m_c and m_b from the R-ratio and pQCD

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I. Introduction & Motivation
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IV. Summary & Conclusion

based on Phys. Rev. D80 (2009), Nucl. Phys. B 778, 192 (2007)
In collaboration with:
K.G. Chetyrkin, J.H. Kühn, A. Maier, P. Maierhöfer, P. Marquard, M. Steinhauser

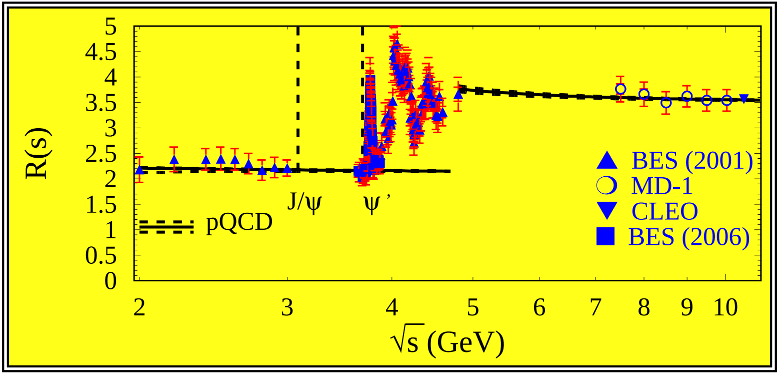
Precise determination of the charm- and bottom-quark masses important:

- Quark masses are fundamental parameters of the Standard Model $\rightsquigarrow$ enter in many physical observables
- Quark masses play an important role in Higgs physics:
  e.g. Higgs decays:
  SM Higgs boson light $\rightsquigarrow$ dominant decay into $b\bar{b}$
  $\Gamma(H \to b\bar{b}) = \frac{G_f M_h}{4\sqrt{2}\pi} m_b^2 \left(1 + \mathcal{O}(\alpha_s) + \ldots\right), \quad \Gamma(H \to c\bar{c}) \sim m_c^2$
- Quark masses relevant in flavor physics:
  e.g. B meson decays: $\Gamma \propto m_b^5, \quad B \to X_u \ell\bar{\nu}, B \to X_c \ell\bar{\nu}$
  Virtual charm quarks: $K \to \pi\nu\bar{\nu}, B \to X_s\gamma$
- Comparison with other methods, e.g. lattice methods
  $\rightsquigarrow$ valuable, mutual cross-checks

II. Method

Experiment: *R*-ratio

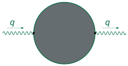
$$R(s) = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

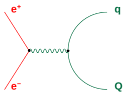
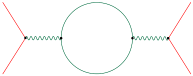


- Heavy quark correlator

$$\Pi^{\mu\nu}(q,j) = i \int dx\, e^{iqx} \langle 0|T j^{\mu}(x) j^{\nu}(0)|0\rangle$$

Here: $j^{\mu}(x)$ electromagnetic heavy quark vector current

$$\Pi^{\mu\nu}(q) = (-g^{\mu\nu} + q^{\mu}q^{\nu}/q^2)\Pi(q^2) \sim$$ 

- $\int d\Pi \left| \right.$  $\left. \right|^2 = 2\,\mathrm{Im}\Big($  $\Big)$

$$R(s) = \frac{\sigma(e^+e^- \to \text{Hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \mathrm{Im}\left[\Pi(q^2 = s + i\varepsilon)\right]$$

- With the help of dispersion-relations:

$$\Pi(q^2) = \Pi(q^2 = 0) + \frac{q^2}{12\pi^2} \int ds \frac{R(s)}{s(s - q^2)}$$

- Exp. moments are related to derivatives of $\Pi(q^2)$ at $q^2 = 0$

II. Method

Relation: Theory $\Longleftrightarrow$ Experiment

- Exp. moments are related to derivatives of $\Pi(q^2)$ at $q^2 = 0$:

$$\frac{12\pi}{n!}\left(\frac{d}{dq^2}\right)^n \Pi(q^2)\Big|_{q^2=0} = \boxed{\mathcal{M}_n^{\text{exp}} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s)}$$

