HCPSS-2010, Introduction to the SM: PS2

Question 1: $\rho = 1$ for a general Higgs

In the SM the Higgs transforms under $SU(2)_L \times U(1)_Y$ as $(2)_{1/2}$. However, any scalar that is charged under the gauge group and acquires a vev will break the SM gauge symmetry.

1. Consider a scalar ϕ that transforms as $(2T+1)_Y$. Since $SU(2)$ is a non Abelian group, $2T+1$ has to be a positive integer, that is, T is a non negative half integer. Since $U(1)$ is Abelian, a priori Y can assume any real value. Yet, we like ϕ to be responsible for the $SU(2)_L \times U(1)_Y \to U(1)_{EM}$ breaking. This requirement restricts the possible values for Y. Find these values.

2. We define
$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1, \qquad \tan\theta_W \equiv \frac{g'}{g}. \tag{1}$$
Show that ρ is given by
$$\rho = \frac{T(T+1) - Y^2}{2Y^2} \tag{2}$$
Hint: Recall that the $2T+1$ dim. representation of SU(2) is given by
$$T_3 = diag\{T, T-1, T-2, \ldots, -T\} \tag{3}$$
$$T_1 = \begin{pmatrix} 0 & a_1 & 0 & \ldots & 0 \\ a_1 & 0 & a_2 & 0 & \vdots \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 & a_{2T} \\ 0 & \ldots & \ldots & a_{2T} & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & ia_1 & 0 & \ldots & 0 \\ -ia_1 & 0 & ia_2 & 0 & \vdots \\ 0 & -ia_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 & ia_{2T} \\ 0 & \ldots & \ldots & -ia_{2T} & 0 \end{pmatrix}$$
where
$$a_i = \frac{\sqrt{T(T+1) - (T-i)(T-i+1)}}{2} \tag{4}$$

3. For $T > 0$ and $Y = 0$ one can see from eq. (2) that $\rho \to \infty$ independent of T. Explain this result using symmetry arguments.

4. Suppose that there exist several Higgs representations ($i = 1, \ldots, N$) whose neutral members acquire VEVs v_i. Find ρ in terms of v_i, T_i and Y_i.

5. Assume that, in addition to the usual Higgs doublet $\{T = 1/2, Y = 1/2\}$ with VEV v_W, there exists one other multiplet $\{T_i, Y_i\}$ which acquires a much smaller VEV v_i. Find $\delta\rho \equiv \rho - 1$ to first order in $(v_i/v_W)^2$.

6. Assume that experimentally $-0.01 \leq \delta\rho \leq +0.005$. Find the constraint on $(v_i/v_W)^2$ for the following multiplets: $(5)_{-1}$ and $(4)_3$.

7. From Eq. (2) it is clear that $\rho = 1$ for all $3Y^2 = T(T+1)$ multiplets. Since experimentally ρ is very close to 1, we assume that the SM Higgs is one of these multiplets. While from the consideration of ρ alone there is no difference which multiplet we take, in the SM we do make a choice and take $T = 1/2$ and $Y = 1/2$. What is the advantage of the SM Higgs compare to the other possible choices?