Comments on Emittance Calculations

"Accelerator physics—a field where often work of the highest quality is buried in lost technical notes or even not published."

—Etienne Forest, J. Phys. A: Math. Gen. **39**, 5321 (2006)

http://puhep1.princeton.edu/~mcdonald/examples/accel/forest_jpa_39_5321_06.pdf

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Overview

A major challenge of a muon collider is "cooling" of the muon beam = reduction of its volume in 6-d phase space.

If/when we succeed in devising a sound concept for this, we will surely know it.

Along the way, we need to evaluate our conceptual progress, for which estimates of 6-d phase volume are helpful.

This leads to several general questions:

- What is phase space? What coordinates can/should we use to describe it?
- How should we account for effects of electromagnetic fields on the beam?
- Under what kinds of beam manipulations is phase volume invariant?
- How can we estimate phase volume numerically?
- Can we describe the evolution of phase volume from the initial pion beam to the decay muon beam?

Hamiltonian Phase Space

The best succinct reference is Chap. 8 of *Mechanics* by Landau and Lifshitz.

The concept of phase space arises in the context of Hamiltonian dynamics, where a particle in 3-space is described by 3 "spatial" coordinates, q_1, q_2, q_3 and their conjugate momenta p_1, p_2, p_3 and an independent variable I will first call t. The equations of motion are

$$\frac{dq_i}{dt} = \frac{\partial H_t}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H_t}{\partial q_i}, \qquad H_t = \mathcal{L}_t - \sum_i q_i p_i, \qquad p_i = \frac{\partial \mathcal{L}_t}{\partial \dot{q}_i},$$

where $\mathcal{L}_t(q_1,p_1,q_2,p_2,q_3,p_3)$ is the Lagrangian and $H_t(q_1,p_1,q_2,p_2,q_3,p_3)$ is the Hamiltonian of the system.

Phase space is the space of the (canonical) coordinates, $(q_1,p_1,q_2,p_2,q_3,p_3)$.

For a particle of mass m and charge e in an electromagnetic field that can be deduced from a scalar potential V and a vector potential $\mathbf{A}$ (in some gauge) , the Lagrangian is

$$\mathcal{L}_t = -mc^2\sqrt{1 - v^2/c^2} + \frac{e}{c}\mathbf{v}\cdot\mathbf{A} - eV, \qquad \text{so,} \qquad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + \frac{e}{c}\mathbf{A} = \mathbf{p}_{\rm mech} + \frac{e}{c}\mathbf{A},$$

$$H_t = c\sqrt{m^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2} + eV = c\sqrt{m^2c^2 + \mathbf{p}_{\rm mech}^2} + eV = E_{\rm mech} + eV = E,$$

$$\frac{dp_i}{dt} = -\frac{\partial H_t}{\partial x_i} = e\sum_j \frac{v_j}{c}\frac{\partial A_j}{\partial x_i} - e\frac{\partial V}{\partial x_i} = \frac{dp_{{\rm mech},i}}{dt} + \frac{e}{c}\frac{dA_i}{dt} = \frac{dp_{{\rm mech},i}}{dt} + \frac{e}{c}\frac{\partial A_i}{\partial t} + e\sum_j \frac{v_j}{c}\frac{\partial A_i}{\partial x_j},$$

$$\frac{dp_{{\rm mech},i}}{dt} = e - \left[\frac{\partial V}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} + e\sum_j \frac{v_j}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\right] = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)_i = F_{{\rm Lorentz},i}.$$

Use of z as the Independent Variable

Along the beamline, we measure particles at fixed position, say z, rather than at a fixed time t. So it would be preferable to have a formalism in which z, rather than t, is the independent variable. This was considered by Courant and Snyder, Ann. Phys. (NY) **3**, 1 (1958), Appendix B.

http://puhep1.princeton.edu/~mcdonald/examples/accel/courant_ap_3_1_58.pdf

It turns out that if we take the momentum conjugate to coordinate t as $p_t = -H_t = -E = -E_{\rm mech} - eV$, then the system is described by the Hamiltonian H_z,

$$H_z = -p_z = -p_{{\rm mech},z} - \frac{e}{c}A_z = -\sqrt{\frac{E_{\rm mech}^2}{c^2} - m^2c^2 - p_{{\rm mech},x}^2 - p_{{\rm mech},y}^2} - \frac{e}{c}A_z$$

$$= -\sqrt{\frac{(p_t + eV)^2}{c^2} - m^2c^2 - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_y - \frac{e}{c}A_y\right)^2} - \frac{e}{c}A_z.$$