- In terms of expansion coefficients:

$$\Pi(q^2) = \frac{3Q_f^2}{16\pi^2}\sum_n \overline{C}_n^v \left(\frac{q^2}{4m^2}\right)^n, \qquad Q_f\text{: charge of quark}, \quad m = m(\mu) : \overline{\text{MS}} \text{ mass}$$

$\overline{C}_n^v$ can be calculated perturbatively

- First and higher derivatives of $\Pi(q^2)$ allow direct determination of the $\overline{\text{MS}}$ charm- and bottom-quark mass:

$$\boxed{\overline{m}(\mu) = \frac{1}{2}\left(Q_f^2 \frac{9}{4}\frac{\overline{C}_n^v}{\mathcal{M}_n^{\text{exp}}}\right)^{1/(2n)}}$$

← Theory ($\overline{C}_n^v$)

← Experiment ($\mathcal{M}_n^{\text{exp}}$)

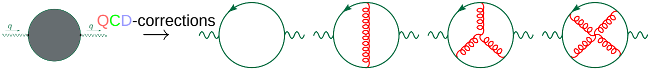
c-quarks: Novikov et al. '78; b-quarks: Reinders et al. '85

$\overline{C}_n^v$ depend on the quark mass through $\log(m(\mu)^2/\mu^2)$

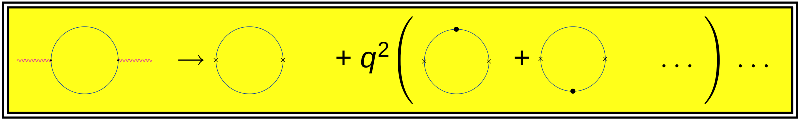
II. Calculation

Pert. calculation of expansion coefficients

- Sample diagrams

$\Pi^{\mu\nu}(q,j) =$ 

- Expansion diagrammatically:

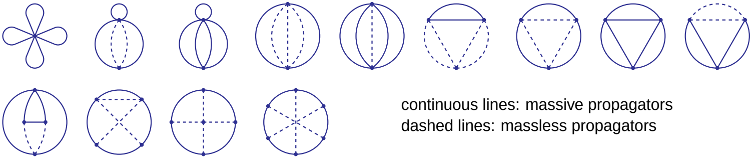


$\hookrightarrow$ One-scale multi-loop integrals in pQCD

- 3-loop(order α_s^2) coefficients $\overline{C}_n$ up to n=8 Chetyrkin,Kühn,Steinhauser 96
  up to higher moments $n \sim 30$ Czakon et al. 06; Maierhöfer, Maier, Marquard 07
  for correlators VV, AA, PP, SS

Computation consists of two steps:

- First step:
  Reduction to a small set of master integrals,
  Integration by parts techniques
- Second step:
  Computation of master integrals
  Here: 13 master integrals



continuous lines: massive propagators
dashed lines: massless propagators

Solution with high precision numeric Y. Schröder, A. Vuorinen
with difference equation method S. Laporta
Subsequently with independent method: ε-finite basis
K.G. Chetyrkin, M. Faisst, C.S., M. Tentyukov

other contributions: D.J. Broadhurst; S. Laporta; B.A. Kniehl, A.V. Kotikov; Y. Schröder, M. Steinhauser

Analytical results in sufficient deep order

II. Calculation

Results at 4-loops

R-ratio method:

– Vector case:
  - first moments $\overline{C}_0$, $\overline{C}_1$
    K. G. Chetyrkin, J. H. Kühn, C.S.'06; R. Boughezal, M. Czakon, T. Schutzmeier'06
  - second moment $\overline{C}_2$ A. Maier, P. Maierhöfer, P. Marquard'08
  - third moment $\overline{C}_3$ A. Maier, P. Maierhöfer, P. Marquard, A.V. Smirnov '09 ← *new*
  - fourth moment $\overline{C}_{4,\ldots,10}$ Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard '09 ← *new*

