3-Loop Static QCD Potential: Computation and Applications

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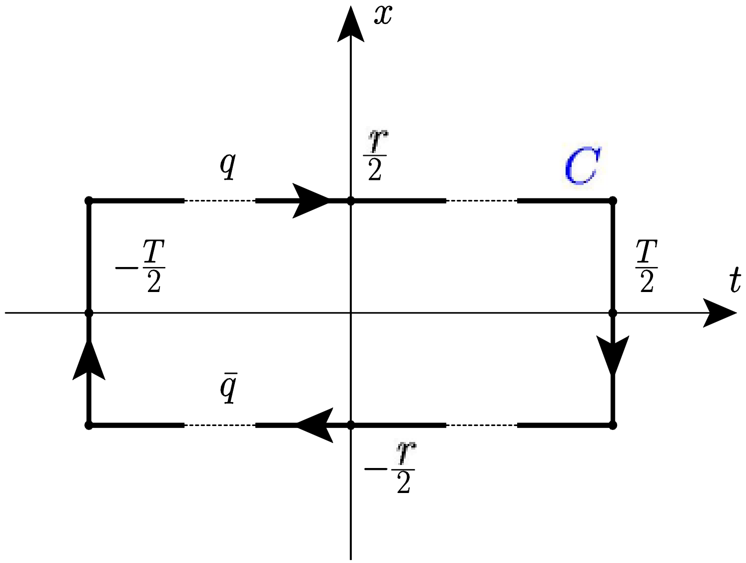
Static QCD potential

Defined from Wilson loop:

$$\left\langle \mathrm{Tr}\,\mathrm{P}\exp\left[ig_s \oint_C dx^\mu A^a_\mu(x) T^a_F\right]\right\rangle$$

$$\sim \exp[-iV_{\rm QCD}(r)T] \quad \text{as} \quad T\to\infty$$



Flow of pert. computation of $V_{\rm QCD}(r)$ [in mom. space]

(1) Diagram generation (GRACE/QGRAF)

(2) Elimination of iterations of lower-order graphs

(3) Reduction of integrals to master integrals

(4) Evaluation of master integrals

(5) Renormalization ($\overline{\rm MS}$ scheme)

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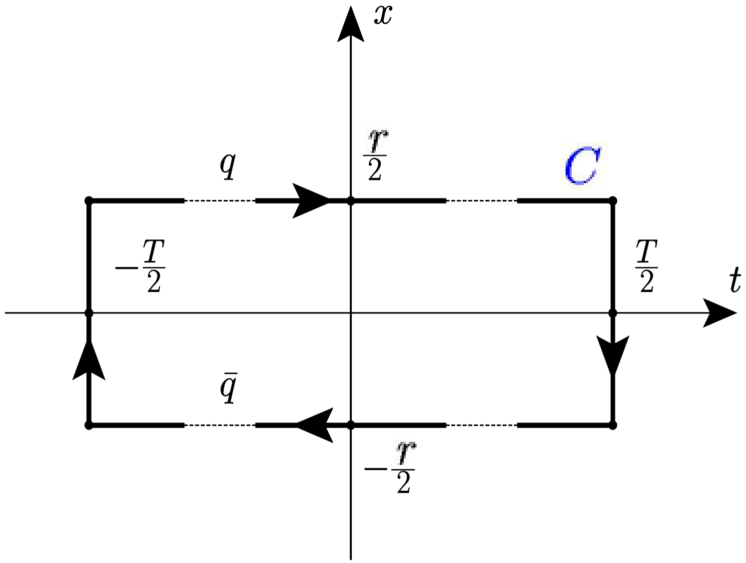
$$\left\langle \mathrm{Tr}\,\mathrm{P}\exp\left[i g_s \oint_C dx^\mu A_\mu^a(x) T_F^a \right] \right\rangle$$

$$\sim \exp[-i V_{\mathrm{QCD}}(r) T] \quad \text{as} \quad T \to \infty$$



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difficult

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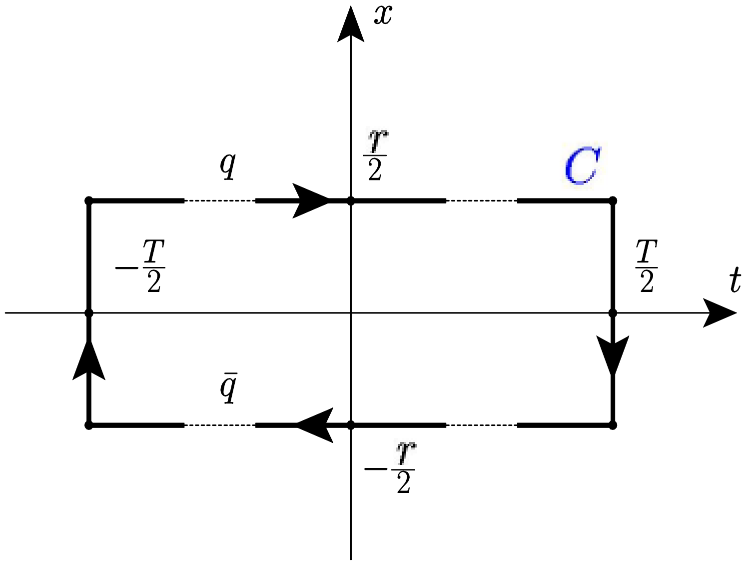

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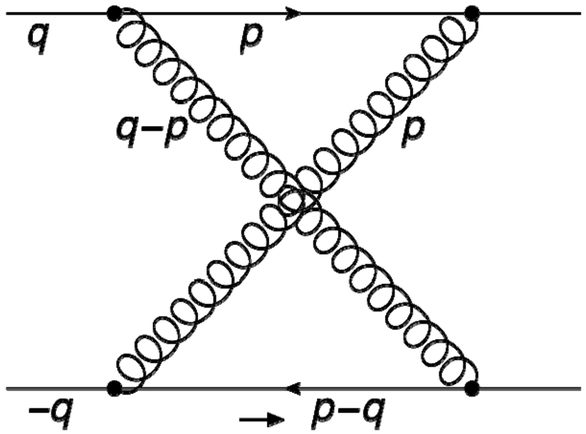
Standard form of loop integrals



$$\left.\begin{array}{l} q^\mu = (0, \vec{q}) \\ v^\mu = (1, \vec{0}) \end{array}\right\} \quad q \cdot v = 0$$

$$\int d^D p \, \frac{1}{[p \cdot v][(p-q) \cdot v][p^2][(q-p)^2]}$$

$$= \int d^D p \, \frac{1}{[p \cdot v]^2 [p^2][(p+q)^2]}$$

$$= J(2,1,1)$$

Express each diagram in terms of standard integrals J

$$J(n_1, \cdots, n_N) \equiv \int d^D p_1 \cdots d^D p_L \, \frac{1}{D_1^{n_1} \cdots D_N^{n_N}}$$

NB: n_i is negative, when representing a numerator.