For what it's worth, the equation of motion for p_t can be rewritten as

$$\frac{dE_{\rm mech}}{dz} = \frac{e}{v_z}\mathbf{v}\cdot\left(-\nabla V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = \frac{e}{v_z}\mathbf{v}\cdot\mathbf{E} = \frac{\mathbf{F}_{\rm Lorentz}\cdot\mathbf{v}}{v_z}.$$

The transformation from coordinates $(x,p_x,y,p_y,,z,p_z)$ to $(x,p_x,y,p_y,,t,p_t)$ is a canonical transformation (*i.e.*, from one set of canonical coordinates to another, such that a Hamiltonian exists for both sets of coordinates).

Of course, evolution in time under Hamiltonian H_t, or evolution in z under Hamiltonians H_z, is also a canonical transformation.

http://puhep1.princeton.edu/~mcdonald/examples/hamiltonian.pdf

Liouville's Theorem

A famous theorem, attributed to Liouville, is that Hamiltonian phase volume is invariant under canonical transformations. *Liouville actually knew nothing about Hamiltonians or phase space. See D. Nolte, The Tangled Tale of Phase Space, Physics Today, **63**, no. 4, 32 (2010),*
http://puhep1.princeton.edu/~mcdonald/examples/mechanics/nolte_pt_63_4_32_10.pdf

A consequence of Liouville's theorem is that phase volume is invariant under evolution in time of a Hamiltonian system.

Similarly, phase volume is invariant under evolution in z of a Hamiltonian system, if z is used as the independent variable.

Since the transformation from t to z as the independent variable is a canonical transformation, phase volume is the same in either coordinates $(x,p_x,y,p_y,,z,p_z)$ or $(x,p_x,y,p_y,,t,p_t)$.

Also, a gauge transformation is a canonical transformation, so phase volume is gauge invariant.

A corollary of Liouville's theorem is that the sums of subvolumes, $dq_1dp_1 + dq_2dp_2 + dq_3dp_3$ and $dq_1\, dp_1\, dq_2\, dp_2 + dq_3\, dp_3$, are also invariant under canonical transformations.

For a beam of n particles, Liouville's theorem applies to the $6n$-dimensional phase space if particle interactions are considered (and a Hamiltonian for the entire system exists), while if the particles are considered to be noninteracting, it applies to the set of n particles in 6-d phase space.

Swann's Theorem

A lesser known theorem is due to W.F.G. Swann, Phys. Rev. 44, 224 (1933), in what is probably the first paper ever to apply Liouville's theorem to a "beam" of charged particles,

http://puhep1.princeton.edu/~mcdonald/examples/accel/swann_pr_44_224_33.pdf

Swann's theorem states the phase volume is the same whether one uses the canonical coordinates (x,p_x,y,p_y,z,p_z) or the more intuitive coordinates $(x,p_{\mathrm{mech},x},y,p_{\mathrm{mech},y},z,p_{\mathrm{mech},z})$.

Similarly, phase volume is the same whether one uses the canonical coordinates $(x,p_x,y,p_y,t,-E)$ or the coordinates $(x,p_{\mathrm{mech},x},y,p_{\mathrm{mech},y},t,-E_{\mathrm{mech}})$.

Thus, we have the freedom to describe our beam in 4 different coordinate systems, and to use any gauge whatsoever, and the phase volume of the beam will be the same (if the beam can be described by a Hamiltonian and the particles are noninteracting).

In practice it is not easy to calculate the phase volume associated with a bunch of particles. We use some numerical approximation. Clearly, we desire to use that coordinate system, and that gauge, for which our numerical approximation to phase volume is the best.

There seems to be no theorem that explains what is the best strategy to deal with this issue.