Lattice method: Replace $\mathcal{M}_n^{exp}$ by lattice sim. HPQCD+ K. Chetyrkin, J. Kühn, M. Steinhauser, C.S. ⟵see talk by C. Davies

– Pseudoscalar case:
  - first moments $\overline{C}_0$, $\overline{C}_1$, $\overline{C}_2$ I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile, J. Shigemitsu, H. Trottier, R.M. Woloshyn, K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S. 08
  - third moment $\overline{C}_3$ A. Maier, P. Maierhöfer, P. Marquard'08
  - fourth moment $\overline{C}_4$ A. Maier, P. Maierhöfer, P. Marquard, A.V. Smirnov '09
  - fifth moment $\overline{C}_{5,\ldots,10}$ Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard '09

– Axial-vector and scalar case:
  - first moments $\overline{C}_0$, $\overline{C}_1$ C. S.'08
  - third moment $\overline{C}_3$ A. Maier, P. Maierhöfer, P. Marquard, A.V. Smirnov '09
  - fourth moment $\overline{C}_{4,\ldots,10}$ Y. Kiyo, A. Maier, P. Maierhöfer, P. Marquard '09

$$\overline{C}_n = \overline{C}_n^{(0)} + \left(\frac{\alpha_s}{\pi}\right)\left(\overline{C}_n^{(10)} + \overline{C}_n^{(11)} l_{m_c}\right)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2\left(\overline{C}_n^{(20)} + \overline{C}_n^{(21)} l_{m_c} + \overline{C}_n^{(22)} l_{m_c}^2\right)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3\left(\overline{C}_n^{(30)} + \overline{C}_n^{(31)} l_{m_c} + \overline{C}_n^{(32)} l_{m_c}^2 + \overline{C}_n^{(33)} l_{m_c}^3\right)$$
$$+ \ldots, \text{with } l_{m_c} = \log(m_c^2/\mu^2)$$

Vector case ($n_f = 4 \leftrightarrow$ charm-quarks):

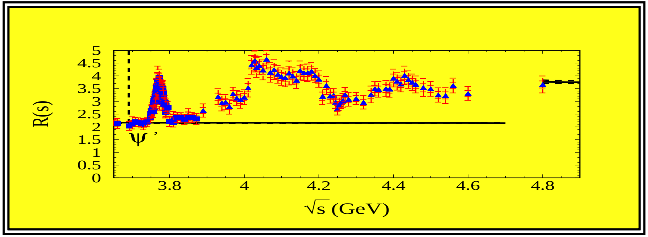
	1-loop	2-loop		3-loop			4-loop			
n	$\overline{C}_n^{(0)}$	$\overline{C}_n^{(10)}$	$\overline{C}_n^{(11)}$	$\overline{C}_n^{(20)}$	$\overline{C}_n^{(21)}$	$\overline{C}_n^{(22)}$	$\overline{C}_n^{(30)}$	$\overline{C}_n^{(31)}$	$\overline{C}_n^{(32)}$	$\overline{C}_n^{(33)}$
1	1.0667	2.5547	2.1333	3.1590	3.44250	0.0889	-7.7624	-0.0599	1.5851	-0.0543
2	0.4571	1.1096	1.8286	3.2319	5.0798	1.9048	-2.6438	4.0100	7.2551	0.1058
3	0.2709	0.5194	1.6254	2.0677	4.5815	3.3185	-1.1745	5.6496	13.4967	2.3967
4	0.1847	0.2031	1.4776	1.2204	3.4726	4.4945	-1.386(10)	3.9381	17.2292	6.2423

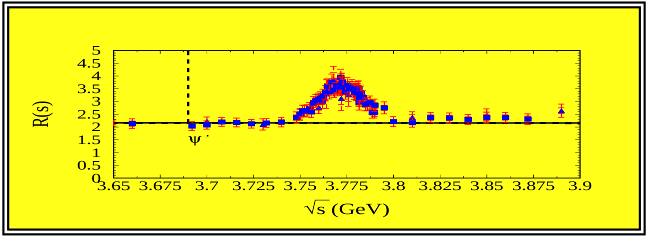
Result available analytically

III. Analysis

R-ratio

<u>Determine</u>: $\mathcal{M}_n^{exp} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s)$





