Calculation of QCD corrections to the electromagnetic vertex in Dimensional Regularization

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A Dimensional Regularization

In the intermediate stages of calculations we must introduce some regularization procedure to control ultraviolet, soft and collinear divergences. The most effective regulator is the method of dimensional regularization which continues the dimension of space-time to $d = 4 - 2\epsilon$ dimensions. This method of regularization has the advantage that the Ward Identities of the theory are preserved at all stages of the calculation. Integrals over loop momenta are performed in d dimensions with the help of the following formula,

$$\int \frac{d^dk}{(2\pi)^d} \frac{(-k^2)^r}{\Big[-k^2 + C - i\varepsilon\Big]^m} =$$
$$\frac{i(4\pi)^\epsilon}{16\pi^2} \left[C - i\varepsilon\right]^{2+r-m-\epsilon} \frac{\Gamma(r + d/2)}{\Gamma(d/2)} \frac{\Gamma(m - r - 2 + \epsilon)}{\Gamma(m)}. \qquad \text{(A.1)}$$

To demonstrate Eq. (A.1), we first perform a Wick rotation of the k_0 contour anti-clockwise. This is dictated by the $i\varepsilon$ prescription, since, for real C, the poles coming from the denominator of Eq. (A.1) lie in the second and fourth quadrant of the k_0 complex plane as shown in Fig. 1. Thus by anti-clockwise rotation of the contour of integration we encounter no poles. After rotation by an angle $\pi/2$, the k_0 integral runs along the imaginary axis in the k_0 plane, $(-i\infty < k_0 < i\infty)$. In order to deal only with real quantities we make the substitution $k_0 = i\kappa_d, k_j = \kappa_j$ for all $j \neq 0$ and introduce $|\kappa| = \sqrt{\kappa_1^2 + \kappa_2^2 \ldots + \kappa_d^2}$. We obtain a d-dimensional Euclidean integral which may be written as,

$$\int d^d\kappa \; f(\kappa^2) \;\; = \;\; \int d|\kappa| \; f(\kappa^2) \; |\kappa|^{d-1} \sin^{d-2}\theta_{d-1} \sin^{d-3}\theta_{d-2} \ldots$$
$$\times \sin\theta_2 \; d\theta_{d-1} d\theta_{d-2} \ldots d\theta_2 d\theta_1. \qquad \text{(A.2)}$$

The range of the angular integrals is $0 \leq \theta_i \leq \pi$ except for $0 \leq \theta_1 \leq 2\pi$. Eq. (A.2) is best proved by induction. Assuming that it is true for an d-

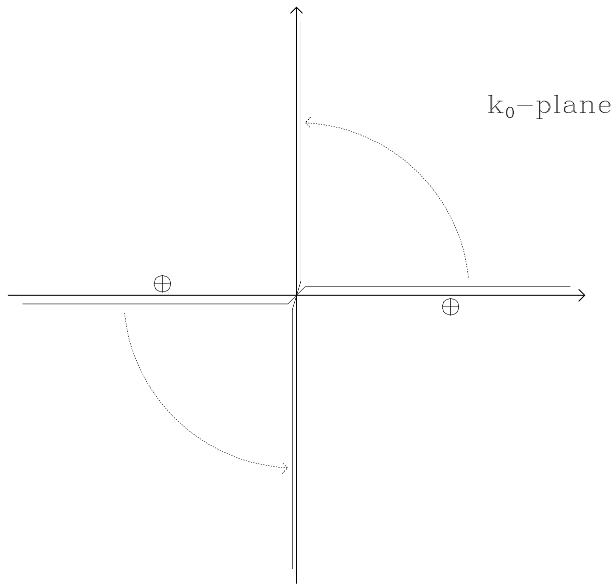

Figure 1: Wick rotation in the complex k_0 plane

dimensional integral, in $(d+1)$ dimensions we can write,

$$\begin{aligned}\int\, d^{d+1}\kappa &= \int\, d\kappa_{d+1}\; d^d\kappa \qquad \text{(A.3)}\\ &= \int\, d\kappa_{d+1}\; d|\kappa|\; |\kappa|^{d-1}\sin^{d-2}\theta_{d-1}\sin^{d-3}\theta_{d-2}\ldots\sin\theta_2\; d\theta_{d-1}d\theta_{d-2}\ldots d\theta_2 d\theta_1 \qquad \text{(A.4)}\end{aligned}$$

The d-dimensional length, κ can be written in terms of the $(d+1)$-dimensional length, ρ

$$\begin{aligned}\kappa_{d+1} &= \rho\cos\theta_d\\ |\kappa| &= \rho\sin\theta_d \qquad \text{(A.5)}\end{aligned}$$

Changing variables to ρ and θ_d we recover the $(d+1)$-dimensional version of Eq. (A.2).

