Pion Condensation in Holographic QCD

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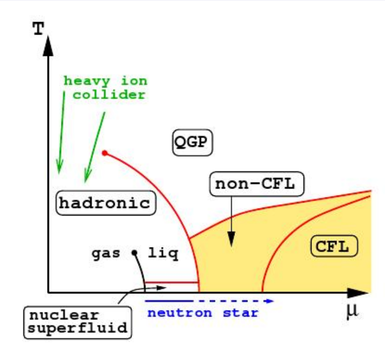
Outline of Talk

1. Motivation.
2. Chiral Lagrangian + Isospin Chemical Potential
3. Holographic QCD + Isospin Chemical Potential
4. Boundary Conditions
5. Conclusions.

Motivation:

QCD phase diagram



(M.G.Alford *et al. 2008*)

- Neutron stars
  - Low temperature, large baryon and isospin number density.
- RHIC & LHC - ALICE
  - High temperature, nonzero baryon and isospin density.

Motivation:

Why isospin chemical potential (μ_I)?
Two main reasons:

- Isospin asymmetric matter exists!
- Can compare to Lattice calculations.

Motivation:

- Son and Stephanov (2000): Chiral Lagrangian with μ_I – pions condense; phase transition is second order (@ $T = 0$).
  $\rightarrow$ based on symmetries.
- Kim, Kim and Lee (2007): No pion condensation in holographic QCD with μ_I.
  - But we expect the chiral Lagrangian is the low energy theory of holographic QCD.
- D.A. and Erlich (2010): Found pion condensation in holographic QCD, but with a first order phase transition.

Chemical Potential

Conserved charge with associated operator N ($[N, H] = 0$).

$$Z = \mathrm{Tr}\left[e^{-\left(\frac{H-\mu N}{T}\right)}\right]$$

For baryon number, the symmetry is $U(1)$. $N_B = \int d^3x\bar{\psi}\gamma^t\psi$.
If we gauge the symmetry, then $\mathscr{L} \supset \bar{\psi}\gamma^t\psi A_t$.

$\Rightarrow$ to add μ_B we change

$$\partial_t \to \partial_t + i\mu_B.$$

Chiral Lagrangian + μ_I

The pattern of symmetry breaking: $SU(2)_L \times SU(2)_R \to SU(2)_V$.
The pions, $\Sigma = \exp[2i\pi^a T^a / f_\pi^2]$, transform as $\Sigma \to U_L \Sigma U_R^\dagger$.

The chiral Lagrangian + isospin chemical potential:

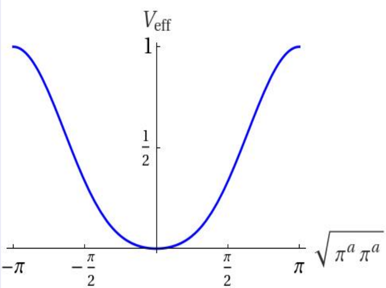
$$\mathscr{L}_{\text{eff}} = \frac{f_\pi^2}{4} \text{Tr}\left(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger\right) + \frac{m_\pi^2 f_\pi^2}{4} \text{Tr}\left(\Sigma + \Sigma^\dagger\right)$$

where $\nabla_t \Sigma = \partial_t \Sigma - i\frac{\mu_I}{2}\left[\sigma^3, \Sigma\right]$ and $\nabla_i = \partial_i$.

(Son & Stephanov, 2000).

Chiral Lagrangian + μ_I

$\mu_I < m_\pi$



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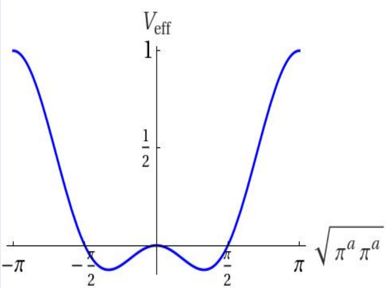

4D Results

Summary of the results (Son & Stephanov, 2000):

- Pions condense when $\mu_I > m_\pi$:

$$\langle \pi^a \pi^a \rangle \simeq 2 f_\pi^2 \left(1 - \frac{m_\pi^4}{\mu_I^4} \right)$$

  assuming $\pi^a \pi^a$ is small.
- Number density

$$n_I = \mu_I f_\pi^2 \left(1 - \frac{m_\pi^4}{\mu_I^4} \right).$$

AdS/CFT

Recipe for model building:

AdS	$\leftrightarrow$	CFT
Fields	$\leftrightarrow$	Operators
Gauge fields in bulk	$\leftrightarrow$	Global Symmetry
KK modes in bulk	$\leftrightarrow$	States of CFT

Holographic QCD

We start with $SU(2)_L \times SU(2)_R$ gauge theory with bifundamental field X:

$$S = \int d^5x \sqrt{g}\,\mathrm{Tr}\left\{ |DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}\left(F_L^2 + F_R^2\right) \right\},$$

where X is dual to $\bar{q}q$.

