 2\hat{x}) \delta(1 - \hat{z}) - \tilde{G}_F(x_B, x_B - x) \frac{1}{\epsilon} \delta(1 - \hat{z}) \right\} + \dots \end{aligned}$$

HP2 contribution brings a collinear singularity $\frac{1}{\epsilon}$.

Virtual correction diagrams in NLO cross section



Z. B. Kang, I. Vitev and H. Xing, Phys. Rev. D 87, 034024 (2013)

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{virtual}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G(x_B, x_B) D(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) \right]$$

NLO virtual cross section

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{virtual}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G(x_B, x_B) D(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) \right]$$

NLO real cross section

$$\begin{aligned} & \frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{real}}}{dx_B dQ^2 dz_h d\phi} \\ = & -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_q e_q^2 \left[C_F \frac{2}{\epsilon^2} G_F^q(x_B, x_B) D^q(z_h) \right. \\ & + \left(-\frac{1}{\epsilon} \right) \left\{ D^q(z_h) \left\{ \int_{x_B}^1 \frac{dx}{x} \left[C_F \frac{1+\hat{x}^2}{(1-\hat{x})_+} G_F^q(x, x) + \frac{N}{2} \left(\frac{(1+\hat{x})G_F^q(x_B, x) - (1+\hat{x}^2)G_F^q(x, x)}{(1-\hat{x})_+} \right. \right. \right. \right. \\ & \left. \left. \left. \left. + \tilde{G}_F^q(x_B, x) \right) \right] - NG_F^q(x_B, x_B) + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1-2\hat{x})G_F^q(x_B, x_B-x) + \tilde{G}_F^q(x_B, x_B-x) \right) \right\} \\ & \left. + G_F^q(x_B, x_B) C_F \int_{z_h}^1 \frac{dz}{z} \frac{1+\hat{z}^2}{(1-\hat{z})_+} D^q(z) \right\} + \dots \end{aligned}$$

NLO virtual cross section

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{virtual}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G(x_B, x_B) D(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) \right]$$

NLO real cross section

$$\begin{aligned} & \frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{real}}}{dx_B dQ^2 dz_h d\phi} \\ = & -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_q e_q^2 \left[C_F \frac{2}{\epsilon^2} G_F^q(x_B, x_B) D^q(z_h) \right] \\ & + \left(-\frac{1}{\epsilon} \right) \left\{ D^q(z_h) \left\{ \int_{x_B}^1 \frac{dx}{x} \left[C_F \frac{1+\hat{x}^2}{(1-\hat{x})_+} G_F^q(x, x) + \frac{N}{2} \left(\frac{(1+\hat{x})G_F^q(x_B, x) - (1+\hat{x}^2)G_F^q(x, x)}{(1-\hat{x})_+} \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. + \tilde{G}_F^q(x_B, x) \right) \right] - NG_F^q(x_B, x_B) + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1-2\hat{x})G_F^q(x_B, x_B-x) + \tilde{G}_F^q(x_B, x_B-x) \right) \right\} \\ & \quad \left. + G_F^q(x_B, x_B) C_F \int_{z_h}^1 \frac{dz}{z} \frac{1+\hat{z}^2}{(1-\hat{z})_+} D^q(z) \right\} + \dots \end{aligned}$$

NLO virtual cross section

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{virtual}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G(x_B, x_B) D(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) \right]$$

NLO real cross section

$$\begin{aligned} & \frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{real}}}{dx_B dQ^2 dz_h d\phi} \\ = & -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_q e_q^2 \boxed{C_F \frac{2}{\epsilon^2} G_F^q(x_B, x_B) D^q(z_h)} \\ & + \left(-\frac{1}{\epsilon} \right) \left\{ D^q(z_h) \left\{ \int_{x_B}^1 \frac{dx}{x} \left[C_F \frac{1+\hat{x}^2}{(1-\hat{x})_+} G_F^q(x, x) + \frac{N}{2} \left(\frac{(1+\hat{x})G_F^q(x_B, x) - (1+\hat{x}^2)G_F^q(x, x)}{(1-\hat{x})_+} \right. \right. \right. \right. \\ & \quad \left. \left. \left. \left. + \tilde{G}_F^q(x_B, x) \right) \right] - NG_F^q(x_B, x_B) + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1-2\hat{x})G_F^q(x_B, x_B-x) + \tilde{G}_F^q(x_B, x_B-x) \right) \right\} \\ & \quad \left. + G_F^q(x_B, x_B) C_F \int_{z_h}^1 \frac{dz}{z} \frac{1+\hat{z}^2}{(1-\hat{z})_+} D^q(z) \right\} + \dots \end{aligned}$$

NLO virtual cross section

$$\frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{virtual}}}{dx_B dQ^2 dz_h d\phi} = -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} G(x_B, x_B) D(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - 8 \right) \right]$$

NLO real cross section

$$\begin{aligned} & \frac{d^4 \langle P_{h\perp} \Delta\sigma \rangle^{\text{real}}}{dx_B dQ^2 dz_h d\phi} \\ = & -\frac{z_h \pi M_N \alpha_{em}^2}{4x_B^2 S_{ep}^2 Q^2} \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_q e_q^2 \boxed{C_F \frac{2}{\epsilon^2} G_F^q(x_B, x_B) D^q(z_h)} \\ & + \left(-\frac{1}{\epsilon} \right) \left\{ D^q(z_h) \left\{ \int_{x_B}^1 \frac{dx}{x} \left[C_F \frac{1+\hat{x}^2}{(1-\hat{x})_+} G_F^q(x, x) + \frac{N}{2} \left(\frac{(1+\hat{x})G_F^q(x_B, x) - (1+\hat{x}^2)G_F^q(x, x)}{(1-\hat{x})_+} \right. \right. \right. \right. \\ & \left. \left. \left. \left. + \tilde{G}_F^q(x_B, x) \right) \right] - NG_F^q(x_B, x_B) + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1-2\hat{x})G_F^q(x_B, x_B-x) + \tilde{G}_F^q(x_B, x_B-x) \right) \right\} \\ & + G_F^q(x_B, x_B) C_F \int_{z_h}^1 \frac{dz}{z} \frac{1+\hat{z}^2}{(1-\hat{z})_+} D^q(z) \right\} + \dots \end{aligned}$$

splitting function $P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$

Subtraction of collinear singularity

We can subtract the collinear singularities by the following renormalization of $G_F(x, x)$.