Extraction of the exp. moments from $R(s)$ (charm quark case)

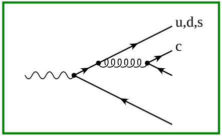
Determine: $\mathcal{M}_n^{exp} = \int \frac{ds}{s^{n+1}} R^{\text{exp}}(s) = \mathcal{M}_n^{res} + \mathcal{M}_n^{thr} + \mathcal{M}_n^{cont}$

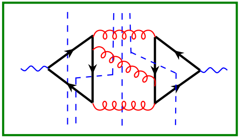
For charm quarks:

$\mathcal{M}_n^{res}$: Contains: $J/\Psi, \Psi(2S)$ treated in narrow width approximation

$$R^{res}(s) = \frac{9\pi M_R \Gamma_{ee}}{\alpha^2} \left(\frac{\alpha}{\alpha(s)}\right)^2 \delta(s - M_R^2)$$

$\mathcal{M}_n^{thr}$: BES data ($\sqrt{s} \geq 3.73$ GeV) subtract background from R_{uds},





$\bar{R}$ from data below 3.73 GeV, $\sqrt{s}$-dependence from theory

$\mathcal{M}_n^{cont}$: pQCD above $\sqrt{s} \geq 4.8$ GeV ,

spare data,

$R(s)$ with full quark mass dependence rhad: R. Harlander, M. Steinhauser '02

- Results $\mathcal{M}_n^{exp}$:

n	$\mathcal{M}_n^{\rm res}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm thresh}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm cont}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm exp}$ $\times 10^{(n-1)}$	$\mathcal{M}_n^{\rm np}$ $\times 10^{(n-1)}$
1	0.1201(25)	0.0318(15)	0.0646(11)	0.2166(31)	−0.0001(2)
2	0.1176(25)	0.0178(8)	0.0144(3)	0.1497(27)	0.0000(0)
3	0.1169(26)	0.0101(5)	0.0042(1)	0.1312(27)	0.0007(14)
4	0.1177(27)	0.0058(3)	0.0014(0)	0.1249(27)	0.0027(54)

- Different relative importance of the various regions
- Consider non-perturbative contributions.

$$\delta\mathcal{M}_n^{\rm np} = \frac{12\pi^2 Q_c^2}{(4m_c^2)^{(n+2)}} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle a_n \left(1 + \frac{\alpha_s}{\pi} \bar{b}_n\right)$$

D.J. Broadhurst, P.A. Baikov, V.A. Ilyin, J. Fleischer, O.V. Tarasov, V.A. Smirnov

- $\mathcal{M}_n^{th} + \mathcal{M}_n^{np} = \mathcal{M}_n^{exp}$ with $\mathcal{M}_n^{th} = \frac{9}{4} Q_q^2 \left(\frac{1}{4m_c^2}\right)^n \overline{C}_n$

$$m(\mu) = \frac{1}{2} \left(Q_f^2 \frac{9}{4} \frac{\overline{C}_n}{\mathcal{M}_n^{exp} - \mathcal{M}_n^{np}} \right)^{1/(2n)}$$

III. Analysis & Results

Determination of the charm quark mass from $R(s)$

- Charm quark mass:

$\mu = (3 \pm 1)$ GeV $\alpha_s(M_Z) = 0.1189 \pm 0.002$

n	$m_c(3\text{ GeV})$	exp	α_s	μ	np	total
1	0.986	0.009	0.009	0.002	0.001	0.013
2	0.976	0.006	0.014	0.005	0.000	0.016
3	0.978	0.005	0.015	0.007	0.002	0.017
4	1.004	0.003	0.009	0.031	0.007	0.033

- Remarkable consistency between $n = 1, 2, 3, 4$
- Result: n=1: $m_c(3\text{ GeV})$=0.986(13) GeV