Each J can be represented by a lattice site in N-dim. space

1 loop $\{D_1, D_2, D_3\} = \{p \cdot v, p^2, (p+q)^2\}$

2 loop $\{D_1, \cdots, D_9\} = \{p_1 \cdot v, p_2 \cdot v, (p_1 + p_2) \cdot v, p_1^2, \cdots\}$

3 loop $\{D_1, \cdots, D_{21}\}$

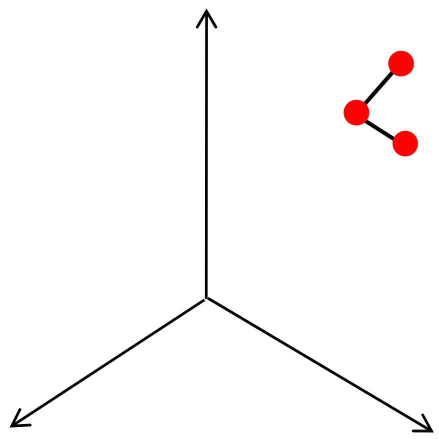
Integration-by-parts (IBP) Identities

Chetyrkin, Tkachov

In dim. reg.

$$0 = \int d^D p_1 \cdots d^D p_L \, \frac{\partial}{\partial X_\mu} \left(\frac{Y_\mu}{D_1^{n_1} \cdots D_N^{n_N}} \right)$$

$$X \in \{p_1, \cdots, p_L\}, \quad Y \in \{q, v, p_1, \cdots, p_L\}$$



Ex. $X = Y = p$ at 1-loop:

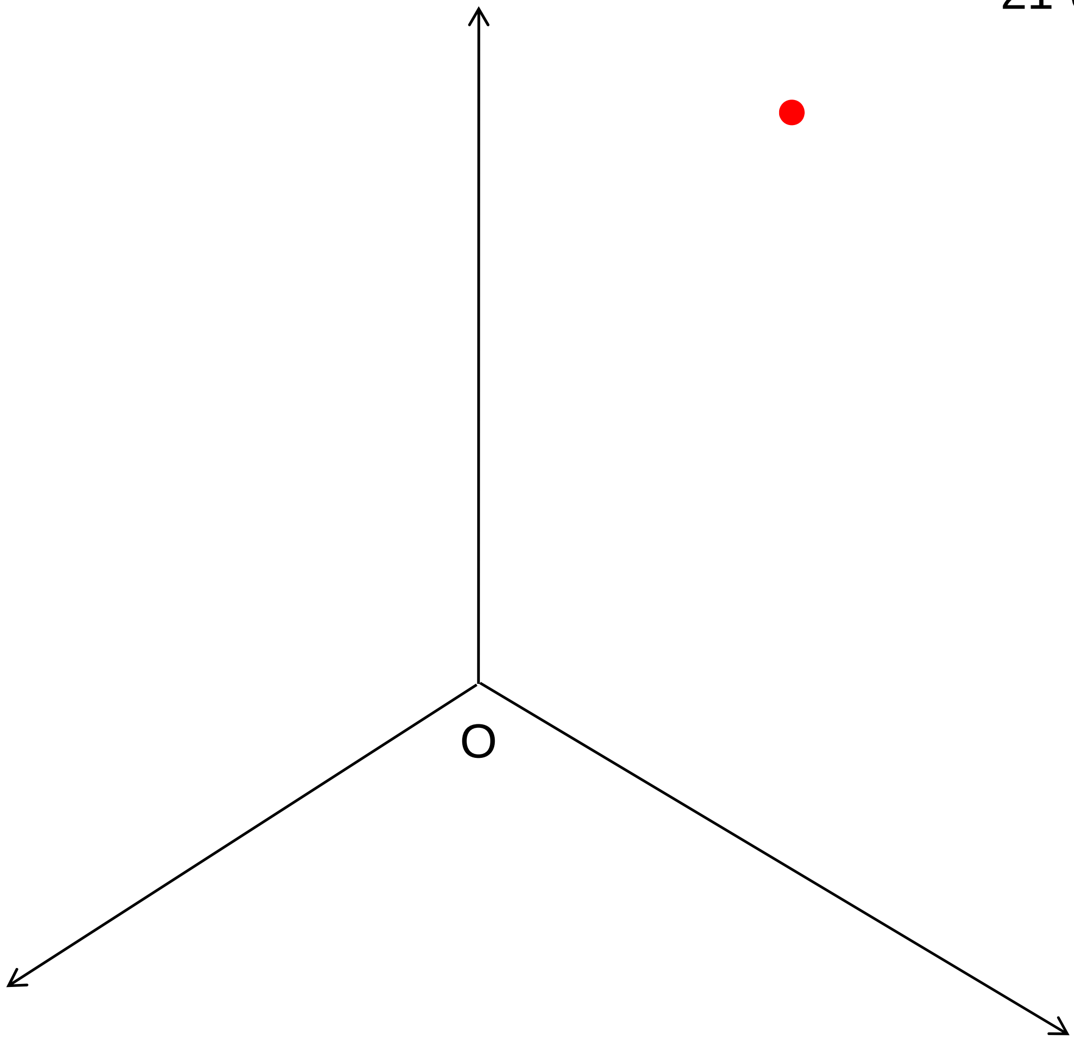
$$0 = \int d^D p \, \frac{\partial}{\partial p_\mu} \frac{p_\mu}{[p\cdot v]^a [p^2]^b [(p+q)^2]^c}$$

$$= \int d^D p \, \frac{1}{[p\cdot v]^a [p^2]^b [(p+q)^2]^c} \left\{ D - a\frac{p\cdot v}{[p\cdot v]} - b\frac{2p^2}{[p^2]} - c\frac{2p\cdot(p+q)}{[(p+q)^2]^c} \right\}$$

$$= (D - a - 2b - c) J(a,b,c) - cJ(a, b-1, c+1) + cq^2 J(a, b, c+1)$$

Reduction by Laporta algorithm

(3-loop)
21-dim. space



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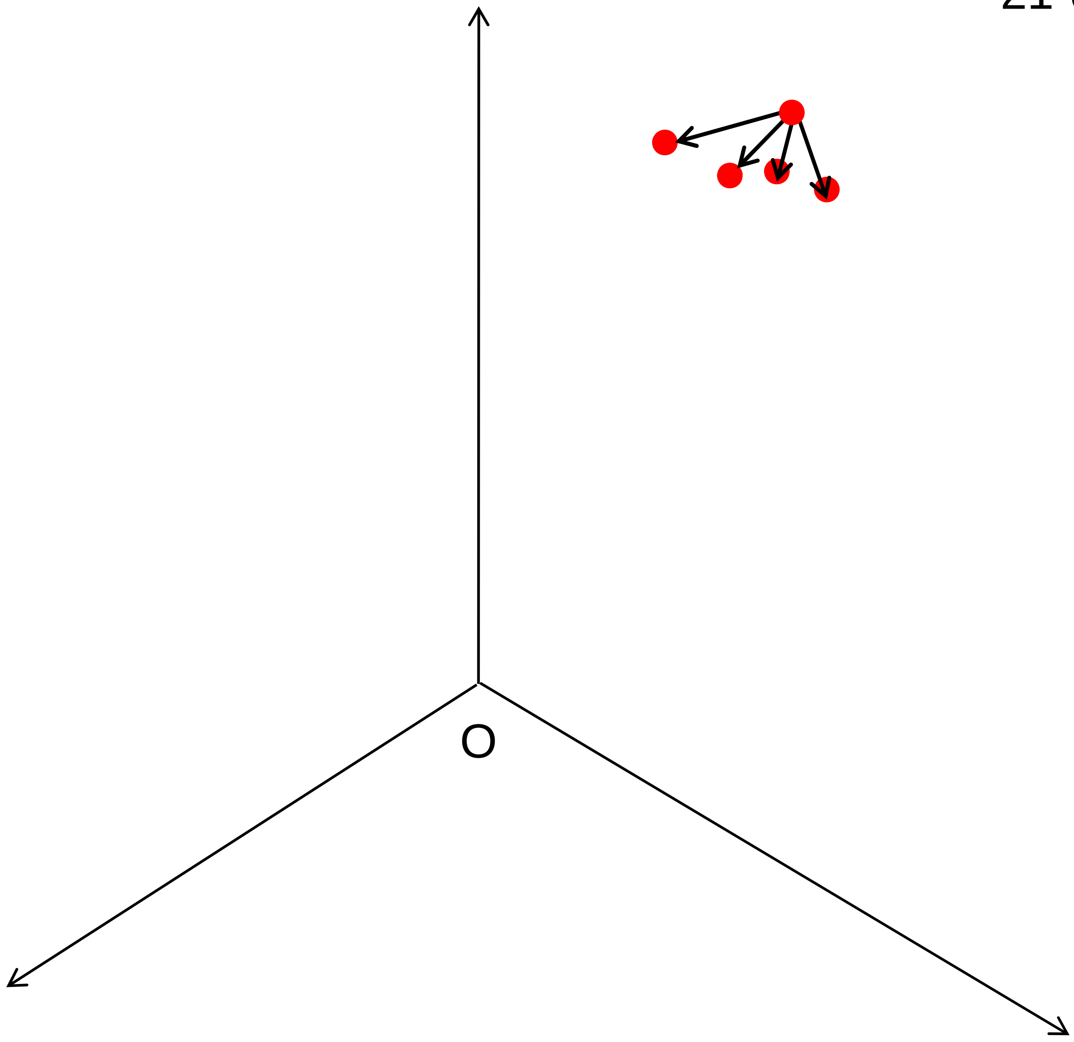