$$\begin{aligned} & G_F(x_B, x_B) \\ = & G_F^{(0)}(x_B, x_B) + \frac{\alpha_s}{2\pi} \left(-\frac{1}{\hat{\epsilon}} \right) \left\{ \int_{x_B}^1 \frac{dx}{x} \left[P_{qq}(\hat{x}) G_F(x, x) \right. \right. \\ & + \frac{N}{2} \left(\frac{(1 + \hat{x}) G_F(x_B, x) - (1 + \hat{x}^2) G_F(x, x)}{(1 - \hat{x})_+} + \tilde{G}_F(x_B, x) \right) \left. \right] - NG_F(x_B, x_B) \\ & \left. + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1 - 2\hat{x}) G_F(x_B, x_B - x) + \tilde{G}_F(x_B, x_B - x) \right) \right\} \end{aligned}$$

with $\overline{\text{MS}}$ scheme $\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi$.

These collinear singularities are the same as those in F-type correlator at 1-loop order.

V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)

J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)

Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

Scale evolution equation for $G_F(x, x)$

$$\frac{\partial}{\partial \ln \mu^2} \frac{d^4 \langle P_{h\perp} \Delta \sigma \rangle^{\text{LO+NLO}}}{dx_B dQ^2 dz_h d\phi} = 0$$

$$\begin{aligned} \rightarrow \quad & \frac{\partial}{\partial \ln \mu^2} G_F(x_B, x_B, \mu^2) = \frac{\alpha_s}{2\pi} \left\{ \int_{x_B}^1 \frac{dx}{x} \left[P_{qq}(\hat{x}) G_F(x, x, \mu^2) \right. \right. \\ & + \frac{N}{2} \left(\frac{(1 + \hat{x}) G_F(x_B, x, \mu^2) - (1 + \hat{x}^2) G_F(x, x, \mu^2)}{(1 - \hat{x})_+} + \tilde{G}_F(x_B, x, \mu^2) \right) \Big] \\ & - NG_F(x_B, x_B, \mu^2) \\ & \left. \left. + \frac{1}{2N} \int_{x_B}^1 \frac{dx}{x} \left((1 - 2\hat{x}) G_F(x_B, x_B - x, \mu^2) + \tilde{G}_F(x_B, x_B - x, \mu^2) \right) \right\} \right. \end{aligned}$$

This completely agrees with the result in different approach.

- V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D80, 114002 (2009)
- J. P. Ma and Q. Wang, Phys. Lett. B715, 157 (2012)
- Z. B. Kang and J. W. Qiu, Phys. Lett. B713 , 273 (2012)

Evolution equation for the twist-3 fragmentation function $H_1^{\perp(1)}(z)$

L. Gamberg, Z. B. Kang, M. Schlegel, K. Tanaka, H. Xing and SY

in progress

List of works on the evolution equations

pion production (spin-0)

- twist-3 naively T -even fragmentation function

A. V. Belitsky and E.A. Kuraev, Nucl.Phys. B499 (1997)

(Double spin asymmetry A_{LT})

Koike, Pitonyak, Takagi, Yoshida(2015)

- twist-3 naively T -odd fragmentation function

Z. B. Kang, Phys.Rev. D83 (2011)

J.P. Ma and G.P. Zhang, arXiv:1701.04141

Λ^\uparrow production (spin- $\frac{1}{2}$)

- twist-3 naively T -odd fragmentation function

Z. B. Kang, Phys.Rev. D83 (2011)

No results from calculation of NLO cross section

Twist-3 fragmentation effect of π

$$p^\uparrow(p, S_\perp) + p(p', \Lambda) \rightarrow \pi(P_h) + X$$

K. Kanazawa et al., Phys. Rev. D93 (2016)

Intrinsic: $H(z)$ Kinematical: $H_1^{\perp(1)}(z)$

Dynamical: $\hat{H}_{FU}(z_1, z_2), \tilde{H}_{FU}(z_1, z_2)$

◦ EOM relation

$$2 \int \frac{dz}{z_1^2} P\left(\frac{1}{1/z - 1/z_1}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) - 2H_1^{\perp(1)}(z) = \frac{H(z)}{z}$$

◦ Lorenz invariant relation

$$\frac{2}{z} \int \frac{dz}{z_1^2} P\left(\frac{1}{(1/z - 1/z_1)^2}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) + \frac{d}{d(1/z)} \frac{H_1^{\perp(1)}(z)}{z} = -\frac{H(z)}{z}$$

Twist-3 fragmentation effect of π

$$p^\uparrow(p, S_\perp) + p(p', \Lambda) \rightarrow \pi(P_h) + X$$

K. Kanazawa et al., Phys. Rev. D93 (2016)

Intrinsic: $H(z)$

Kinematical: $H_1^{\perp(1)}(z)$

$$H_1^{\perp(1)}(z) = z^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_h^2} H_1^\perp(z, z^2 p_\perp^2)$$

Dynamical: $\hat{H}_{FU}(z_1, z_2), \tilde{H}_{FU}(z_1, z_2)$

Collins function

◦ EOM relation

$$2 \int \frac{dz}{z_1^2} P\left(\frac{1}{1/z - 1/z_1}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) - 2H_1^{\perp(1)}(z) = \frac{H(z)}{z}$$

◦ Lorenz invariant relation

$$\frac{2}{z} \int \frac{dz}{z_1^2} P\left(\frac{1}{(1/z - 1/z_1)^2}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) + \frac{d}{d(1/z)} \frac{H_1^{\perp(1)}(z)}{z} = -\frac{H(z)}{z}$$

Twist-3 fragmentation effect of π

$$p^\uparrow(p, S_\perp) + p(p', \Lambda) \rightarrow \pi(P_h) + X$$

K. Kanazawa et al., Phys. Rev. D93 (2016)

Intrinsic: $H(z)$ Kinematical: $H_1^{\perp(1)}(z)$ $H_1^{\perp(1)}(z) = z^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_h^2} H_1^\perp(z, z^2 p_\perp^2)$

