Analytic Forms for an Adiabatic Tapered Solenoid

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To permit more rapid variation in modeling of the front end of the Muon Collider/Neutrino Factory, the axial magnetic field (of the axially symmetric capture/transport system) could be specified analytically. Then, the off-axis magnetic field could be approximated via the series expansions

$$B_z(r,z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n}, \tag{1}$$

and

$$B_r(r,z) = \sum_n (-1)^{n+1} \frac{a_0^{(2n+1)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1}, \tag{2}$$

where

$$a_0^{(n)} = \frac{d^n a_0}{dz^n} = \frac{d^n B_z(0,z)}{dz^n}, \tag{3}$$

as derived in the Appendix.

The magnetic field should vary "slowly", such that the quantity

$$\Phi = \pi \rho^2 B_z = \frac{\pi c^2 p_\perp^2}{e^2 B_z} \tag{4}$$

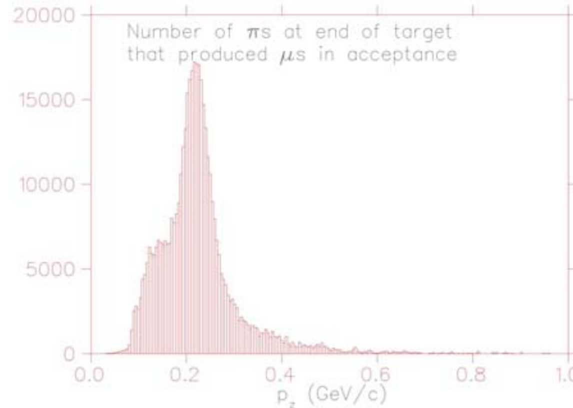
is an adiabatic invariant, where

$$\rho = \frac{p_\perp c}{e B_z} = \frac{p_\perp\,[\mathrm{MeV}/c]}{3 B_z\,[\mathrm{T}]}\ \mathrm{cm} \tag{5}$$

is the radius of the helical trajectory of a particle (muon or pion) of charge e and transverse momentum $p_\perp$, B_z is average axial magnetic field inside the helix, c is the speed of light, and the last form of eq. (4) holds in Gaussian units. By "slowly", we mean that the relative change in B_z is small over one turn of the helical trajectory, whose period λ in z is

$$\lambda = \frac{2\pi p_z c}{e B_z} = \frac{2.1 p_z\,[\mathrm{MeV}/c]}{B_z\,[\mathrm{T}]}\ \mathrm{cm}, \tag{6}$$

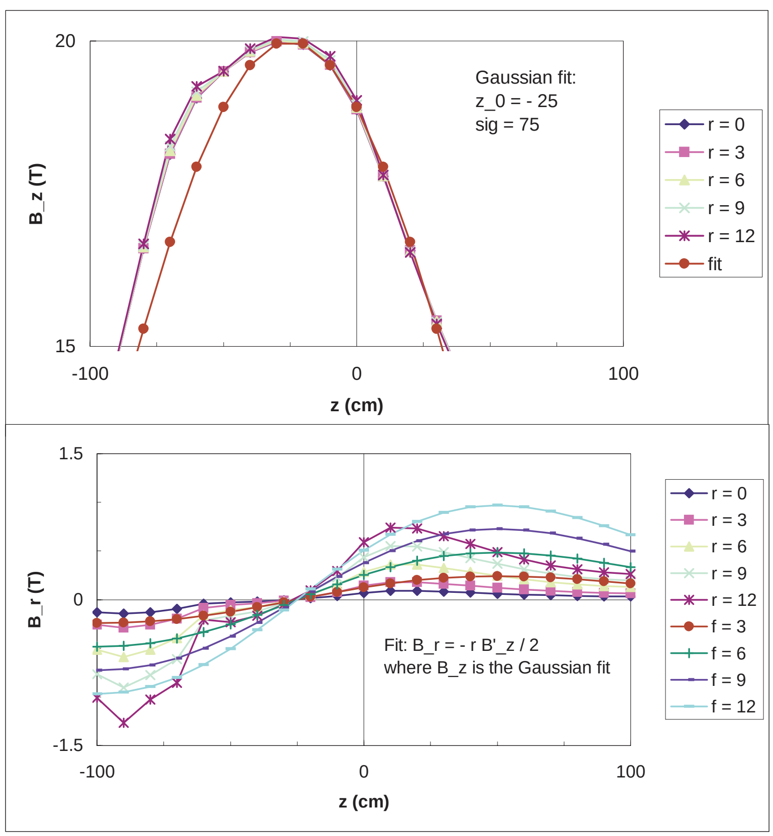
where p_z is axial components of the particles momentum. For example, in the so-called Study 2 Neutrino Factory design [2] we desire to capture pions with $p_z \gtrsim 100$ MeV/c, as shown in the figure on the next page (which is a projection of Fig. 4b of [3]). In this case we are interested in trajectories with period λ as short as ≈ 2 m / B_z [T]. Then, for $B_z \approx 20$ T, the axial field should not vary significantly over $\Delta z = 10$ cm, while for $B_z \approx 2$ T the variation in field should be small over a meter. These criteria are readily satisfied in our designs, such that the quantity (4) will be an adiabatic invariant to a good approximation.