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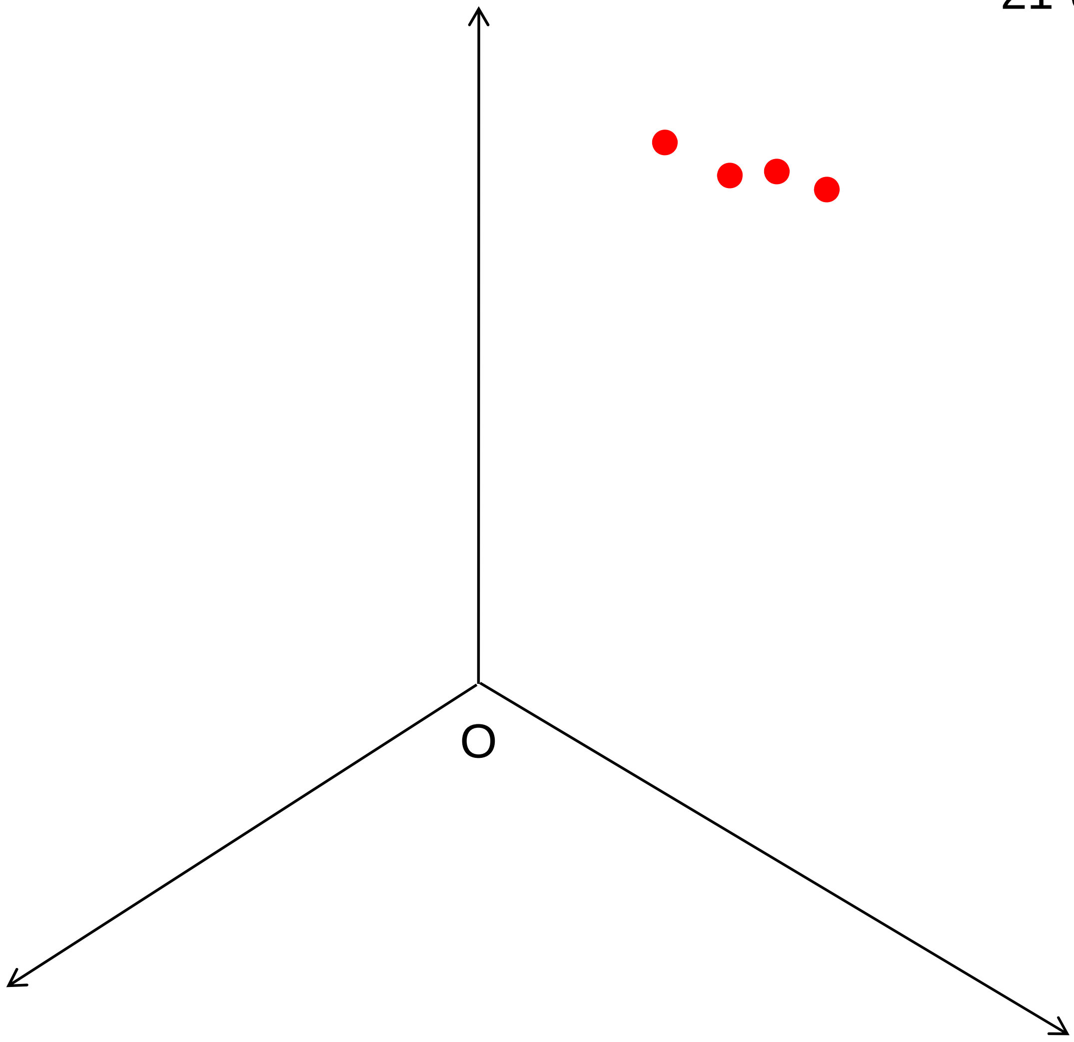

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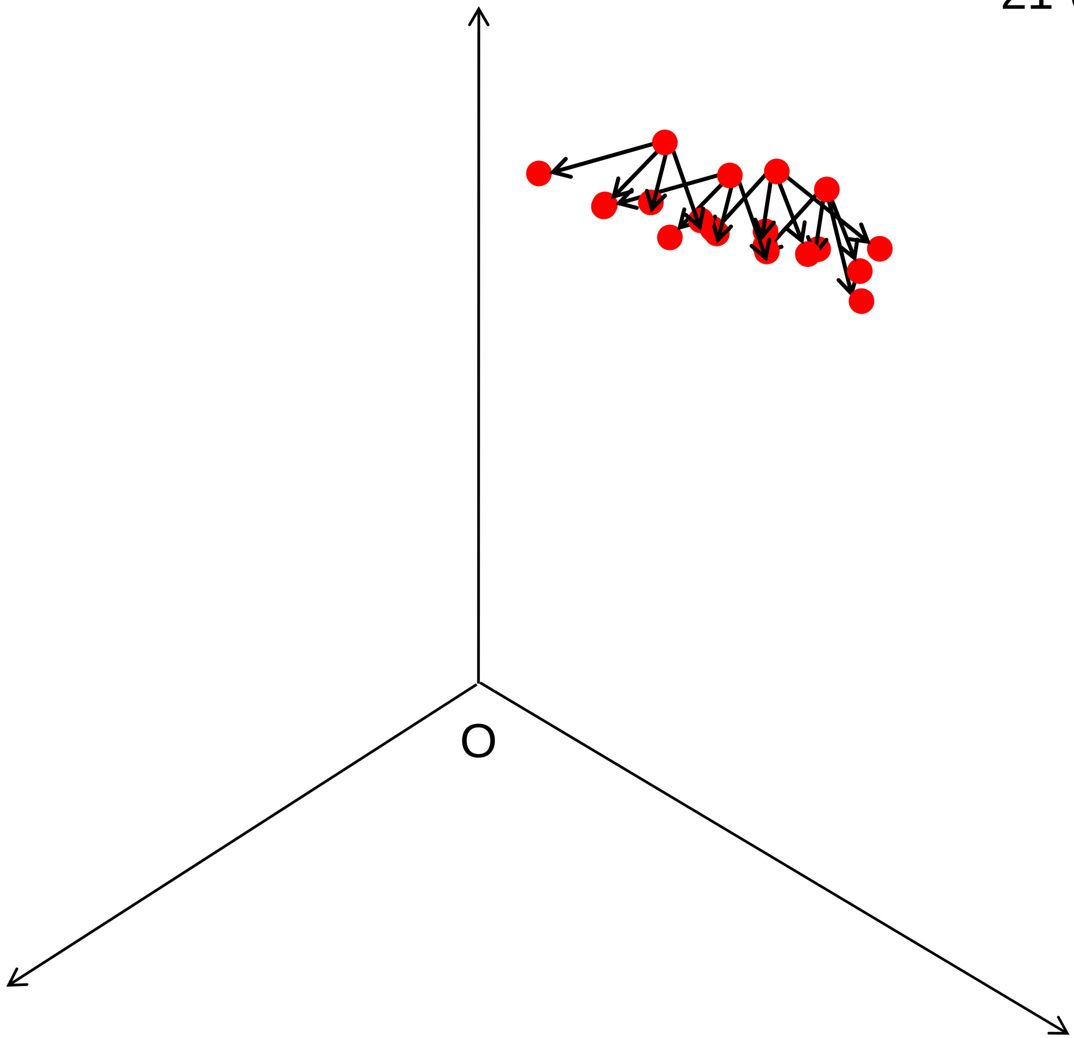