Dynamical: $\hat{H}_{FU}(z_1, z_2), \tilde{H}_{FU}(z_1, z_2)$

Collins function

$$\hat{H}_{FU}(z_1, z_2) = \text{Re}(\hat{H}_{FU}(z_1, z_2)) + i\text{Im}(\hat{H}_{FU}(z_1, z_2))$$

◦ EOM relation

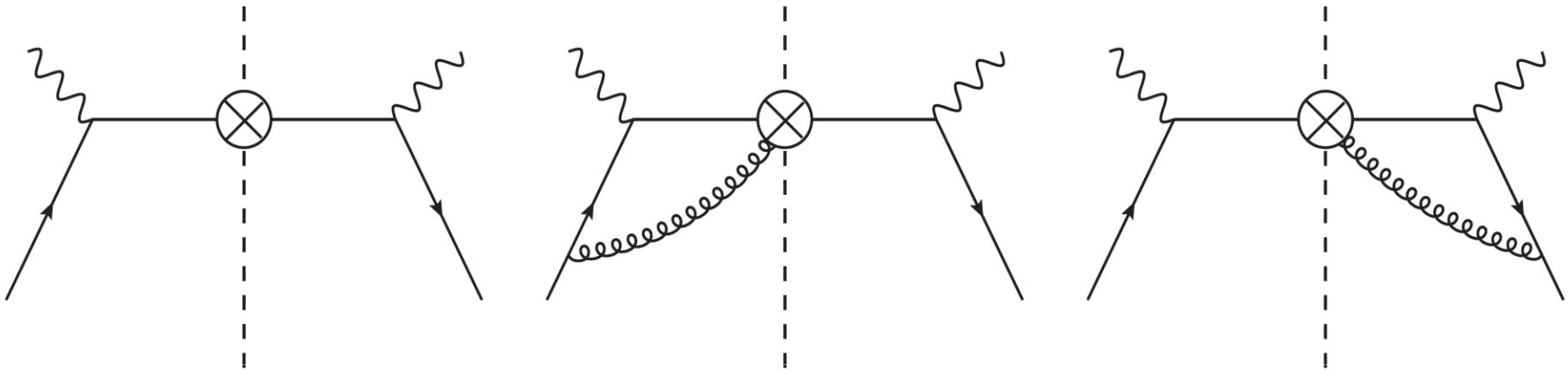
$$2 \int \frac{dz}{z_1^2} P\left(\frac{1}{1/z - 1/z_1}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) - 2H_1^{\perp(1)}(z) = \frac{H(z)}{z}$$

◦ Lorenz invariant relation

$$\frac{2}{z} \int \frac{dz}{z_1^2} P\left(\frac{1}{(1/z - 1/z_1)^2}\right) \text{Im}(\hat{H}_{FU}(z, z_1)) + \frac{d}{d(1/z)} \frac{H_1^{\perp(1)}(z)}{z} = -\frac{H(z)}{z}$$

LO cross section

The imaginary part of twist-3 fragmentation function provide a complex phase.
Pole contribution is not needed. (nonpole contribution)



A. Metz and D. Pitonyak, Phys. Lett. B **723**, 365 (2013)

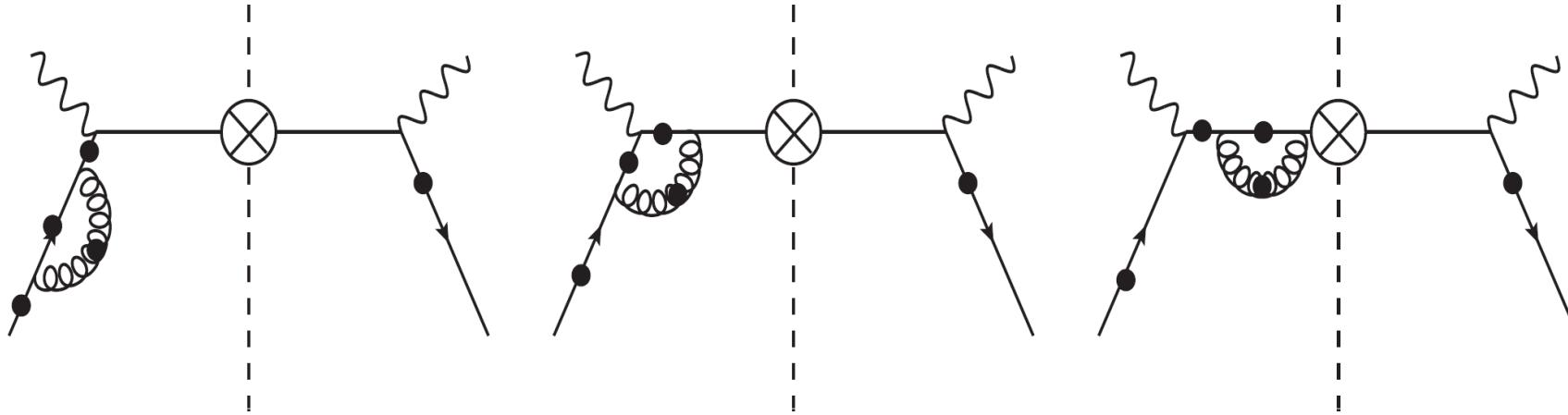
K. Kanazawa and Y. Koike, Phys. Rev. D **88**, 074022 (2013)

K. Kanazawa et al., Phys. Rev. D **93** (2016) no.5, 054024

$$d^4 \langle q_\perp \Delta \sigma \rangle^{\text{LO}} \propto h(x_B) H_1^{\perp(1)}(z_h)$$