RMS "Invariant" Emittance

Our estimate of the phase volume of the bunch is the rms "invariant" emittance,

$$\varepsilon_6 = \frac{\sqrt[6]{\det(\Sigma_{123})}}{m}, \qquad \Sigma_{123,kl} = \langle \Delta x_k \Delta x_l \rangle, \qquad \Delta x_k = x_k - \langle \Delta x_k \rangle, \qquad x_k = (q_1, p_1, q_2, p_2, q_3, p_3).$$

If motion in different indices i is decoupled, we consider the subemittances,

$$\varepsilon_i = \frac{\sqrt{\det(\Sigma_i)}}{m} = \frac{\sqrt{\langle \Delta x_i^2 \rangle \langle \Delta p_i^2 \rangle - \langle \Delta x_i \Delta p_i \rangle^2}}{m}, \quad \Sigma_{i,kl} = \langle \Delta x_k \Delta x_l \rangle, \qquad x_k = (q_i, p_i),$$

$$\varepsilon_\perp = \frac{\sqrt[4]{\det(\Sigma_{xy})}}{m}, \quad \Sigma_{xy,kl} = \langle \Delta x_k \Delta x_l \rangle, \qquad x_k = (q_x, p_x, q_y, p_y).$$

These "invariant" emittances are actually invariant only under "linear" (canonical) transformations.

Unfortunately, propagation of a beam across a field-free drift region is "nonlinear" (even though the particles move along straight lines).

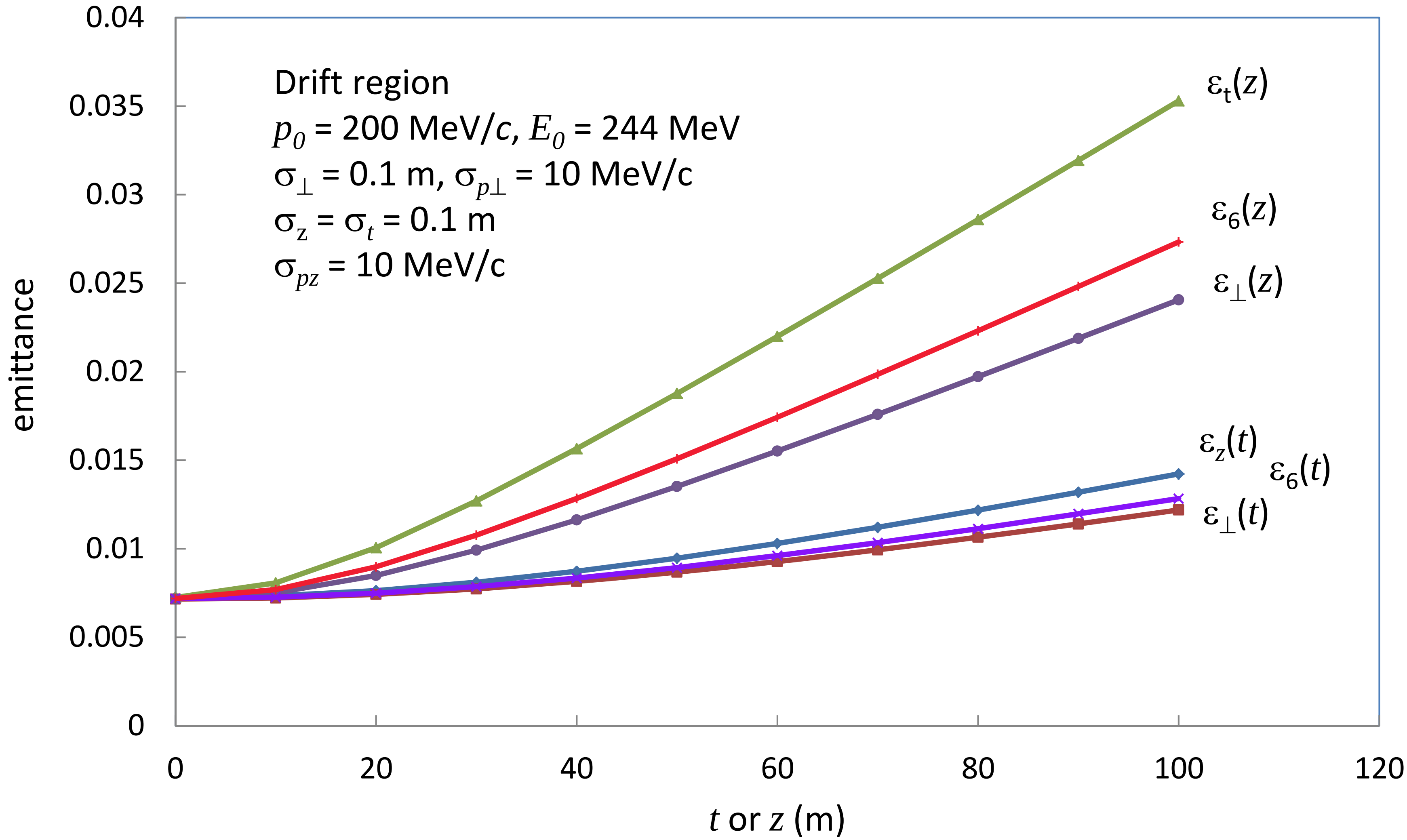
For a beam with $\langle p_z \rangle = p_0$ and initial rms quantities $\sigma_\perp$, $\sigma_{p\perp}$, σ_z, σ_{pz}, the emittances vary with time as