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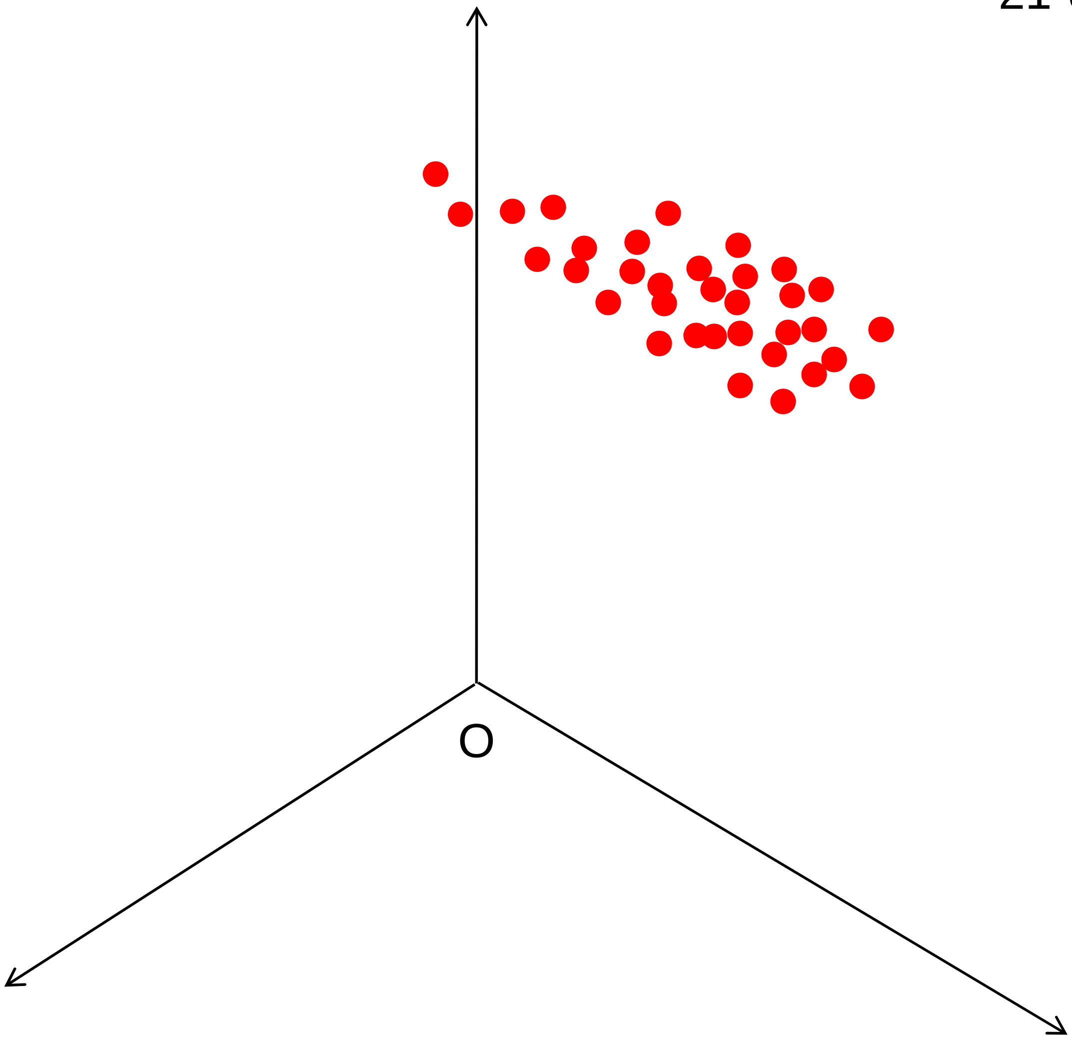

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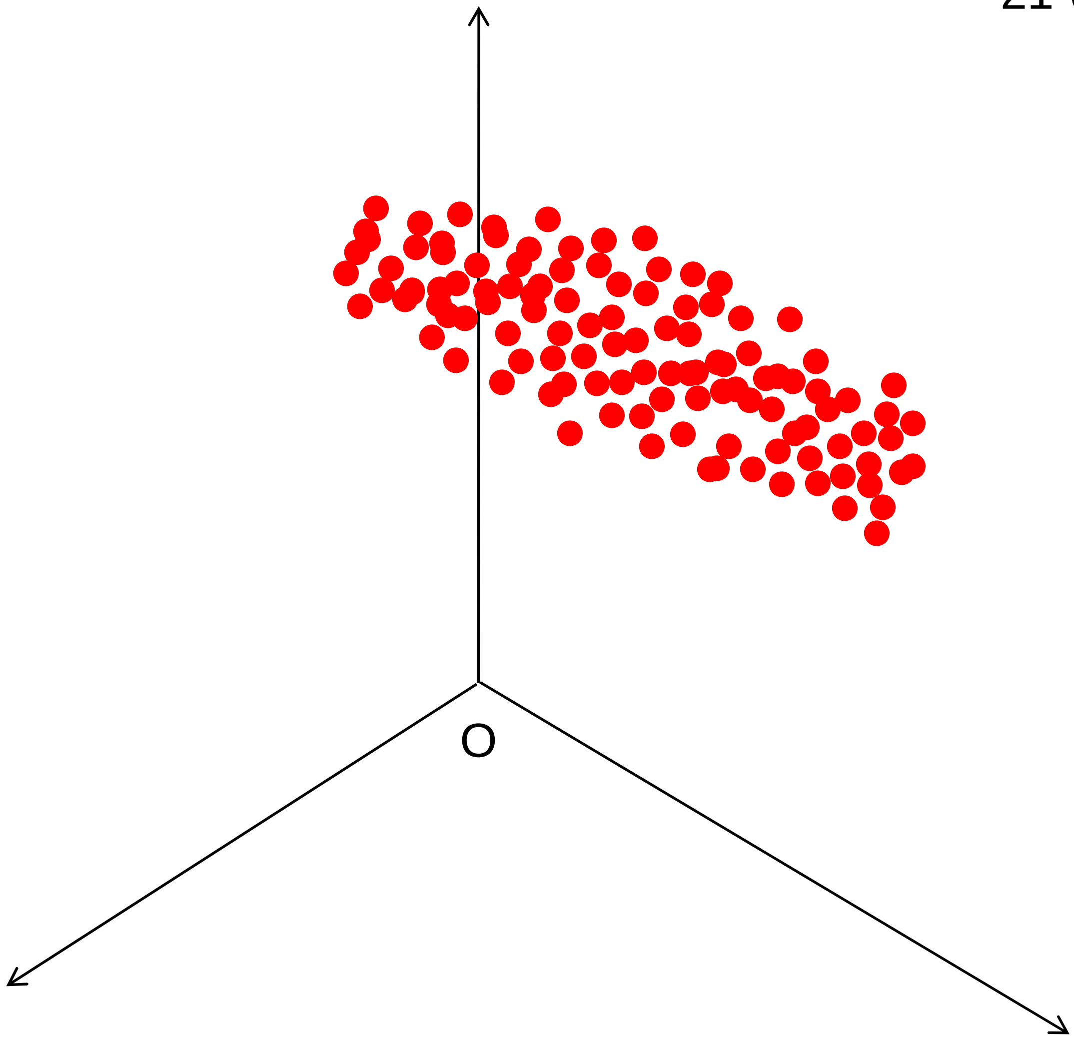