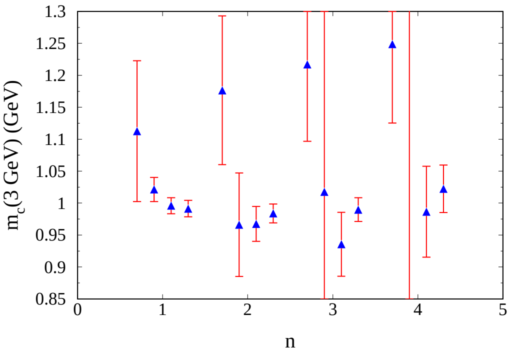
  $m_c(m_c)$=1.279(13) GeV

  Theo. uncertainty by truncation error comparable
- Lattice method: $m_c(3\text{ GeV})$=0.986(6) GeV HPQCD ←see talk by C. Davies

  ↪excellent mutual agreement

Determination of the charm quark mass from $R(s)$

Charm-quarks



$$m_c(3\text{ GeV}) = 0.986(13)\text{ GeV}$$

- Improvement of the stability with increasing order in pQCD
- Preference for lower moments

III. Analysis

Extraction of the exp. moments from $R(s)$ (bottom quark case)

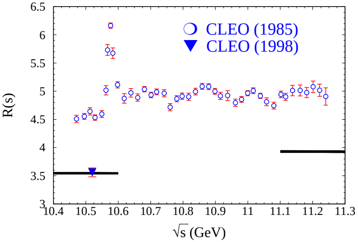
$\mathcal{M}_n^{th}$: analog to charm case, only $n_f = 5$

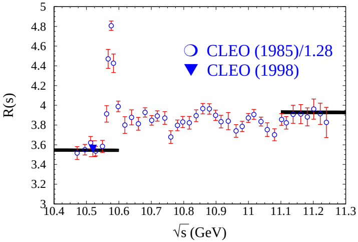
$\mathcal{M}_n^{np}$: negligible

$\mathcal{M}_n^{res}$: $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$, $\Upsilon(4S)$ (PDG)

$\mathcal{M}_n^{thr.}$: BABAR, CLEO data up to 11.24 GeV

Improvements based on the recent BABAR results ← *new*

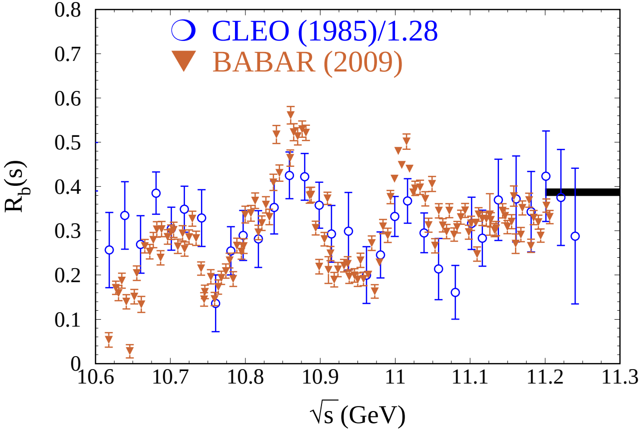




Uncertainties after "renormalization" estimated to be 10%

III. Analysis

Extraction of the exp. moments from $R(s)$ (bottom quark case)



Systematic experimental error $\sim$3.5%

$\mathcal{M}_n^{cont}$: pQCD above 11.24 GeV

III. Analysis & Results

Extraction of the exp. moments from $R(s)$ (bottom quark case)

- $\mathcal{M}_n^{exp}$:

n	$\mathcal{M}_n^{\text{res}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{cont}}$ $\times 10^{(2n+1)}$	$\mathcal{M}_n^{\text{exp}}$ $\times 10^{(2n+1)}$
1	1.394(23)	0.287(12)	2.911(18)	4.592(31)
2	1.459(23)	0.240(10)	1.173(11)	2.872(28)
3	1.538(24)	0.200(8)	0.624(7)	2.362(26)
4	1.630(25)	0.168(7)	0.372(5)	2.170(26)