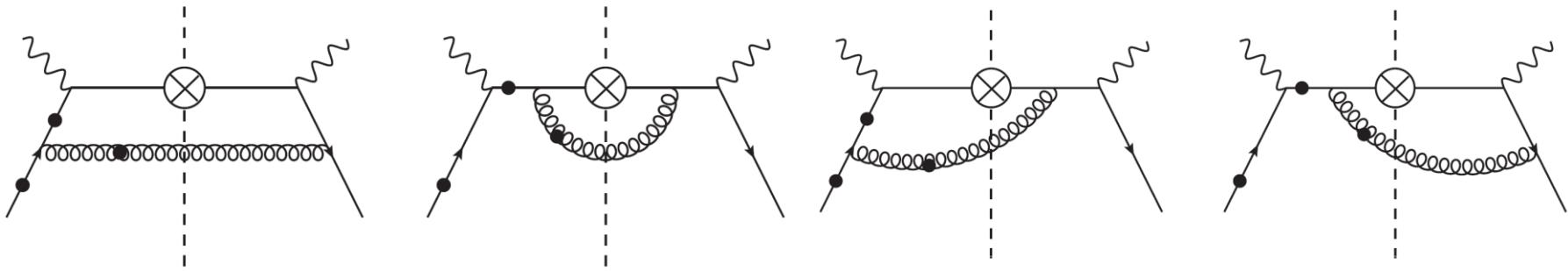
$$H_1^{\perp(1)}(z) \equiv z^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_\Lambda^2} H_1^\perp(z, z^2 p_\perp^2)$$

Virtual correction diagrams in NLO cross section



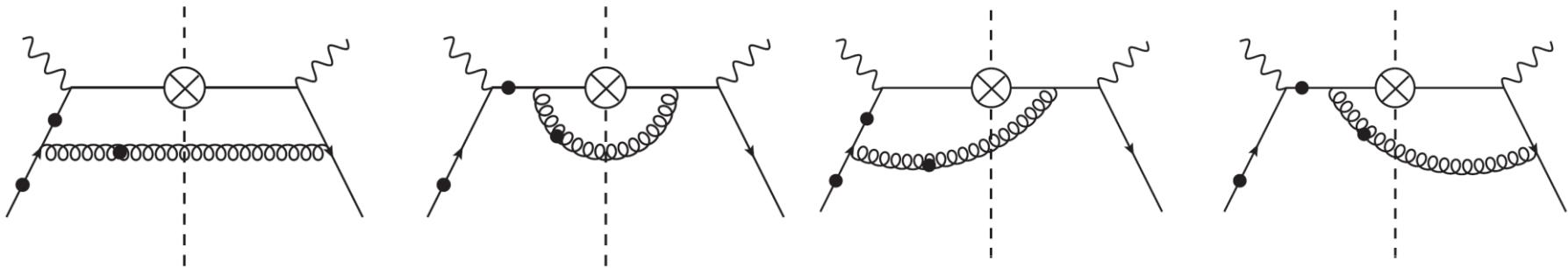
$$d^4 \langle q_\perp \Delta\sigma \rangle^{\text{virtual}} \propto f(x_B) H_1^{\perp(1)}(z_h) \left[C_F \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 \right) \right]$$

Real emission diagrams in NLO cross section



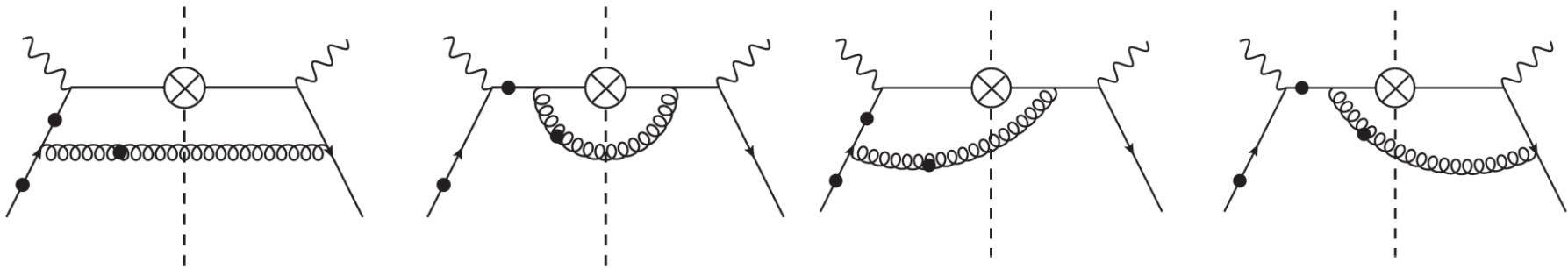
$$\begin{aligned}
 d^4 \langle q_\perp \Delta\sigma \rangle^{\text{real}} \propto & \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_a \left[\frac{2}{\epsilon^2} C_F h_1^a(x_B) H_1^{\perp(1)a}(z_h) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) C_F H_1^{\perp(1)a}(z_h) \int_{x_B}^1 \frac{dx}{x} \frac{2\hat{x}}{(1-\hat{x})_+} h_1^a(x) \\
 & + \left(-\frac{1}{\epsilon} \right) f(x_B) \left(\int_{z_h}^1 \frac{dz}{z} \frac{2\hat{z}}{(1-\hat{z})_+} H_1^{\perp(1)a}(z) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{z}, \frac{1}{z_1}\right) \text{Im} \widehat{H}_{FU}^a(z, z_1) \left[-2\hat{z} C_F \frac{2}{z} P\left(\frac{1}{z}, \frac{1}{z_1}\right) \left(\frac{N_c}{2} (2-\hat{z}) \right) \right. \\
 & \left. \left. + \frac{1}{z} P\left(\frac{1}{z}\right) \left(-\frac{1}{N_c} (1-\hat{z}) \right) + \frac{1}{z^2} P\left(\frac{1}{z}, \frac{1}{z_1}\right) P\left(\frac{1}{z_1}, \frac{1}{z_h}\right) \left(-N_c (1-\hat{z}) \right) \right] \right] + \dots
 \end{aligned}$$