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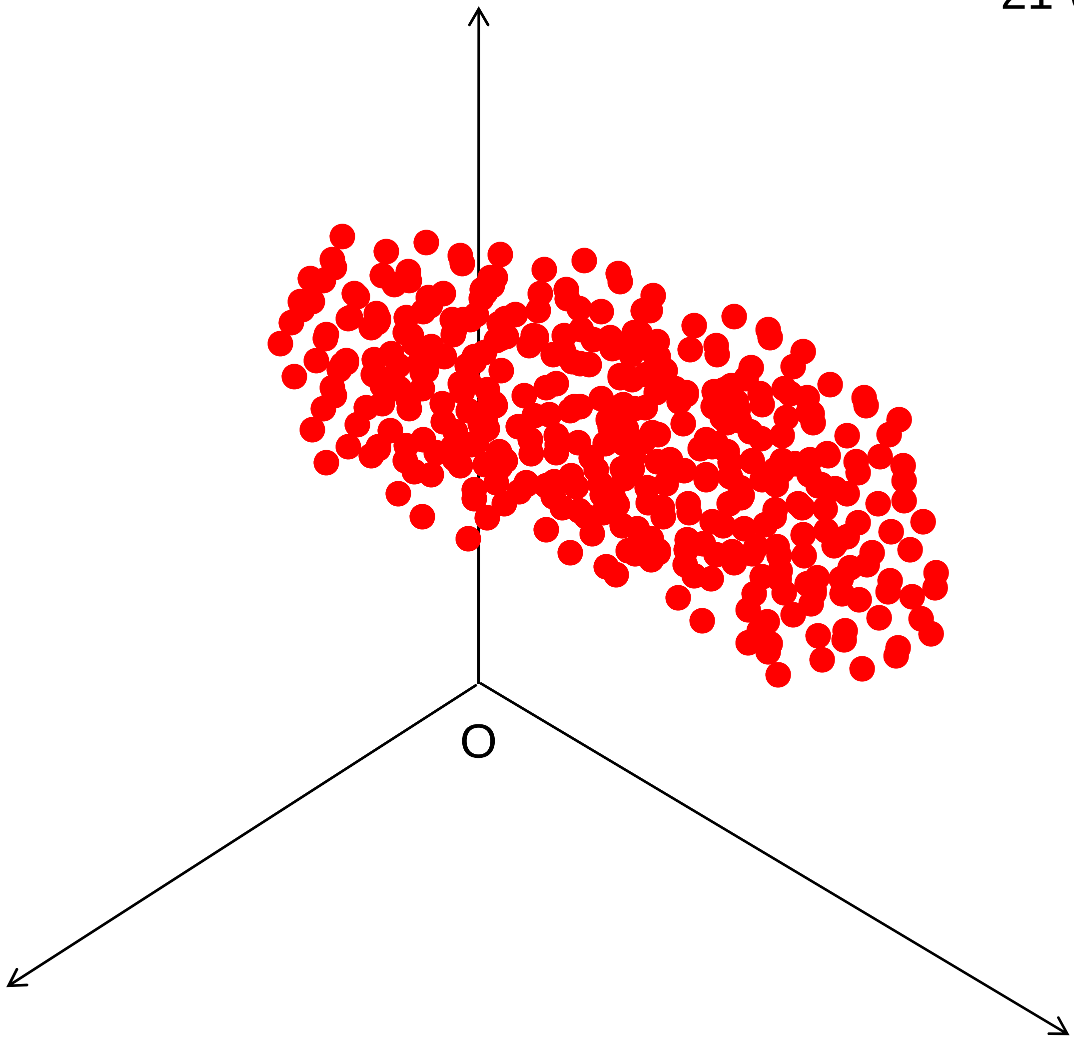

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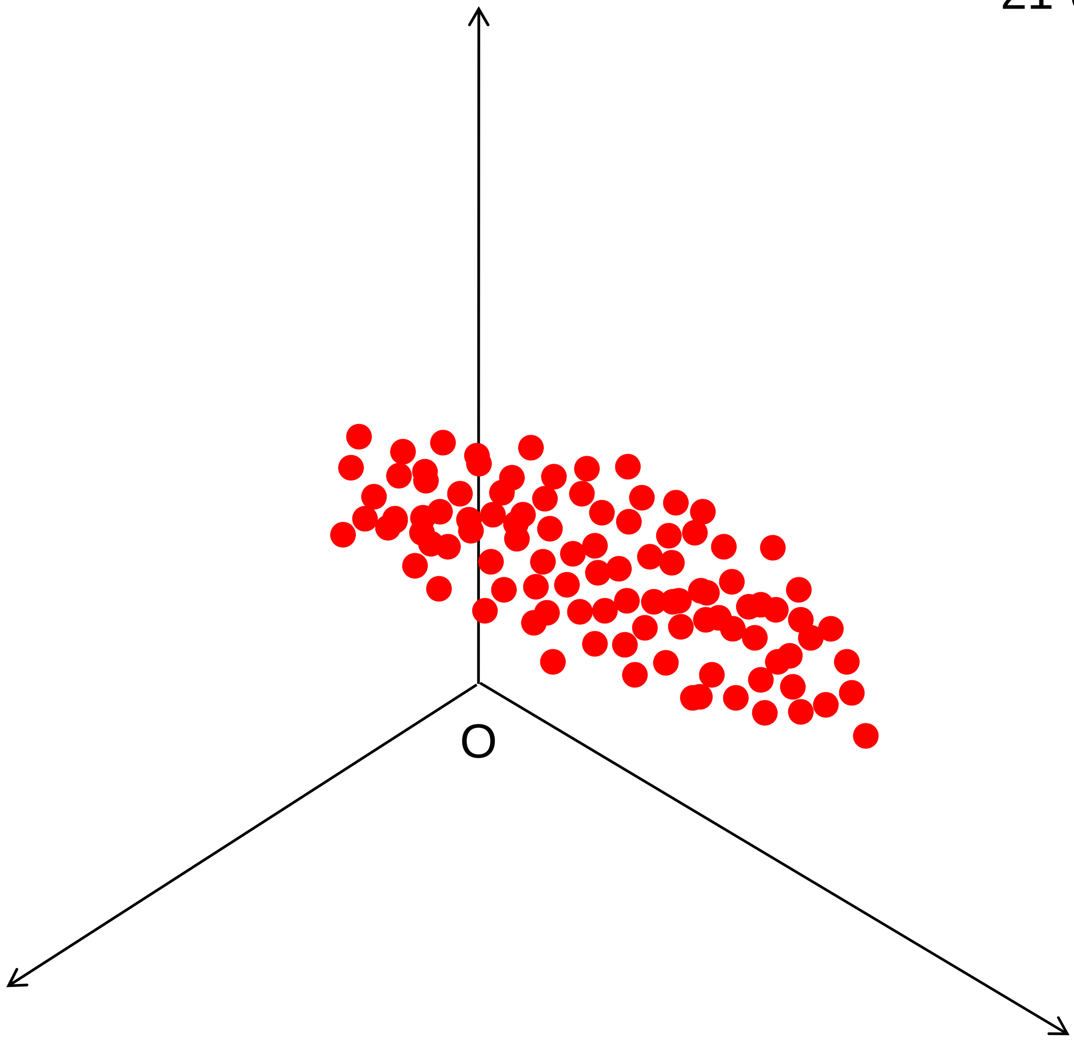