- BABAR $\leftrightarrow$ CLEO

n	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$ BABAR	$\mathcal{M}_n^{\text{thresh}}$ $\times 10^{(2n+1)}$ CLEO
1	0.287(12)	0.296(32)
2	0.240(10)	0.249(27)
3	0.200(8)	0.209(22)
4	0.168(7)	0.175(19)

- Consistency between BABAR and CLEO
- Reduction of experimental error in this region by a factor 3
- Reduction of the total error by about a factor 2/3

III. Analysis & Results

Determination of the bottom quark mass from $R(s)$

- Bottom quark masses:

$\mu = (10 \pm 5)$ GeV; $\alpha_s(M_Z) = 0.1189 \pm 0.002$

n	m_b(10 GeV)	exp	α_s	μ	total
1	3.597	0.014	0.007	0.002	0.016
2	3.610	0.010	0.012	0.003	0.016
3	3.619	0.008	0.014	0.006	0.018
4	3.631	0.006	0.015	0.020	0.026

- Consistency and stability between $n = 1, 2, 3, 4$
- Result: n=2: m_b(10 GeV)= 3.610(16) GeV

  $m_b(m_b)$= 4.163(16) GeV

  Theo. uncertainty by truncation error comparable
- Well consistent with KSS 2007
- Latt. method: m_b(10 GeV)= 3.617(25) GeV HPQCD ←see talk by C. Davies

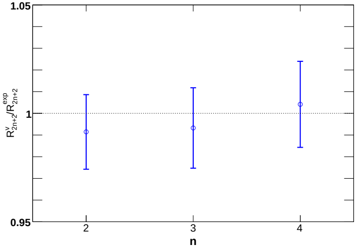
  ↪excellent mutual agreement

III. Comparison: R-ratio $\leftrightarrow$ Lattice moments

I. Allison, E. Dalgic, C.T.H. Davies, E. Follana, R.R. Horgan, K. Hornbostel, G.P. Lepage, C. McNeile,
J. Shigemitsu, H. Trottier, R.M. Woloshyn(HPQCD), K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, C.S.

- Can not only compare final results for quark masses but also exp. moments from R-ratio and moments obtained by latt. sim. of vector current correlator:

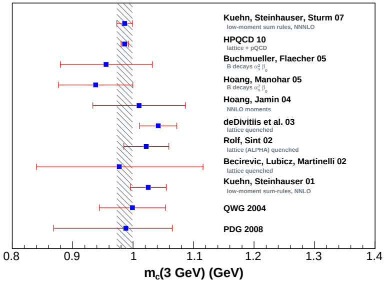
$$\mathcal{R}^{exp}_{2n+2} \leftrightarrow \mathcal{R}^{v}_{2n+2} \quad (n_f = 4)$$



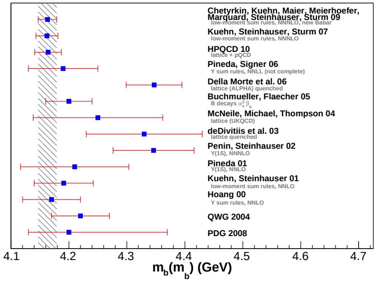
- Within combined errors experimental and simulation results agree within $\sim 2\%$
- Provides excellent cross-check of the analysis with all its details

III. Comparison

charm-quarks



bottom-quarks



IV. Summary & Conclusion

- Precise determination of the charm- and bottom-quark mass can be obtained from the experimentally measured R-ratio in combination with heavy quark current correlators computed in continuum perturbation theory
- Calculation of expansion coefficients of polarization functions up to NNNLO
- Analysis of the R-ratio and extraction of charm- and bottom-quark masses
- Final results

  quark masses :

  - Charm-mass: $m_c(3\text{ GeV}) = 0.986(13)$ GeV e^+e^- + pQCD
  - Bottom-mass: $m_b(10\text{ GeV}) = 3.610(16)$ GeV e^+e^- + pQCD