1 Gaussian Approximation to the Field Near $z = 0$

It has become the convention in simulations of the Muon Collider/Neutrino Factory front end to define $z = 0$ to be the downbeam "end" of the pion-production target. The MARS code is used to simulate pion production in this target (whose position is approximately -60 cm $< z < 0$) and transport these pions to the plane $z = 0$, after which they are modeled using ICOOL or other codes.

The magnetic field near $z = 0$ used in MARS simulations of the Study-2 front end is based on a table with $\Delta r = 3$ cm and $\Delta z = 10$ cm, as shown in the figures below.[1]



[1] http://www.hep.princeton.edu/~mcdonald/mumu/target/taper.xls

The axial field near $z = 0$ is well approximated by the Gaussian form,

$$B_z(r, z) = B_0 e^{-(z-z_0)^2/2\sigma^2}, \tag{7}$$

with $B_0 = 20$ T, $z_0 = -25$ cm and $\sigma = 75$ cm.

Then, the radial field near $z = 0$ is well approximated by the first-order form (33)

$$B_r(r, z) \approx -\frac{r}{2}\frac{\partial B_z(0, z)}{\partial z}, \tag{8}$$

with the derivative based on the Gaussian form (7).

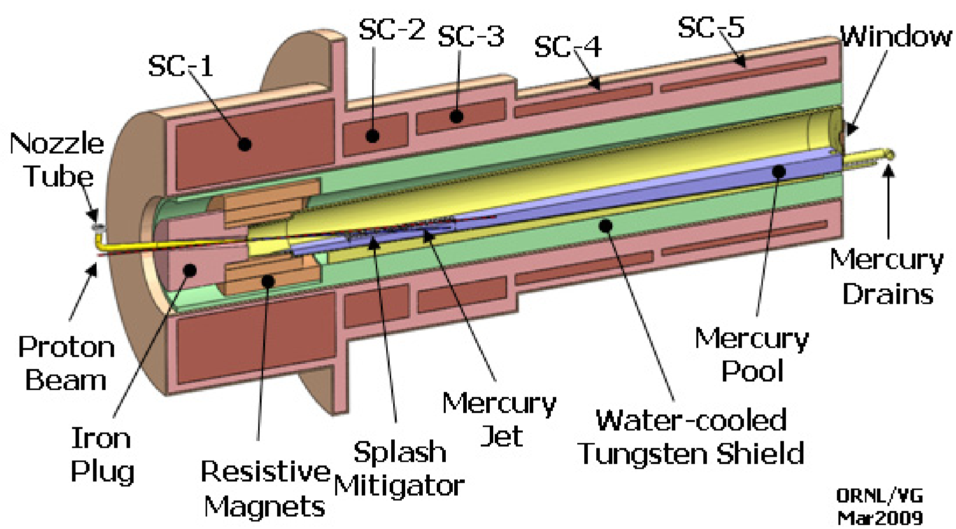
Near $z = 25$ cm the second derivative of the axial field goes to zero, and the Gaussian approximation misestimates the radial field. However, we will use the Gaussian form only at $z = 0$ to match to better analytic forms for the field at $z > 0$.[2]