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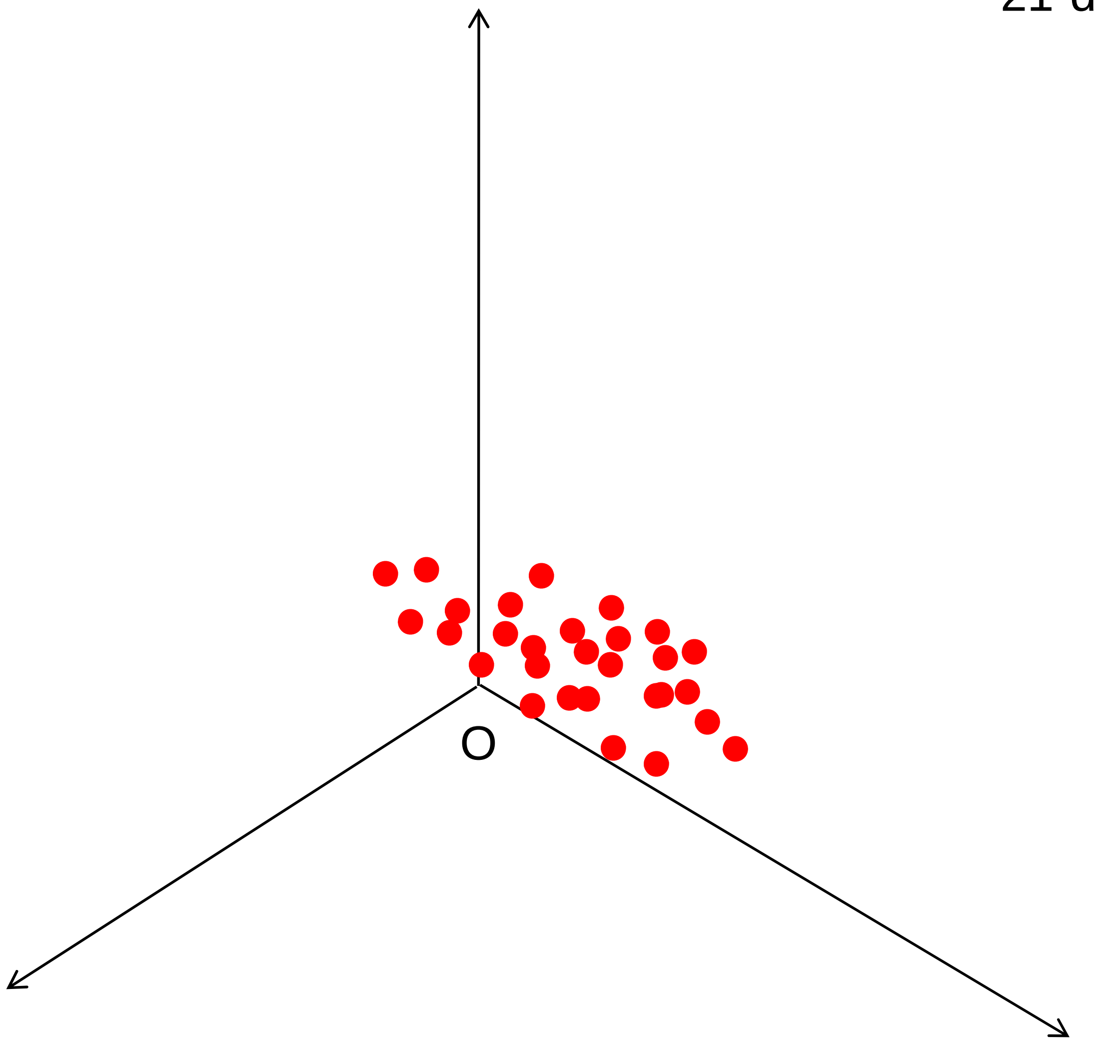

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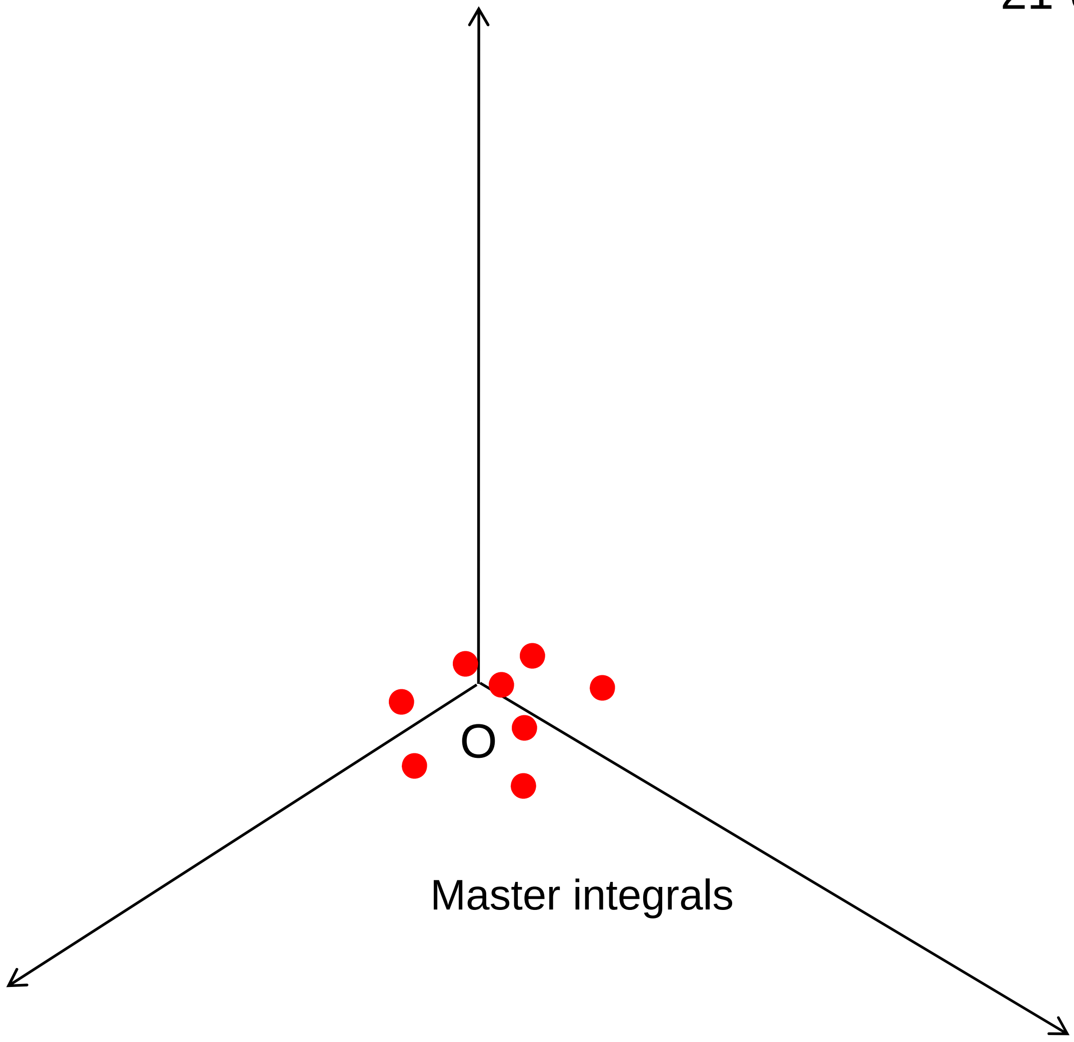
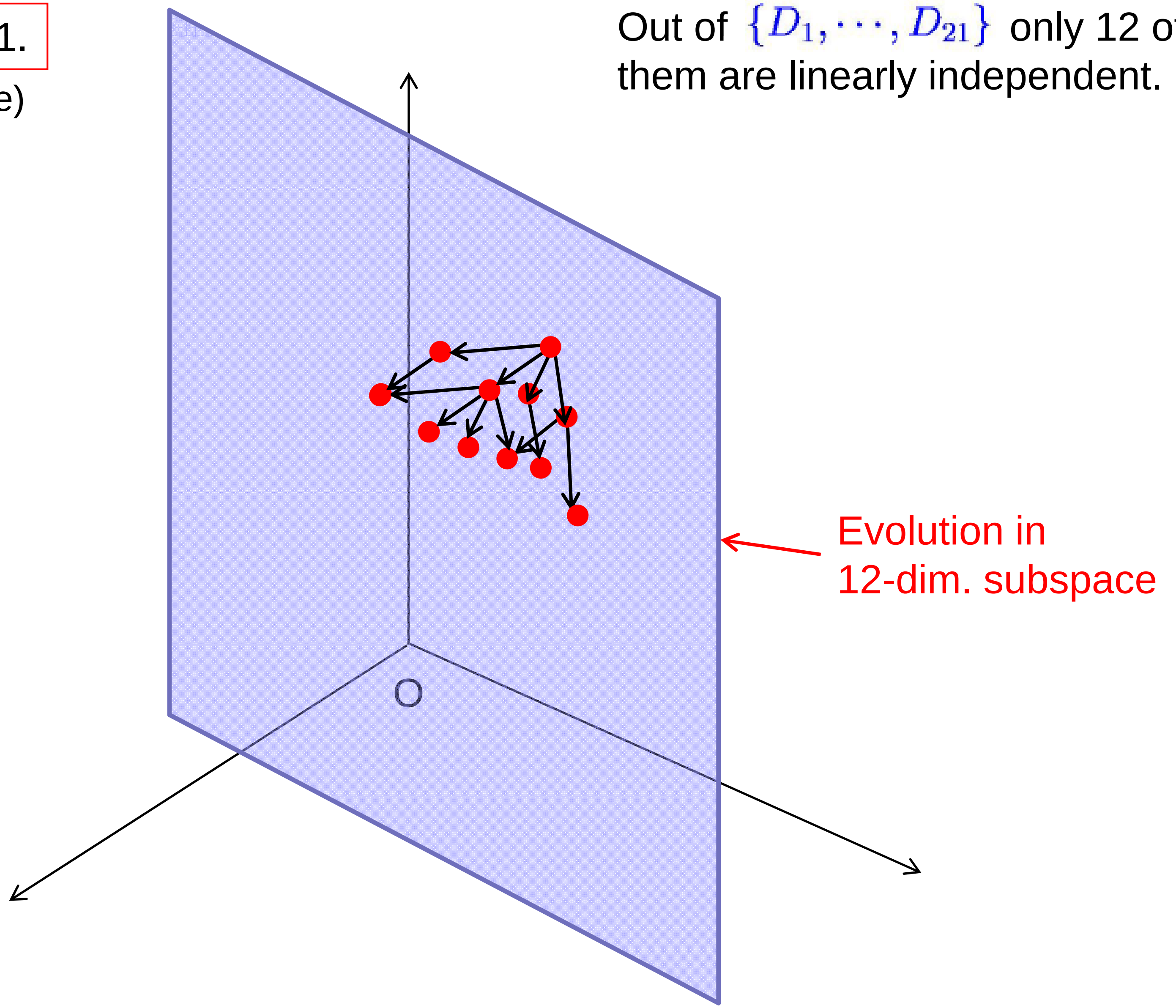