Real emission diagrams in NLO cross section



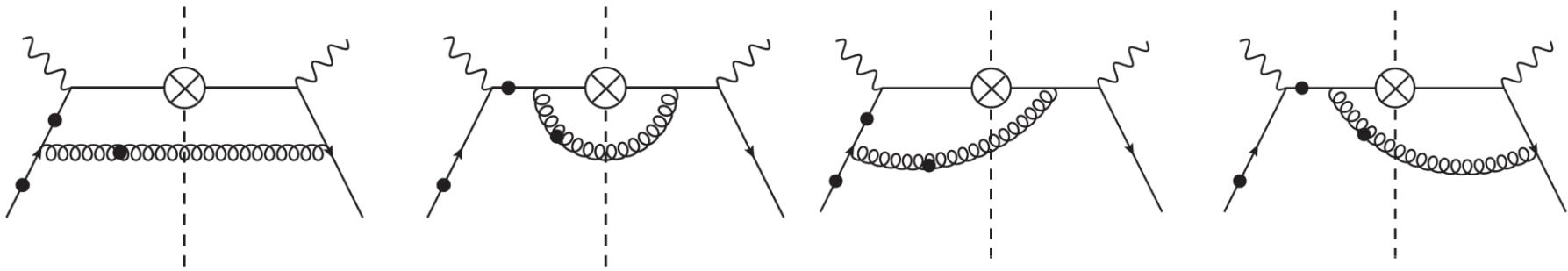
$$\begin{aligned}
 d^4 \langle q_\perp \Delta\sigma \rangle^{\text{real}} \propto & \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_a \boxed{\frac{2}{\epsilon^2} C_F h_1^a(x_B) H_1^{\perp(1)a}(z_h)} \\
 & + \left(-\frac{1}{\epsilon} \right) C_F H_1^{\perp(1)a}(z_h) \int_{x_B}^1 \frac{dx}{x} \frac{2\hat{x}}{(1-\hat{x})_+} h_1^a(x) \\
 & + \left(-\frac{1}{\epsilon} \right) f(x_B) \left(\int_{z_h}^1 \frac{dz}{z} \frac{2\hat{z}}{(1-\hat{z})_+} H_1^{\perp(1)a}(z) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{z - \frac{1}{z_1}}\right) \text{Im} \widehat{H}_{FU}^a(z, z_1) \left[-2\hat{z} C_F \frac{2}{z} P\left(\frac{1}{z - \frac{1}{z_1}}\right) \left(\frac{N_c}{2} (2 - \hat{z}) \right) \right. \\
 & \quad \left. \left. + \frac{1}{z} P\left(\frac{1}{z}\right) \left(-\frac{1}{N_c} (1 - \hat{z}) \right) + \frac{1}{z^2} P\left(\frac{1}{z - \frac{1}{z_1}}\right) P\left(\frac{1}{z_1 - \frac{1}{z_h}}\right) \left(-N_c (1 - \hat{z}) \right) \right] \right) \Big] + \dots
 \end{aligned}$$

Real emission diagrams in NLO cross section



$$\begin{aligned}
 d^4 \langle q_\perp \Delta \sigma \rangle^{\text{real}} \propto & \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_a \boxed{\frac{2}{\epsilon^2} C_F h_1^a(x_B) H_1^{\perp(1)a}(z_h)} \\
 & + \left(-\frac{1}{\epsilon} \right) C_F H_1^{\perp(1)a}(z_h) \int_{x_B}^1 \frac{dx}{x} \frac{2\hat{x}}{(1-\hat{x})_+} h_1^a(x) \\
 & + \left(-\frac{1}{\epsilon} \right) f(x_B) \left(\int_{z_h}^1 \frac{dz}{z} \frac{2\hat{z}}{(1-\hat{z})_+} H_1^{\perp(1)a}(z) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{z - \frac{1}{z_1}}\right) \text{Im} \widehat{H}_{FU}^a(z, z_1) \left[-2\hat{z} C_F \frac{2}{z} P\left(\frac{1}{z - \frac{1}{z_1}}\right) \left(\frac{N_c}{2} (2 - \hat{z}) \right) \right. \\
 & \quad \left. \left. + \frac{1}{z} P\left(\frac{1}{z}\right) \left(-\frac{1}{N_c} (1 - \hat{z}) \right) + \frac{1}{z^2} P\left(\frac{1}{z - \frac{1}{z_1}}\right) P\left(\frac{1}{z_1 - \frac{1}{z_h}}\right) \left(-N_c (1 - \hat{z}) \right) \right] \right] + \dots
 \end{aligned}$$

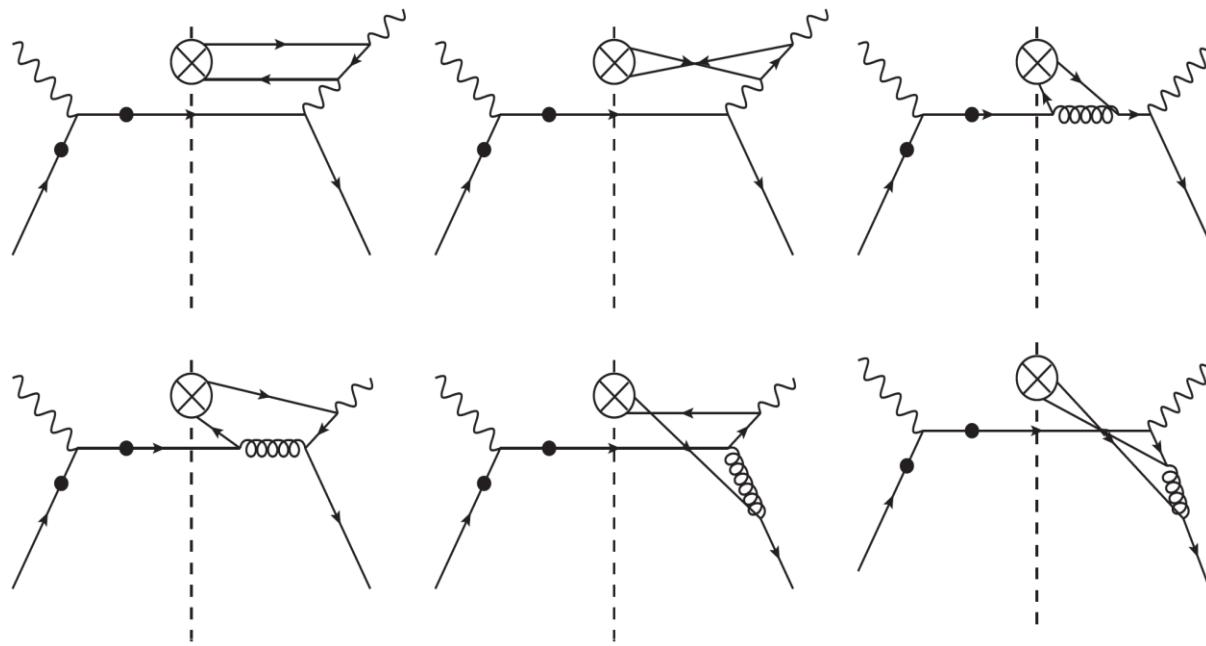
Real emission diagrams in NLO cross section



