HCPSS-2010, Introduction to the SM: PS1

Question 1: Global symmetries

We talked about the fact that global symmetries are accidental in the SM, that is, that they are broken once non-renormalizable terms are included. Write the lowest dimension terms that break each of the global symmetries of the SM.

Answer: The point is to construct operators that are invariant under all the gauge symmetries. Clearly (I hope it is clear) we need an even number of fermions. We also need an even number of SU(2) doublets, and make sure the hypercharge add up to zero. We find that lepton number is broken by a dimension five operator of the form $LLHH$ (all indices are implicit) while Baryon number is broken by a dimension six operators of the form $QQQL$ (where some combinations of Q replaced by U or D is possible).

Question 2: UV divergence in QM

In order to understand the way we treat UV divergences, let us study a simple QM problem that have a similar characteristics. We consider a particle in an n-dimensional box. That is, the potential in each direction is given by

$$V(x_i) = \begin{cases} 0 & \text{for } |x_i| < L, \\ \infty & \text{for } |x_i| > L. \end{cases} \tag{1}$$

We add a small perturbation

$$V = \lambda L^n \delta^{(n)}(x). \tag{2}$$

Our task is to calculate the corrections to the ground state energy due to this perturbation. Since the perturbation is only at the origin, it is clear that we care only about the states that are finite at the origin. They are given by

$$\phi(x_1, x_2, ...) = \phi_{n_1}(x_1)\phi_{n_2}(x_2)..., \qquad \phi_{n_i}(x_i) = \sqrt{\frac{1}{L}}\cos\left(\frac{n\pi x_i}{2L}\right),$$
$$E_n = C(n_1^2 + n_2^2 + ...), \qquad C = \frac{\hbar^2\pi^2}{8mL^2}, \qquad n_i = 1, 3, 5, ... \tag{3}$$

1. We start with the one dimensional problem where everything is finite. First, write the formula for the second order perturbation and evaluate it. You may like to recall that

   $$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2 - 1} = \frac{1}{4}. \tag{4}$$

   Show that the second order perturbation is finite.

   Answer: The 2nd order perturbation is given by

   $$\Delta E = \sum_{n=3}^{\infty} \frac{|\langle 1|\lambda L \delta(x)|n\rangle|^2}{E_1 - E_n} = \frac{\lambda^2}{C}\sum_{n=1}^{\infty}\frac{1}{1-(2n+1)^2} = \frac{-\lambda^2}{4C}, \tag{5}$$

and it is clearly finite.

2. We now move to two dimensions. Again, write the second order perturbation correction and try to evaluate it. Show that it is logarithmically divergent. To do so, look only at the very high energy modes and approximate the sum by an integral. You have to show that the integral is logarithmically divergent. What can you say about the higher dimensional cases?

   Answer:

$$\Delta E = \sum_{n_1,n_2\neq(1,1)} \frac{\left|\langle 1,1|\lambda\delta^{(2)}(x)|n_1,n_2\rangle\right|^2}{E_{1,1}-E_{n_1,n_2}} = \frac{-\lambda^2}{C}\sum_{n_1,n_2\neq(0,0)} \frac{1}{(2n_1+1)^2+(2n_2+1)^2-2} \tag{6}$$

   For the high modes we can convert it into an integral. Lets look at modes such that $(2n_1+1)^2+(2n_2+1)^2 > N^2$ and we get

$$\sum \frac{1}{(2n_1+1)^2+(2n_2+1)^2-2} \approx \int_{x_0}^{\infty} dx \int_{y_0}^{\infty} dy \frac{1}{(2x+1)^2+(2y+1)^2-2} = \int_{r_0}^{\infty} d^2r \frac{1}{r^2-2} = \int_{r_0}^{\infty} \frac{2\pi r dr}{r^2-2} \approx \int_{r_0}^{\infty} \frac{2\pi dr}{r} = \log\Lambda \to \infty \tag{7}$$

   It is easy to see (and here I really think it is easy to see) that the integral is generalized into

$$\int_{r_0}^{\infty} r^{n-3} dr \sim \Lambda^{n-2} \qquad \text{for } n \geq 3 \tag{8}$$

3. We are still in $n=2$. The divergence you found for this case, however, is not physical. Give a physical argument that explain how the sum is cut off in any real physical system. Change the formalism in a way that incorporated the cut-off in it.

   Answer: In the real world there are no delta functions. A real delta function has a finite width. So, when we go the the very high modes, the width of the delta kill the matrix element since then the oscillation of the cos are very rapid. We could change the formalism in many ways. One way is just to put a cut off by hand. Another is to give an explicit representation of the delta function.

4. So far we just regularized the correction. That is, we can make it finite but still the effect depend on the way we do it finite. Yet, as we argued, the final result must be insensitive to the UV physics. In order to see it, we like to ask how do we measure λ. Lets assume that we measured it by looking at the correction to the ground state energy. Then you can determine λ to first order. Express λ to first order based on the measured deviation from the ground state energy. We denote this λ as λ_P. (Note

that the $2,1$ state does not receive corrections so we can really measure the deviation from the zero order result.)
Answer: We get

$$\Delta E_{1,1}^{(1)} = \langle 1,1|\lambda L\delta^{(2)}(x)|1,1\rangle = \lambda_P \tag{9}$$

5. Calculate the correction to the $n = (3,3)$ level only in terms of measured quantities (to the level of perturbation theory we are working at.) To to that, separate the sum into a "low energy" sum that depend on the specific mode, and a "high energy" sum that to a good approximation in universal. Then, show that the final result depends only on the low energy sum and other measured quantities (like C and $\Delta E_{1,1}$.)
Answer: Let us write the correction in the following way

$$\Delta E_{1,1} = \frac{-\lambda^2}{C}(f_{1,1} + g) \tag{10}$$

such that $f_{1,1}$ is finite and g diverge. We can always put more terms into f such that g concentrate on higher mode. The whole point is that g is independent on the low energy mode. That is, for a low energy state with (a,a) we have have

$$g = \sum \frac{1}{(2n_1+1)^2 + (2n_2+1)^2 - 2a^2} \tag{11}$$

Now if a is very small compared to $n_1^2 + n_2^2$ it is clear that we can neglect the a^2 term and we get

$$g \approx \sum \frac{1}{(2n_1+1)^2 + (2n_2+1)^2} \tag{12}$$

and it is universal. Using this we write

$$\Delta E_{3,3} = \frac{-\lambda^2}{C}(f_{3,3} + g) \tag{13}$$

and we get

$$\Delta E_{3,3} \approx \Delta E_{1,1} + \frac{\lambda^2}{C}(f_{3,3} - f_{1,1}) = \Delta E_{1,1} + \frac{\Delta E_{1,1}^2}{C}(f_{3,3} - f_{1,1}) \tag{14}$$

and we can calculate the f as precise as we like. Clearly, this is independent on how we regularized g.

6. Now calculate the correction to the $n = (3,3)$ level and express it in terms of the measured quantities. You can give your result to 10% accuracy. Feel free to use any software you like (like mathematica, matlab or maple) to evaluate any sum or integral you need. Explain how you could get a higher numerical prediction. Also, explain

why it does not make sense to get an extremely high numerical precision.
Answer: All we need to do is to calculate $f_{3,3} - f_{1,1}$. We find that

$$f_{3,3} - f_{1,1} \approx 0.4 \tag{15}$$

(or if we add more terms it is 0.387). Clearly, we do not like to get very precise result since we neglected third order terms.