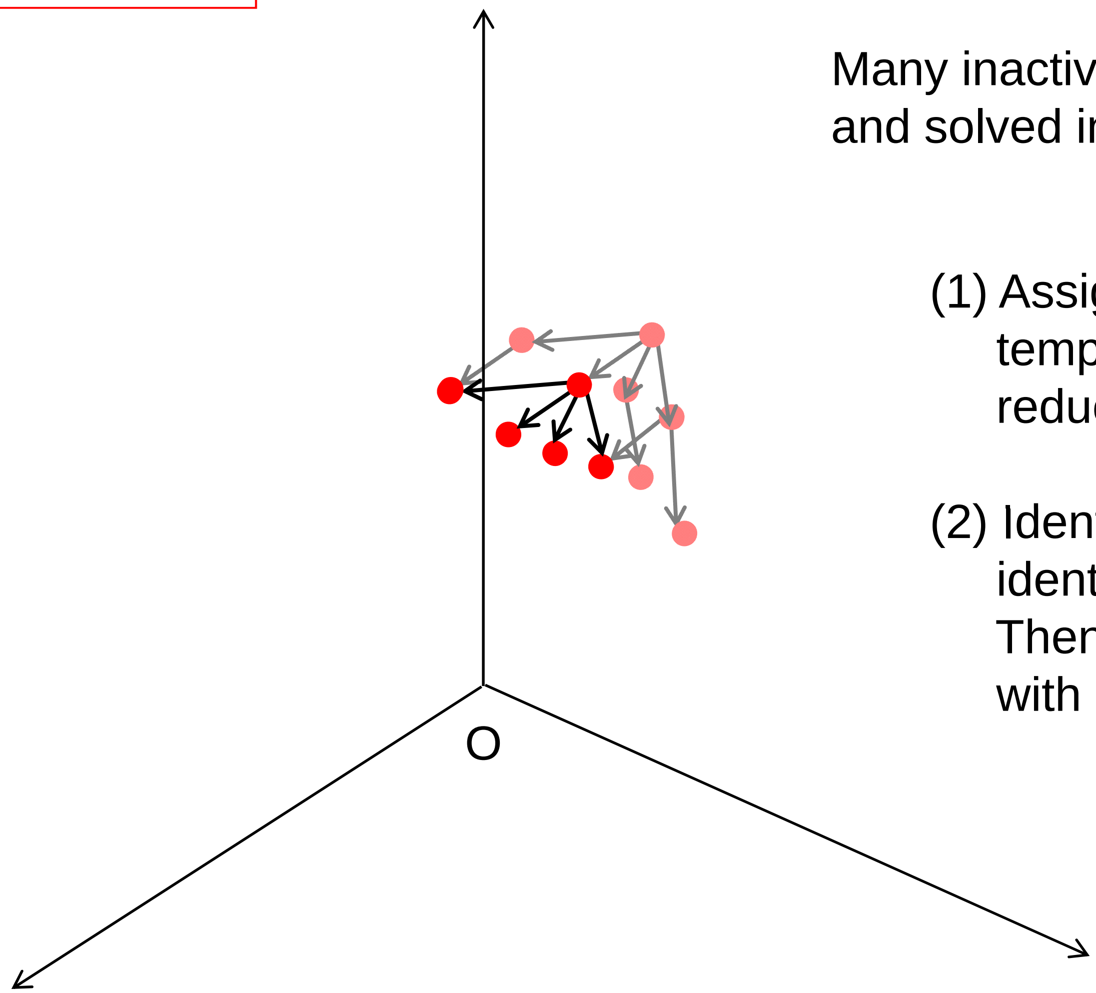
Improvement 1.

(known technique)

Out of $\{D_1, \cdots, D_{21}\}$ only 12 of them are linearly independent.



Improvement 2.



Many inactive IBP id's are generated and solved in Laporta algorithm.

(1) Assign a numerical value to D temporarily and complete reduction.

(2) Identify the necessary IBP identities and reorder them; Then reprocess the reduction with general D .