2 Analytic Forms for an Adiabatic Taper

Here, we consider analytic forms for the axial field, $B_z(0, z)$, that makes a transition from a high axial field B_1 for $z < z_1$ to an approximately constant, weaker axial field B_2 for $z > z_2$. In principle, the axial field and all derivatives (3) should be continuous at z_1 and z_2, such that if the field outside the interval $[z_1, z_2]$ were truly uniform, then all derivatives of B_z would vanish at z_1 and z_2. In a previous study [1] of the front-end system not even the first derivative of the axial field was continuous at z_1 and z_2.

The analytic forms will be based on functions of polynomials. A polynomial of odd order $2n - 1$ has $2n$ constants, which can be chosen to match the axial field, and its first $n - 1$ derivatives at the two points z_1 and z_2. For present purposes it will likely be sufficient to match only the field and its first derivative, so we focus on cubic polynomials. Should matching of higher derivatives be desired later the method can be readily extended to do so.

[2]The radial field used in the Study 2 simulation is slightly nonzero at r = 0, due to the presence of off-axis holes in the upstream iron "plug".



We restrict our attention to the case that the field is constant for $z \geq z_2$, although the forms given here could be generalized to include nonzero derivatives at z_2.

2.1 Cubic Taper

The simplest use of a cubic polynomial approximation to the axial field on the interval $[z_1, z_2]$ can be written as

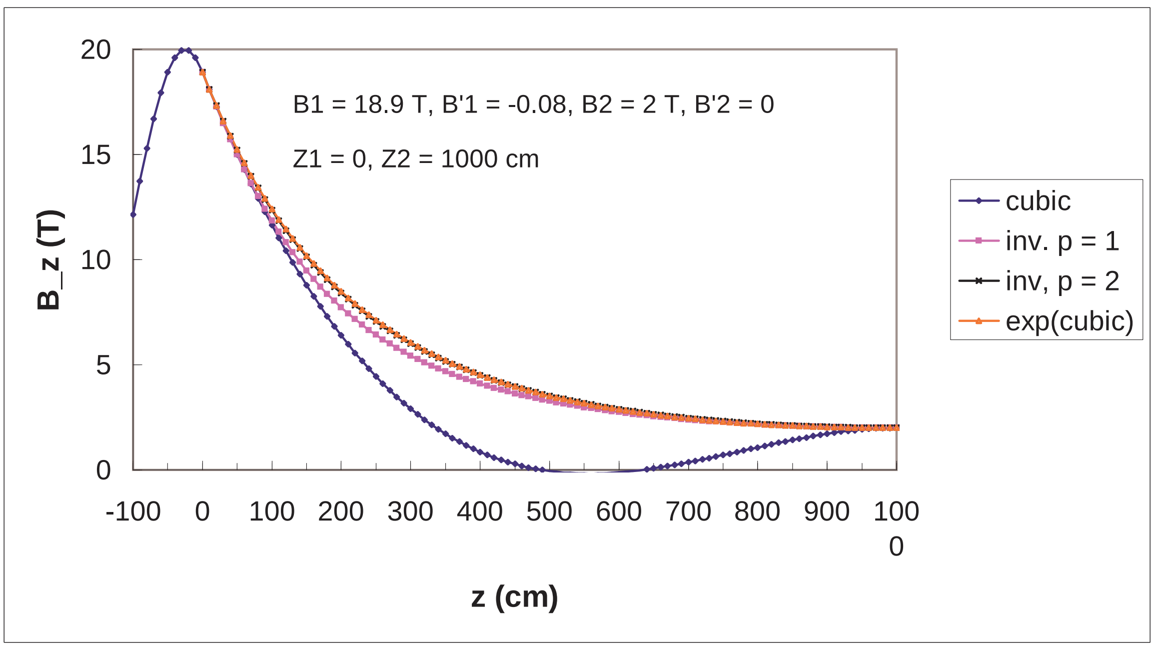
$$B_z(0, z_1 < z < z_2) = B_1 + B_1'(z - z_1) + a_2(z - z_1)^2 + a_3(z - z_1)^3, \tag{9}$$

