All-Orders One-Loop Six-Point Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills Theory

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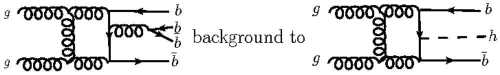
Motivation

- For many years theoretical physicists have attempted to calculate the quantum mechanical probability amplitudes associated to particular scattering processes
- These calculations are of central importance in the LHC era because they are the basic ingredients that one needs to know in order to predict how many events of various types one expects to see at LHC
- It is an inconvenient truth that next-to-leading order (NLO) corrections (and sometimes NNLO) are indispensible for phenomenology



NLO Calculations Are Hard

- Until relatively recently (mid 90's) perturbative scattering amplitudes at NLO were calculated in a very inefficient way
- The study of gluon scattering in a toy model, $\mathcal{N} = 4$ super Yang-Mills theory, played a significant role in the development of better methods for NLO perturbation theory
- In this spirit, we have studied various one-loop $\mathcal{N} = 4$ scattering amplitudes in arbitrary numbers of dimensions ($D = 4 - 2\epsilon$)

Outline

Color-Ordering and the Large N_c Limit

- It is of considerable interest to seperate the color dof from the kinematic dof
- This is accomplished via a decomposition of the amplitude into color-ordered partial amplitudes, $A_1^{L-\text{loop}}(\rho(1), \rho(2), \ldots, \rho(n))$

$$\mathcal{A}_1^{L-\text{loop}} = g^{n-2}\left(\frac{g^2 N_c}{(4\pi)^2}\right)^L \times$$
$$\times \sum_{\rho \in S_n/Z_n} \text{Tr}(T^{a_{\rho(1)}} T^{a_{\rho(2)}} \ldots T^{a_{\rho(n)}}) A_1^{L-\text{loop}}(\rho(1), \rho(2), \ldots, \rho(n))$$

It is only in the planar limit that this simple decomposition holds

Lay of the Land

What about (- - + +), (- - + + +), ($\mp$ + + + + +), ... ?
They're protected by planar $\mathcal{N} = 4$ SUSY!

Status of planar six-gluon scattering in $\mathcal{N} = 4$:

# of loops	MHV(- - + + + +)	NMHV (- - - + + +)
0	FA	FA
1	FA	PA $\to$ FA
2	FA	PA $\to$ FA

FA $\equiv$ full analytic results
PA $\equiv$ partial analytic results

Bern et. al. hep-th/9611127

The MHV amplitude is most naturally expressed as a linear combination of scalar four-, five-, and six-point Feynman integrals:

$$\frac{A_1^{1-\text{loop}}(1^-,2^-,3^+,4^+,5^+,6^+)}{A^{tree}(1^-,2^-,3^+,4^+,5^+,6^+)} = -\frac{1}{2}s_1 s_2 I_4^{(4,5),\ 4-2\epsilon} + \ldots$$

$$\ldots + 2i\varepsilon_{\mu\nu\rho\sigma}k_2^\mu k_3^\nu k_4^\rho k_5^\sigma\ \epsilon\ I_5^{(1),\ 6-2\epsilon} + \cdots + \frac{1}{2}\text{tr}(\not{k}_1\not{k}_2\not{k}_3\not{k}_4\not{k}_5\not{k}_6)\ \epsilon\ I_6^{6-2\epsilon}$$

All of the box (4pt) and pentagon (5pt) integrals in the MHV result are “descendants” of the basic scalar one-loop hexagon integral:

$$I_6^{d-2\epsilon} = -i(4\pi)^{2-\epsilon}\int \frac{d^{d-2\epsilon}p}{(2\pi)^{d-2\epsilon}}\frac{1}{p^2\ldots(p-\sum_{i=1}^5 k_i)^2}$$