$$\varepsilon_\perp(t) \approx \sqrt{\varepsilon_\perp^2(0) + \frac{c^8 \sigma_{p_\perp}^4 t^2}{m^2 E_0^6}\left(4\sigma_{p_\perp}^4 + 2\sigma_{p_\perp}^2 p_0^2 + \sigma_{p_z}^2 p_0^2 + \frac{\sigma_{p_z}^4}{2}\right)},$$

$$\varepsilon_z(t) \approx \sqrt{\varepsilon_z^2(0) + \frac{c^8 t^2}{m^2 E_0^6}\left(\sigma_{p_\perp}^4 \sigma_{p_z}^2 \left(p_0^2 + \sigma_{p_z}^2\right) + \sigma_{p_z}^6 \frac{15 p_0^2 + 3\sigma_{p_z}^2}{2}\right)}.$$

http://puhep1.princeton.edu/~mcdonald/examples/growth.pdf

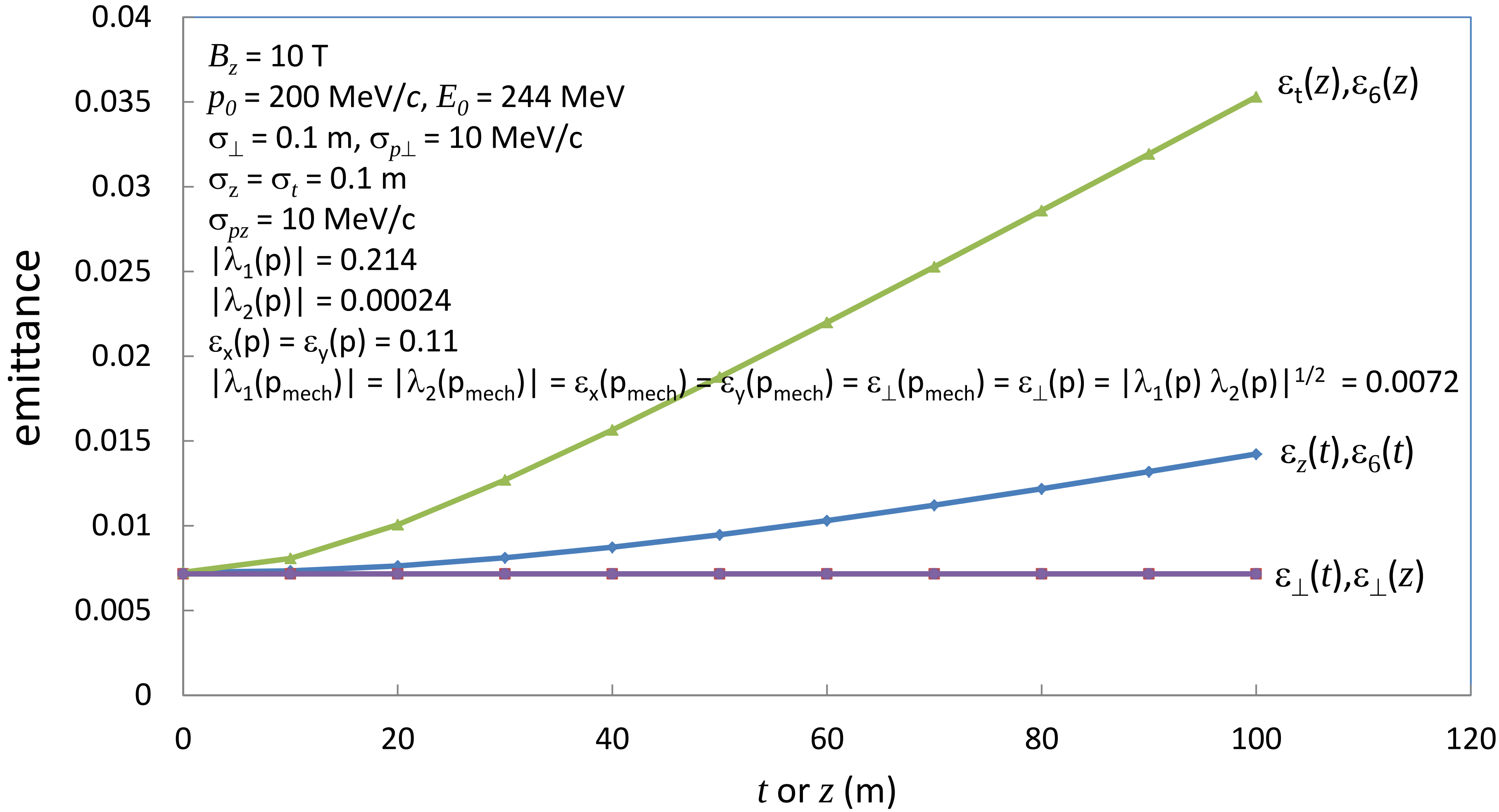
Emittance Growth for Pions in a Drift Region