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A slice of AdS space:

$$ds^2 = \frac{1}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right), \qquad \epsilon \le z \le z_m,$$

where ϵ plays the role of UV cutoff and z_m cuts off the geometry – modeling confinement.

Holographic QCD

Pattern of chiral symmetry breaking: m_q provides explicit breaking, $\langle \bar{q}q \rangle$ spontaneous breaking $\Rightarrow$

$$X_0 = \frac{1}{2}\left(m_q z + \sigma z^3\right).$$

Holographic QCD

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$$X_0 = \frac{1}{2}\left(m_q z + \sigma z^3\right).$$

Similarly, V_μ^a is dual to J_μ^a.
Since μ_I sources number density, $\Rightarrow$

$$V_t^3 = \mu_I.$$

Enter the Pions

We add the pion fluctuations

$$X(x,z) = (X_0 + S(x,z))e^{i2\pi^a(x,z)T^a},$$

where S is the scalar and π^a are the pions.

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KK decomposition: $\pi^a(x,z) = \sum \pi_n^a(x)\psi_n(z)$, and similar for $S(x,z)$.
We integrate out heavy physics, assuming

$$\pi^a(x,z) \to \pi^a(x)\psi(z), \quad \textbf{and} \quad S(x,z) \to 0.$$

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Then

$$X(x,z) \to X_0 e^{i2\pi^a(x)\psi(z)T^a}.$$

Enter the Pions

We impose a consistent set of boundary conditions. $\Rightarrow$
A good Sturm-Liouville problem.

GOR relation:

$$m_\pi^2 f_\pi^2 = 2 m_q \sigma$$

$\rightarrow$ We expect the 4D effective theory to be the chiral Lagrangian.

Comparing to the Chiral Lagrangian

The effective 4D Lagrangian:

$$\mathscr{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - \frac{1}{2}\left(m_\pi^2 - \mu_I^2\right)\left(\pi^1\pi^1 + \pi^2\pi^2\right) - \frac{1}{2}m_\pi^2 \pi^3\pi^3 + \mu_I\left(\partial_t \pi^1 \pi^2 - \partial_t \pi^2 \pi^1\right).$$

Going to quartic order in $\pi^a \Rightarrow \boxed{\langle \pi^a \pi^a \rangle \;\&\; n_I}$:

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Going to quartic order in $\pi^a \Rightarrow \boxed{\langle \pi^a \pi^a \rangle \ \& \ n_I}$:

From 5D

$$\langle \pi^a \pi^a \rangle = \frac{3}{4} 2 f_\pi^2 \left(1 - \frac{m_\pi^4}{\mu_I^4}\right)$$

Standard 4D

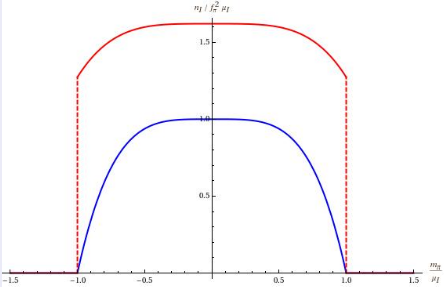
$$\langle \pi^a \pi^a \rangle = 2 f_\pi^2 \left(1 - \frac{m_\pi^4}{\mu_I^4}\right)$$

Comparing to the Chiral Lagrangian

Including the coupling to the gauge fields:

$$n_I = \mu_I f_\pi^2 \frac{1}{\eta}\frac{3}{4}\left(\alpha^2 - \frac{m_\pi^4}{\mu_I^4}\right) \qquad v.s. \qquad n_I = \mu_I f_\pi^2 \left(1 - \frac{m_\pi^4}{\mu_I^4}\right)$$

$n_I/\mu_I f_\pi^2$ v.s. m_π/μ_I



An Important Piont

Looking at the z-derivative term of the 5D Lagrangian,

$$\mathcal{L} \supset \partial_z X \partial_z X^\dagger.$$

If $X \to \exp\left[2i\pi^a(x)\psi(z)T^a\right]$,

$$\partial_z X \partial_z X^\dagger \to (\partial_z \psi(z))^2 \pi^a(x)\pi^a(x)$$

with no higher order pion terms.

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Appears as though the 4D effective theory is

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$\to$ new form for X? Let's try $X = \frac{m_q z}{2} + (\frac{\sigma z^3}{2} + S)e^{2i\pi}$.