The angular integrations, which only give an overall factor, can be performed using

$$\int_0^\pi d\theta\; \sin^d\theta = \sqrt{\pi}\frac{\Gamma\Big(\frac{(d+1)}{2}\Big)}{\Gamma\Big(\frac{(d+2)}{2}\Big)}. \qquad \text{(A.6)}$$

We therefore find that the left hand side of Eq. (A.1) can be written as,

$$\frac{2i}{(4\pi)^{d/2}\Gamma\Big(d/2\Big)}\int_0^\infty d|\kappa|\;\frac{|\kappa|^{d+2r-1}}{\Big[\kappa^2+C\Big]^m}. \qquad \text{(A.7)}$$

$\Gamma(z) = \int_0^\infty \, dt \; e^{-t} t^{z-1}$
$z\Gamma(z) = \Gamma(z+1)$
$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2})$
$\Gamma(n+1) = n!$ for n a positive integer
$\Gamma(1) = 1, \;\; \Gamma(\frac{1}{2}) = \sqrt{\pi}$
$\Gamma\,'(1) = -\gamma_E, \;\; \gamma_E \approx 0.57721566$
$\Gamma\,''(1) = \gamma_E^2 + \frac{\pi^2}{6}$
$B(a,b) = \int_0^1 dx \; x^{a-1}(1-x)^{b-1}$
$B(a,b) = \int_0^\infty dt \; \frac{t^{a-1}}{(1+t)^{a+b}}$ for Re $a, b > 0$
$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Table 1: Useful properties of the Γ and related functions

This last integral can be reduced to a Beta function, (see Table 1)

$$\int_0^\infty dx \frac{x^s}{\Big[x^2 + C\Big]^m} = \frac{\Gamma\big(\frac{(s+1)}{2}\big)}{2} \frac{\Gamma\big(m - s/2 - 1/2\big)}{\Gamma\big(m\big)} C^{s/2+1/2-m} \tag{A.8}$$

which demonstrates Eq. (A.1).

Feynman parameter identities are also useful for calculating virtual diagrams. The general form is,

$$\begin{aligned} \frac{1}{A^\alpha \; B^\beta \cdots E^\epsilon} &= \frac{\Gamma(\alpha+\beta+\cdots\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\epsilon)} \\ &\times \int_0^1 dx \; dy \cdots dz \; \delta(1-x-y\cdots-z) \\ &\times \frac{x^{\alpha-1} \; y^{\beta-1} \cdots z^{\epsilon-1}}{(Ax+By+\cdots+Ez)^{\alpha+\beta+\cdots+\epsilon}} \; . \end{aligned} \tag{A.9}$$

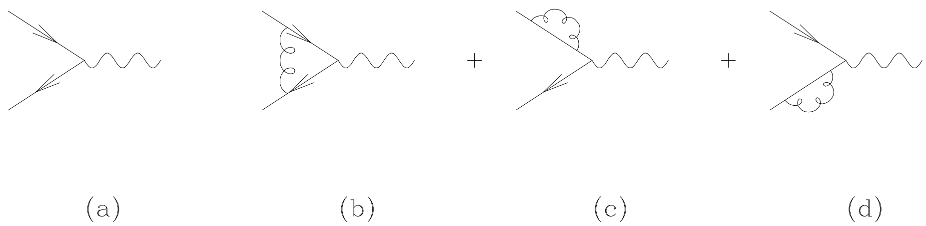

Figure 2: Diagrams for QCD corrections to the electromagnetic vertex

B QCD corrections to the electromagnetic vertex

In this appendix we describe the calculation of the radiative corrections to the electromagnetic vertex. This calculation, which describes the modification of the interaction of a virtual photon with a quark due to strong interactions, is a good illustration of the use of dimensional regularization to control both ultraviolet and infra-red singularities. The appropriate graphs are shown in Fig. 2.