$$\begin{aligned}
 d^4 \langle q_\perp \Delta \sigma \rangle^{\text{real}} \propto & \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \sum_a \left[\frac{2}{\epsilon^2} C_F h_1^a(x_B) H_1^{\perp(1)a}(z_h) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) C_F H_1^{\perp(1)a}(z_h) \int_{x_B}^1 \frac{dx}{x} \frac{2\hat{x}}{(1-\hat{x})_+} h_1^a(x) \\
 & + \left(-\frac{1}{\epsilon} \right) f(x_B) \left(\int_{z_h}^1 \frac{dz}{z} \frac{2\hat{z}}{(1-\hat{z})_+} H_1^{\perp(1)a}(z) \right. \\
 & + \left(-\frac{1}{\epsilon} \right) \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{z}, \frac{1}{z_1}\right) \text{Im} \hat{H}_{FU}^a(z, z_1) \left[-2\hat{z} C_F \frac{2}{z} P\left(\frac{1}{z}, \frac{1}{z_1}\right) \left(\frac{N_c}{2} (2-\hat{z}) \right) \right. \\
 & \quad \left. \left. + \frac{1}{z} P\left(\frac{1}{z_1}\right) \left(-\frac{1}{N_c} (1-\hat{z}) \right) + \frac{1}{z^2} P\left(\frac{1}{z}, \frac{1}{z_1}\right) P\left(\frac{1}{z_1}, \frac{1}{z_h}\right) \left(-N_c (1-\hat{z}) \right) \right] \right] + \dots
 \end{aligned}$$

splitting function $\delta P_{qq}(x) = C_F \left[\frac{2x}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$

Real emission diagrams in NLO cross section



$$\begin{aligned}
 d^4 \langle q_\perp \Delta \sigma \rangle^{\text{real}} \propto & \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon} \right) \sum_a \left[\int_{z_h}^1 dz \int \frac{dz_1}{z_1^2} \text{Im} \tilde{H}_{FU}^a((1/z + 1/z_1)^{-1}, z_1) \right. \\
 & \times \left. \left\{ \frac{1}{z^2} P\left(\frac{1}{z}\right) P\left(\frac{1}{z_1} + \frac{1}{z} - \frac{1}{z_h}\right) \left(\frac{1}{N_c} \frac{(1-\hat{z})^2}{\hat{z}} \right) \right\} \right] + \dots
 \end{aligned}$$

Evolution equation for $H_1^{\perp(1)}(z)$

$$\begin{aligned}
\frac{\partial}{\partial \ln \mu^2} H_1^{\perp(1)}(z_h, \mu^2) = & \frac{\alpha_s}{2\pi} \left\{ \int_{z_h}^1 \frac{dz}{z} \delta P_{qq}(\hat{z}) H_1^{\perp(1)a}(z, \mu^2) \right. \\
& + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \text{Im} \hat{H}_{FU}^a(z, z_1, \mu^2) \left[-2\hat{z}C_F + \frac{2}{z} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \left(\frac{N_c}{2}(2-\hat{z})\right) \right. \\
& \quad \left. \left. + \frac{1}{z} P\left(\frac{1}{\frac{z}{z_h}}\right) \left(-\frac{1}{N_c}(1-\hat{z})\right) + \frac{1}{z^2} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \left(-N_c(1-\hat{z})\right) \right] \right. \\
& \quad \left. + \int_{z_h}^1 dz \int \frac{dz_1}{z_1^2} \text{Im} \tilde{H}_{FU}^a((1/z + 1/z_1)^{-1}, z_1 \mu^2) \left\{ \frac{1}{z_1^2} P\left(\frac{1}{\frac{1}{z_1}}\right) P\left(\frac{1}{\frac{1}{z_1} + \frac{1}{z} - \frac{1}{z_h}}\right) \left(\frac{1}{N_c} \frac{(1-\hat{z})^2}{\hat{z}}\right) \right\} \right)
\end{aligned}$$

$$\frac{\partial}{\partial \ln \mu^2} h(x_B, \mu^2) = \frac{\alpha_s}{2\pi} \int_{x_B}^1 \frac{dx}{x} \delta P_{qq}(\hat{x}) h(x, \mu^2)$$

Chiral-odd functions don't have gluon mixing

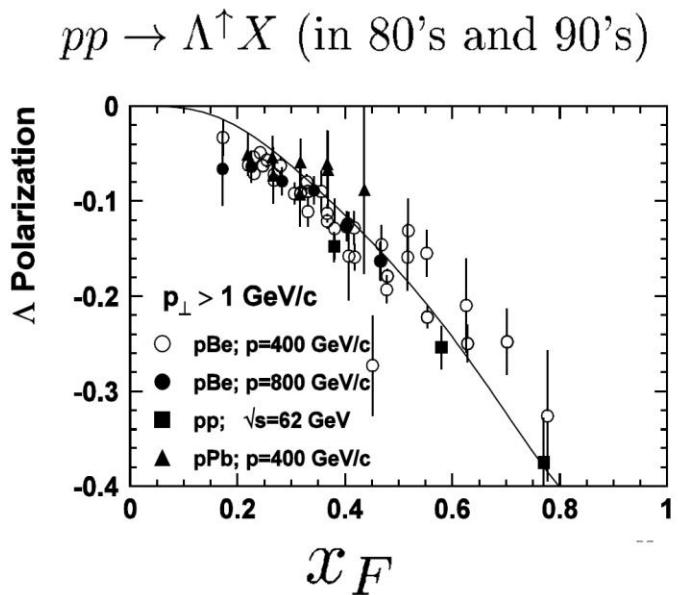
Evolution equation for $H_1^{\perp(1)}(z)$