⇨ Manageable by a contemporary desktop/laptop PC

Our result

$$V_{\rm QCD}(q) = -\frac{4\pi\alpha_s}{q^2}\left[1+\frac{\alpha_s}{4\pi}P_1(L)+\left(\frac{\alpha_s}{4\pi}\right)^2 P_2(L)+\left(\frac{\alpha_s}{4\pi}\right)^3 P_3(L)+\dots\right]$$

$$L = \log\left(\frac{\mu^2}{q^2}\right)$$

$$P_3(L) = {\beta_0}^3 L^3 + \left(\frac{5}{2}\beta_0\beta_1+\dots\right)L^2 + (\beta_2+\dots)L + a_3$$

Known 3-loop contributions by:

Brambilla,Pineda,Soto,Vairo

Kniehl,Penin,Smirnov,Steinhauser

Smirnov,Smirnov,Steinhauser

$$a_3 = \bar{a}_3 + \frac{8}{3}\pi^2 C_A^3\left(\frac{1}{\epsilon}+3\,L\right)$$

$$\bar{a}_3 = n_l^3\,\bar{a}_3^{(3)} + n_l^2\,\bar{a}_3^{(2)} + n_l\,\bar{a}_3^{(1)} + \boxed{\bar{a}_3^{(0)}}$$

last unknown piece

$$\boxed{\bar{a}_3^{(0)} = (502.22(12))\,C_A^3 + (-136.8(14))\,\frac{d_F^{abcd}d_A^{abcd}}{N_A}}$$

Anzai,Kiyo,Y.S.

c.f. $$\bar{a}_3^{(0)} = (502.24(1))\,C_A^3 + (-136.39(12))\,\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

Smirnov,Smirnov,Steinhauser

$$\left(d_R^{a_1,\cdots a_n} = \frac{1}{n!}{\rm Tr}\left[T_R^{a_1}T_R^{a_2}\cdots T_R^{a_n} + (\text{all permutations})\right]\right)$$

New features of QCD pot. at 3-loop

(1) IR div. in pert. expansion

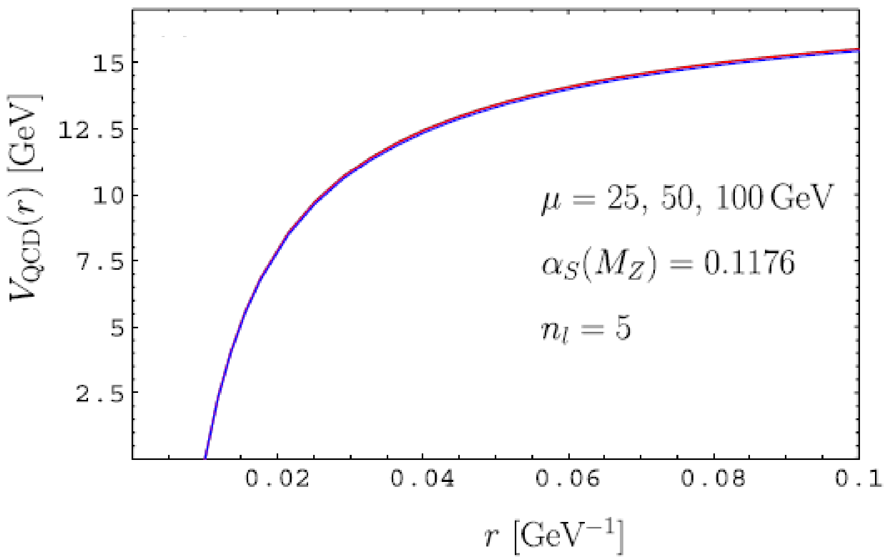
(2) Difference between $V_{\rm QCD}(r)$ and $q\bar{q}$ potential

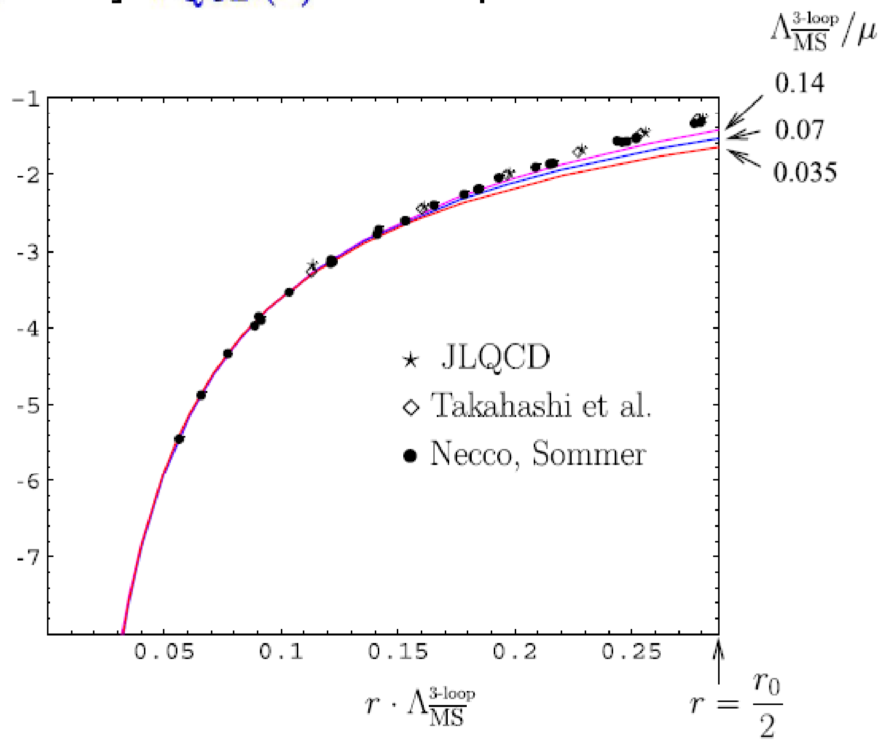
(3) Violation of Casimir scaling

US correction
Appelquist,Dine,Muzunich
Brambilla,Pineda,Soto,Vairo

Anzai,Kiyo,Y.S.

After adding US correction [$\mathcal{O}(\alpha_s^4 \log \alpha_s)$ term], $V_{\rm QCD}(r)$ is computed:





Casimir scaling hypothesis

····· supported by lattice measurements

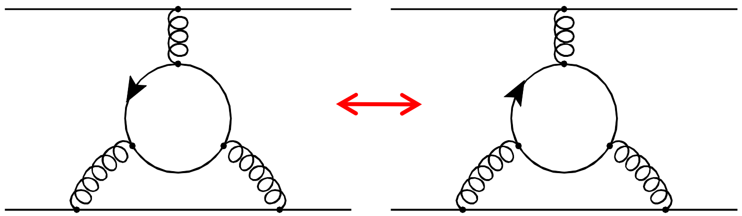
Markum,Faber
Campbell,Jorysz,Michael
Deldar
Bali

$V_R(r) \propto C_R$

2nd Casimir op. for rep. R

cf. $V_R^{\rm tree}(r) = -C_R \frac{\alpha_s}{r}$

2-loop



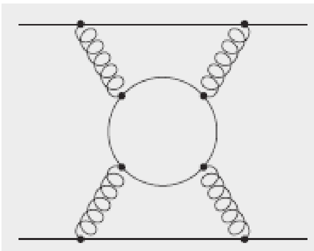
$d_R^{abc} d_F^{abc}$ cancel

$\left(d_R^{a_1 \cdots a_n} = \frac{1}{n!} \mathrm{Tr}\left[T_R^{a_1} T_R^{a_2} \cdots T_R^{a_n} + (\text{all permutations}) \right] \right)$

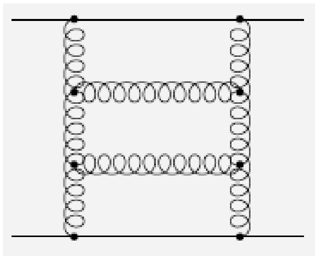
$d_A^{abc} = 0$ by $(T_A^a)^T = -T_A^a$

protected by C-inv.

3-loop



$d_R^{abcd} d_F^{abcd}$



$d_R^{abcd} d_A^{abcd}$

⇨ Casimir scaling violation

Tiny violation predicted, compatible with current lattice data.

Future applications:

- Higher-order computations of properties of heavy quarkonia.
- Precise determinations of heavy quark masses.
- Production of top quarks near threshold @ ILC(/LHC).
- Precise determination of strong coupling constant.