where the first two coefficients have be chose to match the field $B_1 = B_z(0, z_1)$ and its first derivative $B_1' = \partial B_z(0, z_1)/\partial z$ at z_1 at z_1. Then, the polynomial (9) will also match the field B_2 and its derivative B_2' at z_2 by setting

$$a_2 = 3\frac{B_1 - B_2}{(z_2 - z_1)^2} - \frac{2B_1'}{z_2 - z_1}, \qquad \text{and} \qquad a_3 = 2\frac{B_1 - B_2}{(z_2 - z_1)^3} + \frac{B_1'}{(z_2 - z_1)^2}. \tag{10}$$

The characteristic length of the taper is not an independent parameter, but it can be adjusted by varying z_2, the position at which the taper ends.

As an example, the plot below shows the taper according to eq. (9) that matches to the Gaussian form (7) for $z \leq 0$ and which goes to $B_2 = 2$ T at $Z_2 = 10$ m (where the slope B_2' is zero. While the curve satisfies the specified conditions at z_1 and z_2 the axial field is not monotonically decreasing, and actually goes negative for $5 < z < 6$ m. Hence, we consider other forms for the taper, based on functions of cubic polynomials



2.2 Inverse-Cubic Taper

Kevin Paul [1] suggests use of an inverse-cubic form,

$$B_z(0, z_1 < z < z_2) = \frac{B_1}{[1 + a_1(z - z_1) + a_2(z - z_1)^2 + a_3(z - z_1)^3]^p}, \tag{11}$$

for any nonzero power p. This form matches to B_1 and B_1' at z_1 with the choice[3]

$$a_1 = -\frac{B_1'}{pB_1}\,. \tag{12}$$

Matching to B_2 and $B_2' = 0$ at position z_2 determines the coefficients a_2 and a_3 to be

$$B_2 = \frac{B_1}{[1 + a_1(z_2 - z_1) + a_2(z_2 - z_1)^2 + a_3(z_2 - z_1)^3]^p}\,, \tag{13}$$

and

$$0 = a_1 + 2a_2(z_2 - z_1) + 3a_3(z_2 - z_1)^2, \tag{14}$$

which imply that

$$a_2 = 3\frac{(B_1/B_2)^{1/p} - 1}{(z_2 - z_1)^2} - \frac{2a_1}{z_2 - z_1}\,, \qquad \text{and} \qquad a_3 = -2\frac{(B_1/B_2)^{1/p} - 1}{(z_2 - z_1)^3} + \frac{a_1}{(z_2 - z_1)^2}\,. \tag{15}$$

The inverse-cubic tapers with $p = 1$ and 2 for the example of sec. 2.1 is shown in the figure on the previous page. The results are rather satisfactory.

2.3 Exponential Taper

Another function of a cubic polynomial that yields a satisfactory taper is an exponential, A simple form of an adiabatic taper of a characteristic length Δ in z is a Gaussian form.