$$\frac{\partial}{\partial \ln \mu^2} H_1^{\perp(1)}(z_h, \mu^2) = \frac{\alpha_s}{2\pi} \left\{ \int_{z_h}^1 \frac{d\hat{z}}{z} \delta P_{qq}(\hat{z}) H_1^{\perp(1)a}(z, \mu^2) \right. \\ + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \text{Im} \widehat{H}_{FU}^a(z, z_1, \mu^2) \left[-2\hat{z}C_F + \frac{2}{z} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \left(\frac{N_c}{2}(2-\hat{z})\right) \right. \\ \left. \left. + \frac{1}{z} P\left(\frac{1}{\frac{z}{z_h}}\right) \left(-\frac{1}{N_c}(1-\hat{z})\right) + \frac{1}{z^2} P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) P\left(\frac{1}{\frac{z}{z_h} - \frac{1}{z_1}}\right) \left(-N_c(1-\hat{z})\right) \right] \right. \\ \left. + \int_{z_h}^1 dz \int \frac{dz_1}{z_1^2} \text{Im} \widetilde{H}_{FU}^a((1/z + 1/z_1)^{-1}, z_1 \mu^2) \left\{ \frac{1}{z_1^2} P\left(\frac{1}{\frac{z}{z_h} + \frac{1}{z_1} - \frac{1}{z}}\right) \left(\frac{1}{N_c} \frac{(1-\hat{z})^2}{\hat{z}}\right) \right\} \right\}$$

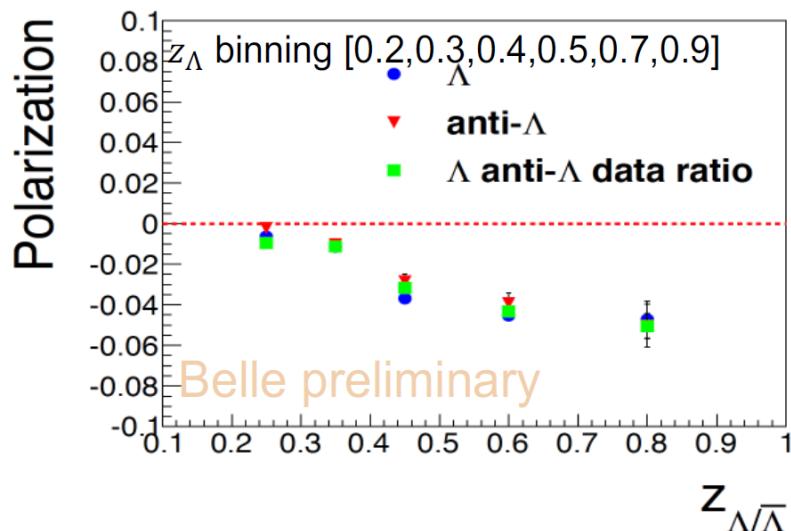
$$\frac{\partial}{\partial \ln \mu^2} h(x_B, \mu^2) = \frac{\alpha_s}{2\pi} \int_{x_B}^1 \frac{dx}{x} \delta P_{qq}(\hat{x}) h(x, \mu^2)$$

Chiral-odd functions don't have gluon mixing

SSA for Λ^\uparrow -production



Belle experiment reported a preliminary data for $e^+e^- \rightarrow \Lambda^\uparrow X$



Guan's talk@SPIN2016

Twist-3 fragmentation effect of Λ^\uparrow

K. Kanazawa et al., Phys. Rev. D93 (2016)

Intrinsic: $D_T(z)$ Kinematical: $D_{1T}^{\perp(1)}(z)$ $D_{1T}^{\perp(1)}(z) = z^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_h^2} D_{1T}^\perp(z, z^2 p_\perp^2)$

Dynamical: $\hat{D}_{FT}(z_1, z_2), \hat{G}_{FT}(z_1, z_2)$

Sivers-fragmentation function

$\tilde{D}_{FT}(z_1, z_2), \tilde{G}_{FT}(z_1, z_2)$

- EOM relation

$$\int \frac{dz_1}{z_1^2} P \frac{1}{1/z - 1/z_1} \left(\text{Im} \hat{D}_{FT}(z, z_1) - \text{Im} \hat{G}_{FT}(z, z_1) \right) = \frac{D_T(z)}{z} + D_{1T}^{\perp(1)}(z)$$

- Lorenz invariant relation

$$-\frac{2}{z} \int \frac{dz_1}{z_1^2} P \frac{1}{(1/z_1 - 1/z)^2} \text{Im} \hat{D}_{FT}(z, z_1) = \frac{D_T(z)}{z} + \frac{d}{d(1/z)} \frac{D_{1T}^{\perp(1)}(z)}{z}$$