NMHV Amplitude

- The B_i coefficients are non-trivial functions of spinor products and Mandelstam invariants Bern et. al. hep-ph/9409265
- My goal was to determine nice formulas for the NMHV G_k coefficients

$$A_1^{1-\text{loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = -\frac{1}{2} B_1 \left(s_4 s_5 I_4^{(2,3)} + \ldots + s_3 t_1 I_4^{(2,6)} \right) - \frac{1}{2} B_2 \left(s_6 s_5 I_4^{(3,4)} + \ldots + s_4 t_2 I_4^{(1,3)} \right) - \frac{1}{2} B_3 \left(s_6 s_1 I_4^{(4,5)} + \ldots + s_5 t_3 I_4^{(2,4)} \right) + G_1 \epsilon \, I_5^{(1),\, 6-2\epsilon} + G_2 \epsilon \, I_5^{(2),\, 6-2\epsilon} + G_3 \epsilon \, I_5^{(3),\, 6-2\epsilon} + G_4 \epsilon \, I_5^{(4),\, 6-2\epsilon} + G_5 \epsilon \, I_5^{(5),\, 6-2\epsilon} + G_6 \epsilon \, I_5^{(6),\, 6-2\epsilon}$$

I recently succeeded in this endeavor and found:

$$G_1 = \tfrac{1}{2}C_1(\tilde{B}_2\ y + \tilde{B}_3\ y^*), G_2 = \tfrac{1}{2}C_2(\tilde{B}_2 + \tilde{B}_3)\ y, G_3 = \tfrac{1}{2}C_3(\tilde{B}_2 + \tilde{B}_3)\ y^*$$
$$G_4 = \tfrac{1}{2}C_4(\tilde{B}_2\ y + \tilde{B}_3\ y^*), G_5 = \tfrac{1}{2}C_5(\tilde{B}_2 + \tilde{B}_3)\ y, G_6 = \tfrac{1}{2}C_6(\tilde{B}_2 + \tilde{B}_3)\ y^*$$

where

$$y = \frac{s_{12}s_{23}s_{45}s_{56}}{t_1} + \langle 1\,2\rangle[2\,3]\langle 3\,4\rangle[4\,5]\langle 5\,6\rangle[6\,1]$$

and Arkani-Hamed et. al. arXiv:0907.5418

$$\tilde{B}_2 = -\frac{i\langle 4|(2{+}3)|1]^3}{\langle 2\,3\rangle\langle 3\,4\rangle t_2\langle 2|3{+}4|5][5\,6][6\,1]} + \frac{i\langle 5\,6\rangle^3[2\,3]^3}{\langle 6\,1\rangle t_2\langle 1|(2{+}3)|4]\langle 5|3{+}4|2][3\,4]}$$
$$\tilde{B}_3 = -\frac{i\langle 6|(1{+}2)|3]^3}{\langle 6\,1\rangle\langle 1\,2\rangle t_3\langle 2|6{+}1|5][3\,4][4\,5]} + \frac{i\langle 4\,5\rangle^3[1\,2]^3}{\langle 3\,4\rangle t_3\langle 3|(1{+}2)|6]\langle 5|6{+}1|2][6\,1]}$$

How to Obtain Results For Arbitrary External States?

Drummond et. al. arXiv:0807.1095

- Avoid specializing to gluon scattering amplitudes (use on-shell superspace)
- Unify all component scattering amplitudes and combine them into a single $\mathcal{N} = 4$ *superamplitude*

Try to make an ansatz for the six-point superamplitude out of building blocks invariant under ALL symmetries in the problem

Supersymmetrized Results

A beautiful, cyclicly symmetric result!