At lowest order the expression for the vertex, Fig 2(a), is

$$\Gamma^\mu_{(a)}(p',p) = -ie\ \bar{u}(p')\gamma^\mu u(p) \tag{B.1}$$

In the dimensional regularization scheme, Figs. 2(c) and Figs. 2(d) give no contribution. For massless, on-shell quarks the dimenensionally regularized integral vanishes

$$\int \frac{d^dk}{k^4} = 0 \tag{B.2}$$

because there is no dimensionful scale to which the integral could be proportional.

The calculation of vertex diagram, Fig 2(b), proceeds as follows. Using the Feynman rules (in the Feynman gauge ($\lambda = 1$)) we have,

$$\Gamma^\mu_{(b)}(p',p) = -i^3(-ig)^2(-ie)\int \frac{d^dl}{(2\pi)^d}\frac{\bar{u}(p')\gamma^\delta t^D(\not{l}+\not{p}')\gamma^\mu(\not{l}+\not{p})\gamma_\delta t^D u(p)}{((l+p')^2+i\varepsilon)((l+p)^2+i\varepsilon)(l^2+i\varepsilon)} \tag{B.3}$$

Collecting terms and applying the color rule $t^D t^D = C_F I$ we get

$$\Gamma^\mu_{(b)}(p',p) = -eg^2 C_F \int \frac{d^dl}{(2\pi)^d}\frac{\bar{u}(p')\gamma^\delta(\not{l}+\not{p}')\gamma^\mu(\not{l}+\not{p})\gamma_\delta u(p)}{((l+p')^2+i\varepsilon)((l+p)^2+i\varepsilon)(l^2+i\varepsilon)} \tag{B.4}$$

Notice the divergence structure of this expression. For large l the integral is logarithmically ultraviolet divergent.

$$\Gamma^\mu_{(b)}(p',p) \to -eg^2 C_F \int \frac{d^dl}{(2\pi)^d}\frac{\bar{u}(p')\gamma^\delta \not{l}\gamma^\mu \not{l}\gamma_\delta u(p)}{(l^2+i\varepsilon)^3}\ . \tag{B.5}$$

For small l (the soft region) the integral is also divergent

$$\Gamma^\mu_{(b)}(p',p) \quad \rightarrow \quad -eg^2 C_F \int \frac{d^dl}{(2\pi)^d} \frac{4p\cdot p' \; \bar{u}(p')\gamma^\mu u(p)}{(2l\cdot p' + i\varepsilon)(2l\cdot p + i\varepsilon)(l^2 + i\varepsilon)} \; . \tag{B.6}$$

In deriving this result we have used the commutation relation

$$\not{a}\gamma^\delta + \gamma^\delta \not{a} = 2a^\delta \tag{B.7}$$

to commute the momenta to positions where they can act on the spinors and use the equation of motion for massless free particles. The integral thus has ultraviolet and infra-red divergences.

Returning to the full expression, Eq. (B.4), it proves useful to introduce Feynman parameters

$$\frac{1}{ABC} = 2\int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \; \frac{\delta(1-\alpha-\beta-\gamma)}{\Big[\alpha A + \beta B + \gamma C\Big]^3} \; . \tag{B.8}$$

So the expression in Eq. (B.4) becomes

$$\begin{aligned} \Gamma^\mu_{(b)}(p',p) &= -2eg^2 C_F \int \frac{d^dl}{(2\pi)^d} \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \; \delta(1-\alpha-\beta-\gamma) \\ &\times \frac{\bar{u}(p')\gamma^\delta(\not{l}+\not{p}')\gamma^\mu(\not{l}+\not{p})\gamma_\delta u(p)}{\Big[l^2 + 2\alpha p\cdot l + 2\beta p'\cdot l + i\varepsilon\Big]^3} \end{aligned} \tag{B.9}$$