New X

With $X = \frac{m_q z}{2} + \left(\frac{\sigma z^3}{2}\right) e^{2i\pi}$, the action has the form:

$$I = \int d^5x \, \mathrm{Tr}\left\{\frac{\sigma^2 z^3}{4}(\partial_\mu U \partial^\mu U^\dagger - \partial_z U \partial_z U^\dagger)\right\} - \int d^4x \, \mathrm{Tr}\left\{\frac{m_q \sigma}{4}(U + U^\dagger)\big|_{z_m}\right\},$$

where $U = e^{2i\pi}$.

- Boundary term looks like the chiral Lagrangian mass term.

Deriving GOR in similar fashion to before gives

$$m_\pi^2 f_\pi^2 = -m_q \sigma.$$

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Deriving GOR in similar fashion to before gives

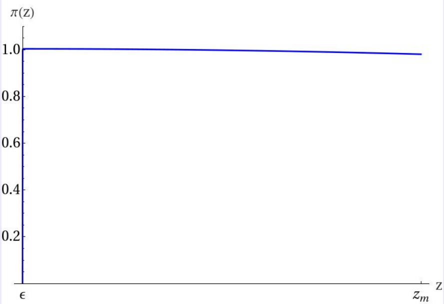
$$m_\pi^2 f_\pi^2 = -m_q \sigma.$$

$\Rightarrow$ we should take $m_q \to -2m_q$.

New X $(m_q \to -2m_q)$

It turns out that $\psi(z) \approx 1$ over the entire interval:

$\psi(z)$



New X ($m_q \to -2m_q$)

Changing to $X = -m_q z + (\frac{\sigma z^3}{2})e^{2i\pi}$, evaluating the action on the linearized EOMs, and performing the z-integrals we get

$$I = \int d^5x \, \mathrm{Tr}\left\{ \frac{\sigma^2 z^3}{4} (\partial_\mu U \partial^\mu U^\dagger - \partial_z U \partial_z U^\dagger) \right\} + \int d^4x \, \mathrm{Tr}\left\{ \frac{m_q \sigma}{2} (U + U^\dagger)\Big|_{z_m} \right\}.$$

$$\Downarrow$$

$$I = \int d^4x \, \mathrm{Tr}\left\{ \frac{f_\pi^2}{4} \partial_\mu U \partial^\mu U^\dagger + \frac{m_\pi^2 f_\pi^2}{4} (U + U^\dagger) \right\}.$$

But why do we need to change the background? (BCs)

An Example

Consider the Lagrangian:

$$\mathscr{L} = \frac{1}{2}\partial_N \phi_1 \partial^N \phi_1 + \frac{1}{2}\partial_N \phi_2 \partial^N \phi_2 - \frac{1}{2} M^2 (\phi_1^2 + \phi_2^2),$$

with boundary conditions in the compact direction

$$\delta\phi_1 = 0, \quad \textbf{and} \qquad \partial_z \phi_2 = 0.$$

Transforming the fields $\phi_\pm = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2) \Rightarrow$

$$\delta\phi_+ = -\delta\phi_-, \quad \textbf{and} \qquad \partial_z \phi_+ = \partial_z \phi_-.$$

What We've Found

Compare the two:

$$X = \left(\frac{1}{2}(m_q z + \sigma z^3) + \tilde{S}\right) e^{2i\tilde{\pi}} \quad \text{v.s.} \quad X = \frac{m_q z}{2} + (\frac{\sigma z^3}{2} + S)e^{2i\pi}.$$

- Go from $(S, \pi) \to \left(\tilde{S}(S,\pi), \tilde{\pi}(S,\pi)\right)$.
- Different boundary conditions – nonlinear, mixing fields.
  - E.g. $\delta S = 0 \Rightarrow \delta\tilde{S} = -\delta f(\tilde{S}, \tilde{\pi})$.
- With these BCs, we need to change the background for both $\mathscr{L}(\tilde{S}, \tilde{\pi})$ and $\mathscr{L}(S, \pi)$; $m_q \to -2m_q$.
- And expect full symmetry is manifest in both forms for X.

What We've Found

Compare the two:

$X = \left(\frac{1}{2}(m_q z + \sigma z^3) + \tilde{S}\right) e^{2i\tilde{\pi}}$ v.s. $X = \frac{m_q z}{2} + (\frac{\sigma z^3}{2} + S)e^{2i\pi}$.

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- With these BCs, we need to change the background for both $\mathscr{L}(\tilde{S}, \tilde{\pi})$ and $\mathscr{L}(S, \pi)$; $m_q \to -2m_q$.
- And expect full symmetry is manifest in both forms for X.

$\Rightarrow$ Choose $X = -m_q z + (\frac{\sigma z^3}{2} + S)e^{2i\pi}$.

To Conclude

- Original form (or actually any form) for X with some consistent set of boundary conditions will not always match to QCD.
- Holographic QCD matches on to QCD with a particular choice of non-obvious, nonlinear boundary conditions.
- New form for X works in a more transparent way.