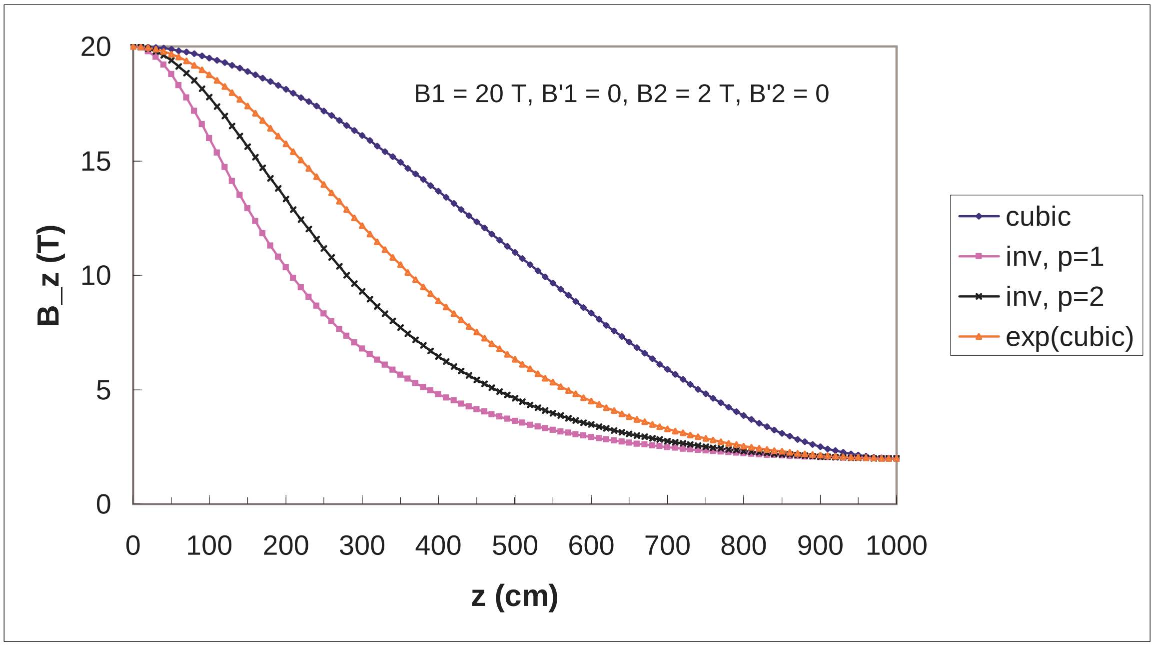
$$B_z(0, z_1 < z < z_2) = B_1 e^{a_1(z-z_1)+a_2(z-z_1)^2+a_3(z-z_3)^3}, \tag{16}$$

where matching to B_1' at z_1 and to B_2 and $B_2' = 0$ at z_2 determines the coefficients to be

$$a_1 = \frac{B_1'}{B_1}\,, \quad a_2 = \frac{3\ln(B_2/B_1)}{(z_2 - z_1)^2} - \frac{2a_1}{z_2 - z_1}\,, \quad \text{and} \quad a_3 = -\frac{2\ln(B_2/B_1)}{(z_2 - z_1)^3} + \frac{a_1}{(z_2 - z_1)^2}\,. \tag{17}$$

In the example on the previous page, the exponential of a cubic polynomial yields a curve that differs only slightly from the inverse polynomial with $p = 1$ and is essentially identical to that for $p = 2$. A greater difference between the inverse cubics and exponential of a cubic is seen in an example where both B_1' and B_2' are zero, as shown in the figure on the next page. There, the inverse cubic leads to a more rapid taper than does the exponential form, and all of these are more rapid that the symmetric taper based on basic cubic form (9).

[3]Actually, parameter a_1 was set to zero in many implementations of the form (11), so the slope B' was not continuous at z_1 in those studies.



A Appendix: Expansion of an Axially Symmetric Magnetic Field in Terms of Its Axial Field

Suppose a magnetic field in a current-free region is rotationally symmetric about the z-axis. Then,

$$\mathbf{B} = B_r(r,z)\hat{\mathbf{r}} + B_z(r,z)\hat{\mathbf{z}} \tag{18}$$

in cylindrical coordinates. If we write

$$B_z(r,z) = \sum_{n=0}^{\infty} a_n(z) r^n, \qquad \text{and} \qquad B_r(r,z) = \sum_{n=0}^{\infty} b_n(z) r^n \ , \tag{19}$$

then $a_0(z) = B_z(0,z)$. Since the divergence of the magnetic field vanishes, the proposed expansions (19) obey