Evolution equation for $D_{1T}^{\perp(1)}(z)$

$$\begin{aligned}
\frac{\partial}{\partial \ln \mu^2} D_{1T}^{\perp(1)}(z_h, \mu^2) = & \frac{\alpha_s}{2\pi} \left\{ \int_{z_h}^1 \frac{dz}{z} P_{qq}(\hat{z}) D_{1T}^{\perp(1)}(z) + C_F D_{1T}^{\perp(1)}(z_h) \right. \\
& + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P \frac{1}{(1/z_1 - 1/z)} \text{Im} \widehat{D}_{FT}(z, z_1) \left[-\frac{2}{z} P \frac{1}{1/z_1 - 1/z} \left(-\frac{1}{2N} \frac{2 - \hat{z} + \hat{z}^2}{2\hat{z}} - C_F \frac{1 + 3\hat{z}}{2} \right) \right. \\
& \quad \left. + \frac{1}{z} P \frac{1}{1/z_1} \left(-\frac{1}{2N} \frac{2 - \hat{z}}{\hat{z}} \right) + \frac{1}{z^2} P \frac{1}{(1/z_1 - 1/z)(1/z_1 - 1/z_h)} \left(-\frac{N}{2} \frac{1 - \hat{z}^2}{\hat{z}} \right) \right] \\
& + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} P \frac{1}{(1/z_1 - 1/z)} \text{Im} \widehat{G}_{FT}(z, z_1) \left[\frac{1}{z} P \frac{1}{1/z_1 - 1/z} \left(-\frac{N}{2} (1 - \hat{z}) \right) \right. \\
& \quad \left. - \frac{1}{2N} \frac{1}{z} P \frac{1}{1/z_1} + \frac{1}{z^2} P \frac{1}{(1/z_1 - 1/z)(1/z_1 - 1/z_h)} \left(\frac{N}{2} \frac{(1 - \hat{z})^2}{\hat{z}} \right) \right] \\
& + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} \text{Im} \widetilde{D}_{FT}((\frac{1}{z} + \frac{1}{z_1})^{-1}, z_1) \left[P \frac{1}{1/z_1} \left(-\frac{1}{2N} \frac{1 - \hat{z} + (1 - \hat{z})^2}{\hat{z}} \right) \right. \\
& \quad \left. + P \frac{1}{1/z_1(1/z_1 - 1/z_h + 1/z)} \left(-\frac{1}{2N} \frac{1 - \hat{z} + (1 - \hat{z})^2}{\hat{z}} \right) \right] \\
& + \int_{z_h}^1 \frac{dz}{z} \int \frac{dz_1}{z_1^2} \text{Im} \widetilde{G}_{FT}((\frac{1}{z} + \frac{1}{z_1})^{-1}, z_1) \left[\left(P \frac{1}{1/z_1} \right) \left(-\frac{1}{2N} (1 - \hat{z}) \right) \right. \\
& \quad \left. + \frac{1}{z} P \frac{1}{1/z_1(1/z_1 - 1/z_h + 1/z)} \left(-\frac{1}{2N} (1 - \hat{z}) \right) \right] \} .
\end{aligned}$$

$D_{1T}^{\perp(1)}$ has gluon mixing

Summary

$$\circ p^\dagger p \rightarrow \pi X$$

- Evolution equations were derived both for the distribution part and the fragmentation part.
- Evolution equations for QS function match between two different methods.
- Evolution equations for kinematical twist-3 function have not matched yet between two different methods.

$$\circ pp \rightarrow \Lambda^\dagger X$$

- Evolution equation for kinematical twist-3 function was derived.
- 3-gluon fragmentation effect has to be included for flavor singlet evolution.

A complete set of the functions EOM relation LIR relation

Backup

Twist-3 fragmentation functions

K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak and M. schlegel, Phys. Rev. D93 (2016)

- Intrinsic twist-3 functions

$$\frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{z}} \langle 0 | \psi_i(0) | h(P_h) X \rangle \langle h(P_z) X | \bar{\psi}_j(\lambda w) | 0 \rangle = \frac{M_N}{2z} (i\gamma_5 \sigma_{\lambda\alpha})_{ij} \epsilon^{\lambda\alpha w P_h} H(z) + \dots,$$

- Kinematical twist-3 function

$$\begin{aligned} & \int d^2 k_T k_T^\rho \left[\frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d^2 \xi_T}{(2\pi)^2} e^{-i\frac{\lambda}{z} - i\xi_T \cdot k_T} \langle 0 | [\pm\infty w, 0] \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}(\lambda w + \xi_T) \pm \infty w + \xi_T | 0 \rangle \right] \\ &= -\frac{M_N}{z} (i\gamma_5 \not{P}_h \gamma_\lambda)_{ij} \epsilon^{\lambda\alpha w P_h} H_1^{\perp(1)}(z) \quad H_1^{\perp(1)}(z) = z^2 \int d^2 p_\perp \frac{\vec{p}_\perp^2}{2M_h^2} \underline{H_1^\perp(z, z^2 p_\perp^2)} \end{aligned}$$

Collins function

- Dynamical twist-3 functions

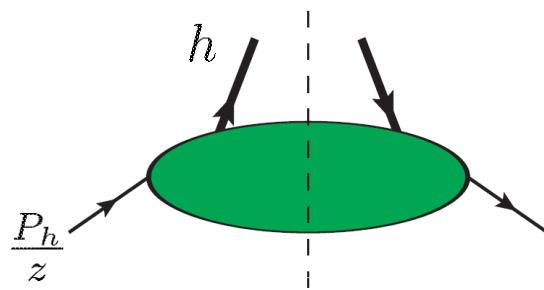
$$\begin{aligned} & \frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | g F^{\alpha w}(\mu w) \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}_j(\lambda w) | 0 \rangle \\ &= \frac{M_N}{z_2} (\gamma_5 \not{P}_h \gamma_\lambda)_{ij} \epsilon^{\lambda\alpha w P_h} \hat{H}_{FU}(z_1, z_2) + \dots, \\ & \frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | g F^{\alpha w}(\mu w) \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}_j(\lambda w) | 0 \rangle \\ &= \frac{M_N}{z_2} (\gamma_5 \not{P}_h \gamma_\lambda)_{ij} \epsilon^{\lambda\alpha w P_h} \hat{H}_{FU}(z_1, z_2) + \dots, \end{aligned}$$

Twist-3 fragmentation functions

K. Kanazawa, Y. Koike, A. Metz, D. Pitonyak and M. schlegel, Phys. Rev. D93 (2016)

- Intrinsic twist-3 functions

$$\frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{z}} \langle 0 | \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}_j(\lambda w) | 0 \rangle$$



- Dynamical twist-3 functions

$$\frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | g F^{\alpha w}(\mu w) \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}_j(\lambda w) | 0 \rangle$$

$$\frac{1}{N_c} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d\mu}{2\pi} e^{-i\frac{\lambda}{z_1}} e^{-i\mu(\frac{1}{z_2} - \frac{1}{z_1})} \langle 0 | \bar{\psi}_j(\lambda w) \psi_i(0) | P_h X \rangle \langle P_h X | g F^{\alpha w}(\mu w) | 0 \rangle$$

- Kinematical twist-3 function

$$\int d^2 k_T k_T^\rho \left[\frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} \int \frac{d^2 \xi_T}{(2\pi)^2} e^{-i\frac{\lambda}{z} - i\xi_T \cdot k_T} \langle 0 | [\pm \infty w, 0] \psi_i(0) | P_h X \rangle \langle P_h X | \bar{\psi}(\lambda w + \xi_T) \pm \infty w + \xi_T | 0 \rangle \right]$$