The emittances grow quadratically with t or z, and the emittances with z as the independent variable grow more rapidly than those with t as the independent variable.

The integrations in t and z were analytic in this and the next slide.

Stabilization of Transverse Emittance by an Axial Magnetic Field



The transverse emittance is completely stabilized by an axial magnetic field. It makes no difference whether canonical momentum $\boldsymbol{p}$ or mechanical momentum $\boldsymbol{p}_{\text{mech}}$ is used in the calculation of $\varepsilon_\perp$, although $\varepsilon_x = \varepsilon_y$ are large when using $\boldsymbol{p}$ (but do not grow with t or z).

Is this effect documented in the literature?

Eigenemittances aka Courant-Snyder Invariant Eigenvalues

A. Dragt argues that we learn more if we calculate the so-called invariant eigenemittances.

These are the absolute values of the 3 distinct eigenvalues, λ_1, λ_2, λ_3, of the matrix

$$J\Sigma_{xyz} = \begin{pmatrix} \langle \Delta x \Delta p_x \rangle & \langle \Delta p_x^2 \rangle & \langle \Delta y \Delta p_x \rangle & \langle \Delta p_x \Delta p_y \rangle & \langle \Delta z \Delta p_x \rangle & \langle \Delta p_x \Delta p_z \rangle \\ -\langle \Delta x^2 \rangle & -\langle \Delta x \Delta p_x \rangle & -\langle \Delta x \Delta y \rangle & -\langle \Delta x \Delta p_y \rangle & -\langle \Delta x \Delta z \rangle & \langle \Delta x \Delta p_z \rangle \\ \langle \Delta x \Delta p_y \rangle & \langle \Delta p_x \Delta p_y \rangle & \langle \Delta y \Delta p_y \rangle & \langle \Delta p_y^2 \rangle & \langle \Delta z \Delta p_y \rangle & \langle \Delta p_y \Delta p_z \rangle \\ -\langle \Delta x \Delta y \rangle & -\langle \Delta y \Delta p_x \rangle & -\langle \Delta y^2 \rangle & -\langle \Delta y \Delta p_y \rangle & -\langle \Delta y \Delta z \rangle & \langle \Delta y \Delta p_z \rangle \\ \langle \Delta x \Delta p_z \rangle & \langle \Delta p_x \Delta p_z \rangle & \langle \Delta y \Delta p_z \rangle & \langle \Delta p_y \Delta p_z \rangle & \langle \Delta z \Delta p_z \rangle & \langle \Delta p_z^2 \rangle \\ -\langle \Delta x \Delta z \rangle & -\langle \Delta z \Delta p_x \rangle & -\langle \Delta y \Delta z \rangle & -\langle \Delta z \Delta p_y \rangle & -\langle \Delta z^2 \rangle & -\langle \Delta z \Delta p_z \rangle \end{pmatrix}.$$