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \frac{1}{r}\frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} = \sum_n \left[(n+1)b_n r^{n-1} + a_n^{(1)} r^n\right] = 0, \tag{20}$$

where $a^{(m)}(z) \equiv d^m a/dz^m$. For this to be true at all r, the coefficients of r^n must separately vanish for all n. Hence,

$$b_0 = 0, \tag{21}$$

$$b_n = -\frac{a_{n-1}^{(1)}}{n+1} \ . \tag{22}$$

Since the curl of the magnetic field also vanishes (outside the source currents),

$$(\boldsymbol{\nabla}\times\mathbf{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \sum_n \left(b_n^{(1)} r^n - n a_n r^{n-1}\right) = 0 \ , \tag{23}$$

Again, the coefficient of r^n must vanish for all n, so that

$$b_n^{(1)} = (n+1)a_{n+1}. \tag{24}$$

Using eq. (24) in eq. (22), we find

$$b_n = -\frac{b_{n-2}^{(2)}}{(n+1)(n+3)}\,. \tag{25}$$

Since b_0 vanishes, b_{2n} vanishes for all n, and from eq. (24), a_{2n+1} vanishes for all n. Then, using eq. (25) in eq. (24), we find

$$a_{2n} = -\frac{a_{2n-2}^{(2)}}{4n^2}\,. \tag{26}$$

Repeatedly applying this to itself gives

$$a_{2n} = (-1)^n \frac{a_0^{(2n)}}{2^{2n}(n!)^2}\,. \tag{27}$$

Inserting this in eq. (22), we get

$$b_{2n+1} = (-1)^{n+1} \frac{a_0^{(2n+1)}}{2^{2n+1}(n+1)(n!)^2}\,. \tag{28}$$

Combining eqs. (27)-(28) with eq. (19), we arrive at the desired forms,

$$B_z(r,z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n}, \tag{29}$$

and

$$B_r(r,z) = \sum_n (-1)^{n+1} \frac{a_0^{(2n+1)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1}, \tag{30}$$

for the field components, where

$$a_0^{(n)} = \frac{d^n a_0}{dz^n}\,. \tag{31}$$

These results are overly detailed for some purposes. If one is interested only in the leading behavior at small r, then eqs. (29)-(30) simplify to

$$B_z(r,z) \approx B_z(0,z), \qquad B_r(r,z) \approx -\frac{r}{2}\frac{\partial B_z(0,z)}{\partial z}\,. \tag{32}$$

The result for B_r also follows quickly from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, according to eq. (20),

$$B_r(r,z) = -\int_0^r r\frac{\partial B_z(r,z)}{\partial z} dr \approx -\int_0^r r\frac{\partial B_z(0,z)}{\partial z} dr = -\frac{r}{2}\frac{\partial B_z(0,z)}{\partial z}\,. \tag{33}$$

It is also instructive that the approximation (33) can be deduced quickly from the integral form of Gauss' law (without the need to recall the form of $\boldsymbol{\nabla} \cdot \mathbf{B}$ in cylindrical coordinates). Consider a Gaussian pillbox of radius r and thickness dz centered on $(r = 0, z)$. Then,

$$\begin{aligned} 0 &= \int \mathbf{B} \cdot d\mathbf{S} \approx \pi r^2 [B_z(0, z + dz) - B_z(0,z)] + 2\pi r\ dz\ B_r(r,z) \\ &\approx \pi r^2\ dz\ \frac{\partial B_z(0,z)}{\partial z} + 2\pi r\ dz\ B_r(r,z)\,, \end{aligned} \tag{34}$$

which again implies eqs. (32).

References

[1] K. Paul and C. Johnstone, *Optimizing the Pion Capture and Decay Channel*, MUC0289 (9 Feb. 2004),
http://www.hep.princeton.edu/~mcdonald/mumu/target/Paul/paul_muc0289.pdf

[2] S. Ozaki, R. Palmer and M. Zisman, eds., *Feasibility Study-II of a Muon-Based Neutrino Source*, BNL-52623 (2001), http://www.cap.bnl.gov/mumu/studyii/FS2-report.html

[3] J. Gallardo *et al.*, *30-T-on-Target Neutrino Factory/Muon Collider Front End* (Dec. 18, 2009),
http://www.hep.princeton.edu/~mcdonald/mumu/target/Gallardo/front-end-paper.pdf