$$\cdots + \frac{i}{6}\epsilon\, \mathcal{A}^{tree}_{6;2} \sum_{i=1}^{6} C_i \Bigg(\frac{1}{2}\left(2s_{i+1}s_{i-2} - t_i t_{i+1}\right) t_{i-1} \left(R_{i+2\,i-1\,i+1} + R_{i-1\,i+2\,i-2}\right) -$$
$$\Big([i\,i+1]\langle i+1\,i+2\rangle[i+2\,i+3]\langle i+3\,i+4\rangle[i+4\,i+5]\langle i+5\,i\rangle - \langle i\,i+1\rangle \times$$
$$[i+1\,i+2]\langle i+2\,i+3\rangle[i+3\,i+4]\langle i+4\,i+5\rangle[i+5\,i]\Big) \left(R_{i+2\,i-1\,i+1} - R_{i-1\,i+2\,i-2}\right)$$
$$+\frac{1}{2}\left(2s_{i-1}s_{i+2} - t_{i-1}t_{i+1}\right) t_i \left(R_{i+3\,i\,i+2} + R_{i\,i+3\,i-1}\right) + \frac{1}{2}\left(2s_{i+3}s_i - t_i t_{i-1}\right) t_{i+1} \times$$
$$\left(R_{i+1\,i-2\,i} + R_{i-2\,i+1\,i-3}\right) - \Big([i\,i+1]\langle i+1\,i+2\rangle[i+2\,i+3]\langle i+3\,i+4\rangle \times$$
$$[i+4\,i+5]\langle i+5\,i\rangle - \langle i\,i+1\rangle[i+1\,i+2]\langle i+2\,i+3\rangle[i+3\,i+4]\langle i+4\,i+5\rangle \times$$
$$[i+5\,i]\Big) \left(R_{i+1\,i-2\,i} - R_{i-2\,i+1\,i-3}\right) \Bigg) I_5^{(i),\, D=6-2\epsilon}$$

A Mysterious Relation

Stieberger et. al. hep-th/0609175, arXiv:0711.4354

- At tree-level in open superstring theory compactified from 10 to 4 dimensions, all scattering amplitudes with gluonic external states, both MHV and NMHV, are known through six-points
- Stieberger and Taylor noticed a remarkable connection between their results at the MHV level and analogous dimensionally shifted one-loop $\mathcal{N} = 4$ amplitudes: Bern et. al. hep-th/9611127

$$A^{tree}_{str}(1^-, 2^-, 3^+, .., 6^+)\Big|_{\mathcal{O}(\alpha'^2)} = -6\zeta(2) A_1^{1-\text{loop}}(1^-, 2^-, 3^+, .., 6^+)\Big|_{\epsilon \to \epsilon - 2}$$

Two obvious questions:

Does the relation generalize beyond the MHV level?
Does the relation generalize beyond $\mathcal{O}(\alpha'^2)$?

From One-Loop $\mathcal{N}=4$ Super Yang-Mills to Tree-Level Open Superstring Theory

Dimensionally shifted ($4\to 8$ or $4\to 10$) $\mathcal{N}=4$ amplitudes are related to $\left(\mathcal{O}(\alpha'^2) \text{ or } \mathcal{O}(\alpha'^3)\right)$ gluon open superstring amplitudes for all helicity configurations! RMS *to appear*

- $\mathcal{N}=4$ amplitudes develop UV divergences when considered in higher dimensions and the $d=8$ and $d=10$ counterterm Lagrangians are fixed by $\mathcal{N}=4$ SUSY (but not $d=12$)

  Dunbar et. al. hep-th/0203104

- The non-Abelian Born-Infeld action is fixed by $\mathcal{N}=4$ SUSY at $\mathcal{O}(\alpha'^2)$ and at $\mathcal{O}(\alpha'^3)$ (but not $\mathcal{O}(\alpha'^4)$) Bilal hep-th/0106062, Koerber et. al. hep-th/0108169, Collinucci et. al. hep-th/0205150, Drummond et. al. hep-th/0305202

$$A^{tree}_{str}(1,2,...,n)\Big|_{\mathcal{O}(\alpha'^3)} = 60\zeta(3)A_1^{1-\text{loop}}(1,2,...,n)\Big|_{\epsilon\to\epsilon-3}$$

Future Directions

- In spite of much progress, two loops is still mostly uncharted territory in gauge theories with $\mathcal{N} < 4$. What can we learn from $\mathcal{N} = 4$?
- What about the moduli space of $\mathcal{N} = 4$? (*i.e.* giving a vacuum expectation value (VEV) to some number of the six scalars)

  Alday et. al. arXiv:0908.0684

  Known examples look very similar to all-orders results in dimensional regularization.

For example, consider

$$A_{str}^{tree}(1, 2, ..., n)\Big|_{\mathcal{O}(\alpha'^2)} = -6\zeta(2)A_1^{1-\text{loop}}(1, 2, ..., n)\Big|_{\epsilon\to\epsilon-2}$$

Field theory side maps to large VEV limit of amplitudes in $D = 4$:

$$\frac{1}{v^4} \to \alpha'^2$$