We now perform the shift $l = l' + \alpha p + \beta p'$ so that the expression becomes

$$\Gamma^\mu_{(b)}(p',p) = -2eg^2 C_F \int \frac{d^dl}{(2\pi)^d} \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \; \frac{\delta(1-\alpha-\beta-\gamma)\, N^\mu}{\Big[l'^{\,2} + \alpha\beta q^2 + i\varepsilon\Big]^3} \tag{B.10}$$

where the numerator is

$$N^\mu = \bar{u}(p')\gamma^\delta(\not{l}' + (1-\beta)\not{p}' - \alpha\not{p})\gamma^\mu(\not{l}' + (1-\alpha)\not{p} - \beta\not{p}')\gamma_\delta u(p) \tag{B.11}$$

and $q = p' - p$. Terms odd in l' can be dropped so that the numerator becomes

$$N^\mu = \bar{u}(p')\gamma^\delta \not{l}'\gamma^\mu \not{l}'\gamma_\delta u(p) + \bar{u}(p')\gamma^\delta((1-\beta)\not{p}' - \alpha\not{p})\gamma^\mu((1-\alpha)\not{p} - \beta\not{p}')\gamma_\delta u(p) \tag{B.12}$$

Using $l'_\mu l'_\nu = g_{\mu\nu} l'^{\,2}/n$ and $\gamma^\beta\gamma^\alpha\gamma_\beta = -2(1-\epsilon)\gamma^\alpha$, the first term in the numerator can be simplified to give,

$$N^\mu_{(1)} = \frac{4\,(1-\epsilon)^2}{n} \; l'^{\,2} \; \bar{u}(p')\gamma_\delta u(p) \; . \tag{B.13}$$

The second term in the numerator can also be simplified by using the equations of motion

$$\bar{u}(p')\,\not{p}' = 0, \;\; \not{p}\, u(p) = 0. \tag{B.14}$$

to give

$$N^{\mu}_{(2)} = 4\, p \cdot p' \; (1-\alpha)(1-\beta)\; \bar{u}(p')\gamma^{\mu} u(p) \; . \tag{B.15}$$

At first sight the term

$$+\alpha\beta\; \bar{u}(p')\gamma^{\delta} \not{p} \gamma^{\mu} \not{p}' \gamma_{\delta} u(p) \tag{B.16}$$

might appear to give a contribution, but since it is free from IR divergences it can be evaluated in four dimensions and hence vanishes using the equation of motion. The l' integral can easily be performed using Eq. (A.1)

$$\begin{aligned} \Gamma^{\mu}_{(b)}(p',p) &= -ie\; \bar{u}(p')\gamma^{\mu} u(p)\; \frac{g^2 C_F}{16\pi^2} \Big(\frac{4\pi\mu^2}{-q^2}\Big)^{\epsilon}\; \Gamma(1+\epsilon) \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \\ \times \quad & \delta(1-\alpha-\beta-\gamma)\; \Big[2\frac{(1-\epsilon)^2}{\epsilon}(\alpha\beta)^{-\epsilon} - 2(1-\alpha)(1-\beta)(\alpha\beta)^{-1-\epsilon}\Big] \end{aligned} \tag{B.17}$$

The integrals can be performed using the results in Table 1,

$$\begin{aligned} \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma\; \delta(1-\alpha-\beta-\gamma)\; \frac{(1-\alpha)(1-\beta)}{(\alpha\beta)^{1+\epsilon}} &= \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \Big[\frac{1}{\epsilon^2} + \frac{1}{2(1-\epsilon)}\Big] \\ \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma\; \delta(1-\alpha-\beta-\gamma)\; (\alpha\beta)^{-\epsilon} &= \frac{1}{2(1-\epsilon)} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} \end{aligned} \tag{B.18}$$

Using the identity

$$\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} = \frac{1}{\Gamma(1-\epsilon)} + O(\epsilon^3) \tag{B.19}$$

Collecting terms we may write the final answer for the lowest order term plus the radiative correction as,

$$\Gamma^{\mu}(p',p) = -ie\bar{u}(p')\gamma^{\mu} u(p) \Big\{ 1 + \frac{g^2 C_F}{16\pi^2 \Gamma(1-\epsilon)} \Big(\frac{4\pi\mu^2}{-q^2 - i\varepsilon}\Big)^{\epsilon} \Big[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + O(\epsilon)\Big] \Big\} \tag{B.20}$$

Notice that for q^2 negative the result is real. For $q^2 > 0$ there is a branch cut extending from $q^2 = 0$ to $q^2 = \infty$. The path around the cut is indicated by the symbol ε.