HCPSS-2010, Introduction to the SM: PS2

Question 1: $\rho = 1$ for a general Higss

In the SM the Higgs transforms under $SU(2)_L \times U(1)_Y$ as $(2)_{1/2}$. However, any scalar that is charged under the gauge group and acquires a vev will break the SM gauge symmetry.

1. Consider a scalar ϕ that transforms as $(2T+1)_Y$. Since $SU(2)$ is a non Abelian group, $2T+1$ has to be a positive integer, that is, T is a non negative half integer. Since $U(1)$ is Abelian, a priori Y can assume any real value. Yet, we like ϕ to be responsible for the $SU(2)_L \times U(1)_Y \to U(1)_{EM}$ breaking. This requirement restricts the possible values for Y. Find these values.
   Answer: Since we want EM to be a good symmetry the neutral component of the Higgs is the one that acquires a vev. Therefore, we require that there is a neutral component, namely that there exist T_3 such that $T_3 + Y = 0$. Since $-T \le T_3 \le T$ we find that $Y \le |T|$ and it is integer if T is integer and half integer if T is half integer.

2. We define
$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1, \qquad \tan\theta_W \equiv \frac{g'}{g}. \tag{1}$$
   Show that ρ is given by
$$\rho = \frac{T(T+1) - Y^2}{2Y^2} \tag{2}$$
   Hint: Recall that the $2T+1$ dim. representation of SU(2) is given by
$$T_3 = diag\{T, T-1, T-2, \dots, -T\} \tag{3}$$
$$T_1 = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ a_1 & 0 & a_2 & 0 & \vdots \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 & a_{2T} \\ 0 & \dots & \dots & a_{2T} & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & ia_1 & 0 & \dots & 0 \\ -ia_1 & 0 & ia_2 & 0 & \vdots \\ 0 & -ia_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 & ia_{2T} \\ 0 & \dots & \dots & -ia_{2T} & 0 \end{pmatrix}$$
   where
$$a_i = \frac{\sqrt{T(T+1) - (T-i)(T-i+1)}}{2} \tag{4}$$

   Answer: The mass term for the W and the Z is obtained from the kinetic term of the Higgs
$$|(gT_a W_a^\mu + g' Y B^\mu)\phi|^2 \tag{5}$$
   ϕ has $2T+1$ components. The neutral one which acquires a vev is in the $C = T - Y + 1$ line from above. At that line $T_3 = -Y$. Thus, we see that we care only about the

3×3 sub-matrix

$$\left| \begin{pmatrix} g(Y-1)W_3^\mu + g'YB^\mu & a_{C-1}(W_1^\mu + iW_2^\mu) & 0 \\ a_{C-1}(W_1^\mu - iW_2^\mu) & gY\,W_3^\mu + g'YB^\mu & a_C(W_1^\mu + iW_2^\mu) \\ 0 & a_C(W_1^\mu - iW_2^\mu) & g(Y+1)\,W_3^\mu + g'YB^\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \right|^2 \quad (6)$$

We then get

$$v^2 \left[Y^2 \left(gW_3^\mu - g'B^\mu \right)^2 + g^2 \left(T(T+1) - Y^2 \right) W^{+\mu} W^{-\mu} \right] \quad (7)$$

Since

$$Z = \frac{gW_3 - g'B}{\sqrt{g^2 + g'^2}} \quad (8)$$

we get

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = \frac{T(T+1) - Y^2}{2Y^2} \quad (9)$$

where we used

$$\cos^2 \theta_W = \frac{g^2}{g^2 + g'^2} \quad (10)$$

3. For $T > 0$ and $Y = 0$ one can see from eq. (2) that $\rho \to \infty$ independent of T. Explain this result using symmetry arguments.

<u>Answer:</u> The symmetry reason is that for $Y = 0$ the Higgs is not charged under $U(1)_Y$ and therefore, this symmetry remains exact. The symmetry breaking pattern in this case is $SU(2)_L \times U(1)_Y \to U(1)_{EM} \times U(1)_Y$. Since there are two exact local $U(1)$ symmetries, we should have two massless gauge bosons, the photon and the Z. This is indeed what we find.

4. Suppose that there exist several Higgs representations ($i = 1, \ldots, N$) whose neutral members acquire VEVs v_i. Find ρ in terms of v_i, T_i and Y_i.

<u>Answer:</u> It can be seen from the Lagrangian that in this case the m^2 of the gauge bosons are the sum of all the m^2 from the different Higgs fields. Thus,

$$\rho = \frac{\sum v_i^2 [T_i(T_i+1) - Y_i^2]}{2 \sum v_i^2 Y_i^2} \quad (11)$$

5. Assume that, in addition to the usual Higgs doublet $\{T = 1/2, Y = 1/2\}$ with VEV v_W, there exists one other multiplet $\{T_i, Y_i\}$ which acquires a much smaller VEV v_i. Find $\delta\rho \equiv \rho - 1$ to first order in $(v_i/v_W)^2$.

<u>Answer:</u>

$$\delta\rho = 2 \left(\frac{v_i}{v_W} \right)^2 [T_i(T_i+1) - 3Y_i^2] \quad (12)$$

6. Assume that experimentally $-0.01 \leq \delta\rho \leq +0.005$. Find the constraint on $(v_i/v_W)^2$ for the following multiplets: $(5)_{-1}$ and $(4)_3$.
Answer: For $(5)_{-1}$ we have $T = 2, Y = -1$, and thus $\delta\rho > 0$ and we get $(v_i/v_W)^2 < 8.33 \times 10^{-4}$. For $(4)_3$ we have $T = 3/2, Y = 3$, and thus $\delta\rho < 0$ and we get $(v_i/v_W)^2 < 2 \times 10^{-4}$.

7. From Eq. (2) it is clear that $\rho = 1$ for all $3Y^2 = T(T+1)$ multiplets. Since experimentally ρ is very close to 1, we assume that the SM Higgs is one of these multiplets. While from the consideration of ρ alone there is no difference which multiplet we take, in the SM we do make a choice and take $T = 1/2$ and $Y = 1/2$. What is the advantage of the SM Higgs compare to the other possible choices?
Answer: In the SM the Higgs mechanism generates masses for the gauge bosons and for the fermions. However, since the fermions are in doublets and singlets of $SU(2)_L$ only a Higgs doublet can generate masses for them. Higher multiplets can only generate masses to the gauge bosons.