Eigen-Emittance Basics
+
some thoughts on symplectic methods

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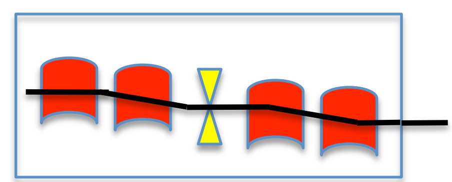
What are eigen-emittances?

- Eigen-emittances, λ_j , are the generalization of the usual rms emittances, ε_j , to systems where there may be correlations between the phase space planes.
- How do eigen-emittances differ from rms emitttances?
  - Eigen-emittances are derived from beam 2nd moment matrix, Σ. Namely, the eigenvalues of $J\Sigma$ are $\pm i\ \lambda_j$
  - mean squared emittances, ε_i^2 = determinant of the 2x2 submatrices of Σ, i.e., ε_i^2, = $\langle q_i^2\rangle\langle p_i^2\rangle - \langle q_i p_i\rangle^2$

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

- If there are no correlations (or if they are removed at some location in a beamline), the eigen-emittances *are* the rms emittances
- Eigen-emittances are invariant under linear symplectic transformations, but they can be *exchanged* among the phase planes, i.e., they are not tied to a specific plane
  - Emittance exchangers exemplify this

$$M_{eex} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{bmatrix}$$



Are eigen-emittances important?

- It is easier to ask, “when are they not important?” Answer:
  - if rms emittance is not important, or
  - if the evolving beam has no (or weak) correlations among the phase planes
- Most accelerator design problems involve producing a beam with certain properties, usually defined by the rms emittances, at certain locations (interaction region in a collider, wiggler entrance in a light source,…)
  - In these situations, if there is strong coupling among the phase planes, then eigen-emittances are an essential design tool.
- Ignoring non-Hamiltonian effects, eigen-emittances tell the designer, at a given location in the beamline, what it is possible to achieve (in regard to rms emittance) elsewhere in the beamline, in the linear approximation.

eigen-emittances vs determinant of Σ

- The product of the eigen-emittances = det(Σ)
- If we only care about det(Σ), we don't need to compute eigen-emittances, we can just compute the determinant
- Why should we care about more than det(Σ)?
  - often a design requires optimizing *certain* rms emittances, e.g., small *transverse* emittance in a linac for a light source
  - in such cases one needs to know the eigen-emittances, not just det(Σ)
  - Even when we know det(Σ) – think of it as describing an ellipsoid in 6D phase space – symplectic dynamics does not allow the ellipsoid to be transformed into an arbitrary ellipsoid of equal volume (Gromov's theorem). Can't "turn a symplectic cigar into a symplectic ball." See Ch 6 of Alex Dragt's textbook, downloadable from http://www.physics.umd.edu/dsat/
    - in other words, even if we know det(Σ), we can't hope to stretch and squeeze the phase space ellipse arbitrarily while keeping det(Σ) constant. The eigen-emittances must be preserved in the linear approximation. Det(Σ) doesn't tell the whole story of what can be accomplished.

What about highly nonlinear beamlines?

- Even in nonlinear beamlines, accelerator designers have a concept of what they would like the particle beam to do
  - As a first step in the design, one tries to achieve this with linear beam optics
  - The eigen-emittances are conserved quantities that describe what can be achieved by the designer in this approximation.
- Having produced a linear design, one can perform nonlinear tracking studies to see the importance of nonlinear effects
- It's also possible to do nonlinear design to cancel or minimize certain nonlinear effects (.e.g. sextupoles to change chromaticity in a ring)
- In principle one could perform numerical optimization to minimize some target function involving the eigen-emittances

Should we use symplectic methods?

- Symplectic methods are the mainstay of circular machine design
  - Symplectic does not equate to high accuracy, but since they “make phase errors” they are ideally suited to computing dynamic aperture in circular machines
- The importance of symplectic methods for linac modeling is less clear
  - personal opinion: since non-symplectic methods generally exhibit secular growth in particle amplitude, and since we already have mature circular machine codes that are applicable to linacs too, it is better to use a symplectic method for linac modeling, unless the symplectic method requires significantly more computational effort, and if computational effort is an issue
    - with the advent of parallel codes, effort associated with single-particle dynamics is less important than it used to be, because that portion of the calculation is trivially parallel
- Why use a symplectic method if the design also involves cooling?
  - If we see eigen-emittance growth in a non-symplectic code, we don’t know if it’s from physics (nonlinearities; non-Hamiltonian effects like cooling) or if it’s numerical. By using symplectic methods for the symplectic portion of the calculation, it helps isolate where the eigen-emittance evolution is coming from: it is either due to nonlinearities in the beamline or due to the cooling portion of the simulation.

What about the fact that a drift is nonlinear in canonical variables?

- We need to distinguish beamline *design* from beamline *evaluation*
- For design purposes, it does not matter that a drift is nonlinear, we simply treat linear and nonlinear effects consistently.
  - People do remarkable designs (e.g. high order achromats) and it does not matter that a drift is nonlinear.
- For evaluation purposes, if we want to track particles and see where they go based on the design, we can still track particles through a drift *exactly* regardless of the fact that a drift is nonlinear
  - it simply requires evaluating a square root, no big deal