Are solenoids more robust against errors?

Trying to understand whether one of the arguments in favor of solenoids is well-founded.

Positioning error

Solenoid entrance (hard-edge) transfer matrix[a] :

$$M_{in}(K_s) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K_s/2 & 0 \\ 0 & 0 & 1 & 0 \\ +K_s/2 & 0 & 0 & 1 \end{pmatrix}$$

with

$$K_s \equiv \frac{e}{p} B_s$$

The exit transfer matrix is $M_{out} = M_{in}(-K_s)$.

[a] similar to a thin *skew* quadrupole but $M_{41} = -M_{23}$

Trajectory kick due to a horizontal and vertical offset δ_x and δ_y

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K_s/2 & 0 \\ 0 & 0 & 1 & 0 \\ +K_s/2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\delta_x \\ 0 \\ -\delta_y \\ 0 \end{pmatrix} = \begin{pmatrix} -\delta_x \\ -K_s\delta_y/2 \\ -\delta_y \\ K_s\delta_x/2 \end{pmatrix}$$

That is $\Delta x' = -K_s\delta_y/2$ and $\Delta y' = K_s\delta_x/2$. If the solenoid is very short so that $\Delta x = \Delta y = 0$, the kicks at the exit and entrance *cancel* each other.

The kick due to a misaligned quadrupole is (thin lens)

$$\Delta x' = K\ell\delta_x \quad \text{and} \quad \Delta y' = K\ell\delta_y$$

Angle error

The solenoid itself introduce a large coupling, but a roll angle around $\hat{s}$ has no consequences for the solenoid, while it produces spurious linear $x - y$ coupling for the quadrupole.

Solenoid transfer matrix

$$M = M_{out} M_c M_{in}$$

with

$$M_c = \begin{pmatrix} 1 & \frac{\sin\chi}{K_s} & 0 & -\frac{(1-\cos\chi)}{K_s} \\ 0 & \cos\chi & 0 & -\sin\chi \\ 0 & \frac{(1-\cos\chi)}{K_s} & 1 & \frac{\sin\chi}{K_s} \\ 0 & \sin\chi & 0 & \cos\chi \end{pmatrix}$$

with

$$K_s \equiv \frac{e}{p} B_s \qquad \chi \equiv K_s \ell_s$$

The effect of the entrance of a solenoid tilted by $-\delta'_x$ and $-\delta'_y$ is [a]

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 = M_{in} \begin{pmatrix} 0 \\ \delta'_x \\ 0 \\ \delta'_y \end{pmatrix} = \begin{pmatrix} 0 \\ \delta'_x \\ 0 \\ \delta'_y \end{pmatrix}$$

[a] assuming no offset at the entrance

In addition, the effect of the center is

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = M_c \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 = \begin{pmatrix} \frac{\sin\chi}{K_s}\delta'_x - \frac{(1-\cos\chi)}{K_s}\delta'_y \\ \cos\chi\delta'_x - \sin\chi\delta'_y \\ \frac{(1-\cos\chi)}{K_s}\delta'_x + \frac{\sin\chi}{K_s}\delta'_y \\ \sin\chi\delta'_x + \cos\chi\delta'_y \end{pmatrix}$$

Finally, the effect of the whole solenoid is

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_3 = M_{out} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = \begin{pmatrix} \frac{\sin\chi}{K_s}\delta'_x - \frac{(1-\cos\chi)}{K_s}\delta'_y \\ \frac{1}{2}\delta'_x + \frac{1}{2}\cos\chi\delta'_x - \frac{1}{2}\sin\chi\delta'_y \\ \frac{(1-\cos\chi)}{K_s}\delta'_x + \frac{\sin\chi}{K_s}\delta'_y \\ \frac{1}{2}\delta'_y + \frac{1}{2}\cos\chi\delta'_y + \frac{1}{2}\sin\chi\delta'_x \end{pmatrix}$$