Any function of the λ_1, λ_2, λ_3 is also invariant under "linear" transformations,

Examples: $$\varepsilon_6 = \frac{\sqrt{|\lambda_1 \lambda_2 \lambda_3|}}{m} = \frac{\sqrt[6]{\det(\Sigma_{xyz})}}{m}, \qquad \frac{\sqrt{|\lambda_1 \lambda_2|}}{m}, \qquad \text{and} \qquad \frac{\sqrt{|\lambda_3|}}{m}.$$

If the x-y and z motions are decoupled, the method of eigenemittances reveals that

$$\varepsilon_\perp = \frac{\sqrt[4]{\det(\Sigma_{xy})}}{m} = \frac{\sqrt{|\lambda_1 \lambda_2|}}{m}, \qquad \text{and} \qquad \varepsilon_z = \frac{\sqrt{\det(\Sigma_z)}}{m} = \frac{\sqrt{|\lambda_3|}}{m}.$$

are invariant under "linear" transformations. *($|\lambda_1|$ and $|\lambda_2|$ are listed on slide 9.)*

Even if x, y and z are coupled, there is no "emittance exchange" between $\varepsilon_\perp$ and ε_z under "linear" transformations, if the emittances are defined in terms of eigenemittances.

Perhaps we should check for "cooling" of the λ_1, λ_2, λ_3 as well as of the emittances.

A Beam of Pions and Muons

Before we have a muon beam we have a pion beam.

Presumably, the pion beam has a phase volume/emittance which has some relation to the phase volume/emittance of the muon beam it decays into.

To date, we largely ignore the phase volume/emittance of the pion beam, although this can be manipulated in the target/decay region. Indeed, the magnetic taper from 20 down to 1.5 T provides a coupling of longitudinal and transverse phase space.

The decay of pions to muons is not describable by a Hamiltonian, and phase volume is altered during the decay.

Somewhat unintuitively, the decay "heats" rather than "cools" the phase volume although energy is lost during the decay.

For example, the decay of the pion bunch considered on slides 8 and 9 roughly triples the transverse and longitudinal emittances, both in zero field and in 10-T field. *However, the initial emittance of this bunch is smaller than that we will consider for a muon collider.*

It is an open question whether there could be favorable coupling between a tapering magnetic field in the decay region and the unwanted emittance growth during $\pi \to \mu$ decay.

B. Autin made some comments on emittance growth during $\pi \to \mu$ decay in
http://puhep1.princeton.edu/~mcdonald/examples/accel/autin_nim_a503_363_03.pdf

Emittance Calculations Including RF Cavities

Our beamline includes rf accelerating cavities, and we may wish to perform emittance calculations for transport through these cavities.

If we use canonical coordinates in the emittance calculations, we need to know the scalar and vector potentials of an rf cavity, which means choosing a gauge.

In the Lorenz gauge (and also in the Coulomb gauges and in the Poincaré gauge) the potentials are nonzero outside a closed cavity where the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are zero.

It may be preferable to use the Hamiltonian gauge, in which the scalar potential is zero everywhere, and the vector potential is (for time dependence $e^{-i\omega t}$ and wave number $k = \omega/c$) simply

$$\mathbf{A} = -\frac{i}{k}\mathbf{E}.$$

This vector potential is zero where $\boldsymbol{E}$ is zero.

http://puhep1.princeton.edu/~mcdonald/examples/cylindrical.pdf

http://puhep1.princeton.edu/~mcdonald/examples/EM/jackson_ajp_70_917_02.pdf

In the summer of 1987, while simulating transverse emittance in the first BNL rf gun, I found that the numerical results were more stable when the vector potential (Hamiltonian gauge) was included in the momentum:

http://puhep1.princeton.edu/~mcdonald/atf/four_cavity_studies.pdf

http://puhep1.princeton.edu/~mcdonald/atf/vector.pdf