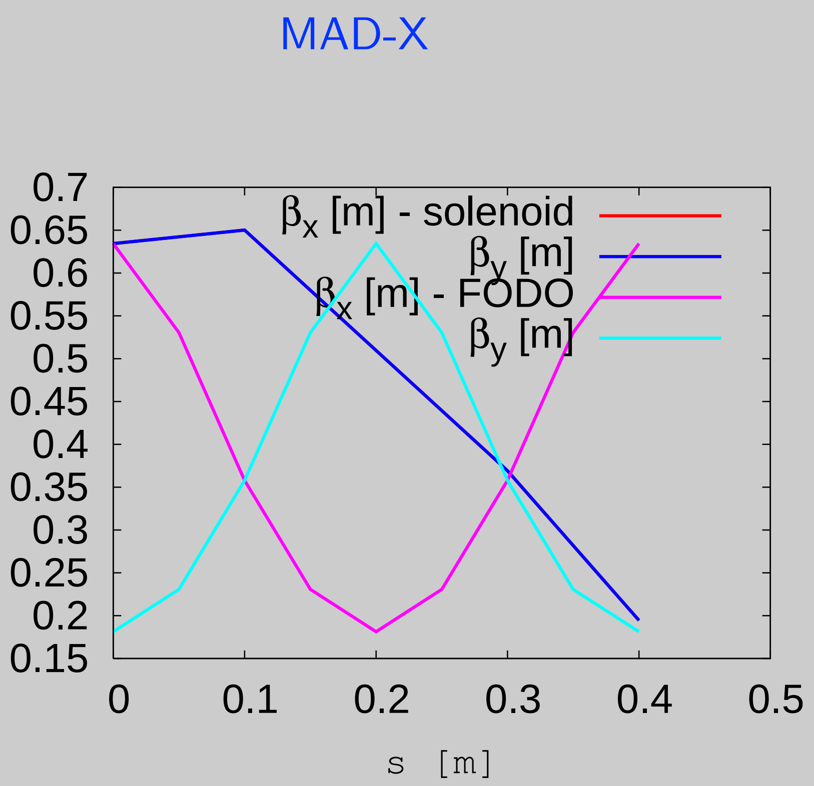
In the limit $\chi \equiv K_s \ell_s \longrightarrow 0$ keeping the 1th order terms

$$\Delta x' = -\frac{1}{2}K_s \ell_s \delta'_y \quad \text{and} \quad \Delta y' = \frac{1}{2}K_s \ell_s \delta'_x$$

A tilt angle around $\hat{x}$ or $\hat{y}$ produce no kick for the quadrupole, in thin lens approximation.

As the size of the effects depend upon the strength, it is fair to compare for an *equivalent* focusing effect. This means that a *couple* of quadrupoles must be compared to one solenoid. For instance:

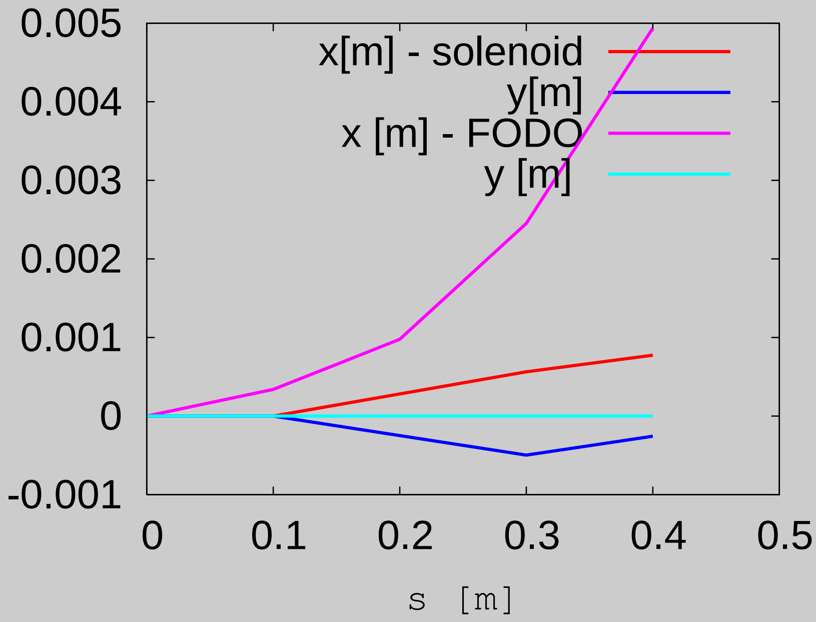
- 2 quads, ℓ_q=0.1 m $K_q = \pm$72.1 m^{-2}
- one solenoid ℓ_s=0.2 m and $K_s =$ 8.5 m^{-1}



Position errors:

- $\delta_x^{q1} = -\delta_x^{q1} = 1$ mm
- $\delta_x^{s} = 1$ mm

MAD-X



Roll errors:

- $\delta_\phi^{q1} = -\delta_\phi^{q1} = 1$ mrad (rotation around $\hat{x}$)
- $\delta_\phi^{s} = 1$ mrad

MAD-X