

# $N^3LO$ computations for deep-inelastic scattering

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Snowmass EF05/EF06 meeting:  $NNLO$  and  $N^3LO$  computations for PDF analyses, Zoom, Sep 23, 2020

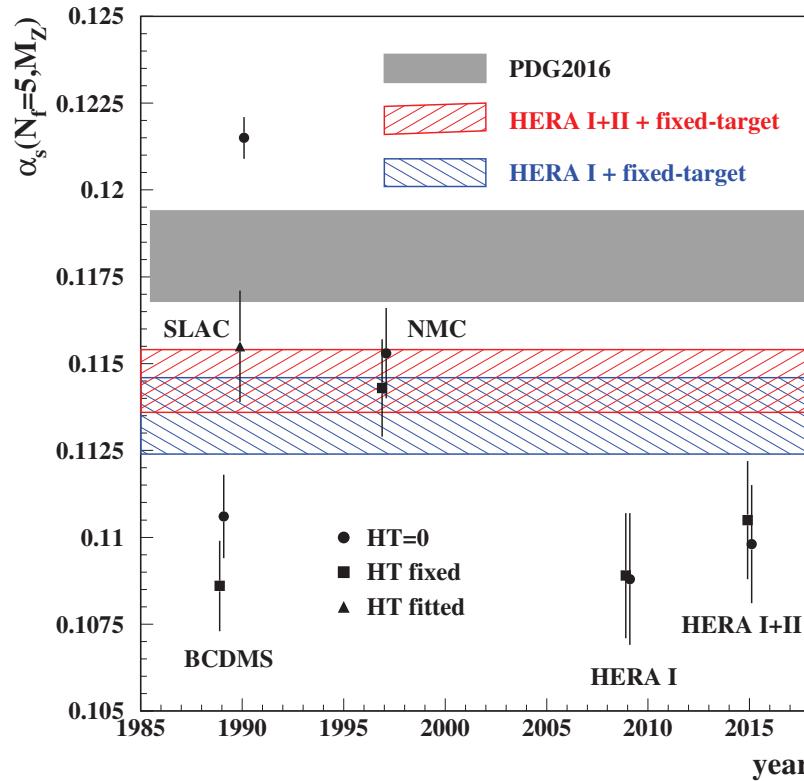
## *Based on work done in collaboration with:*

- *Approximate four-loop QCD corrections to the Higgs-boson production cross section*  
G. Das, S. M., and A. Vogt [arXiv:2004.00563](#)
- *Soft corrections to inclusive deep-inelastic scattering at four loops and beyond*  
G. Das, S. M., and A. Vogt [arXiv:1912.12920](#)
- *Five-loop contributions to low- $N$  non-singlet anomalous dimensions in QCD*  
F. Herzog, S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt  
[arXiv:1812.11818](#)
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*  
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1805.09638](#)
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*  
S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt [arXiv:1707.08315](#)
- Many more papers of MVV and friends ...  
[2001 – ...](#)

# *Motivation*

# Theory considerations in $\alpha_s$ determinations

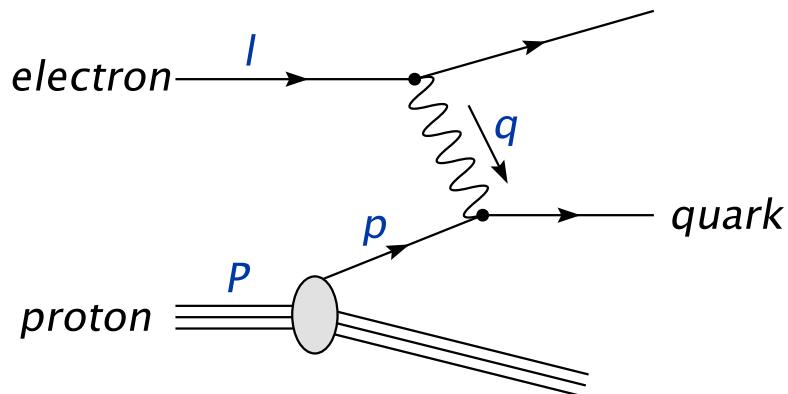
- Correlation of errors among different data DIS sets
- Target mass corrections (powers of nucleon mass  $M_N^2/Q^2$ )
- Higher twist  $F_2^{\text{ht}} = F_2 + \text{ht}^{(4)}(x)/Q^2 + \dots$
- Variants with no higher twist give larger  $\alpha_s$  values Alekhin, Blümlein, S.M. '17



- Theoretical uncertainty of  $\alpha_s$  at NNLO from DIS data  $\gtrsim \mathcal{O}(1\dots 2)\%$

## *Theoretical framework*

# Deep-inelastic scattering



## Kinematic variables

- momentum transfer  $Q^2 = -q^2$
- Bjorken variable  $x = Q^2/(2p \cdot q)$

- Structure functions (up to order  $\mathcal{O}(1/Q^2)$ )

$$F_a(x, Q^2) = \sum_i [C_{a,i}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes PDF(\mu^2)](x)$$

- Coefficient functions up to  $N^3LO$

$$C_{a,i} = \alpha_s^n \left( c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \dots \right)$$

- Evolution equations up to  $N^3LO$

- non-singlet ( $2n_f - 1$  scalar) and singlet ( $2 \times 2$  matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) = [P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)](x)$$

- splitting functions  $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \dots$

# Evolution equations

- Parton distribution functions  $q_i(x, \mu^2)$ ,  $\bar{q}_i(x, \mu^2)$  and  $g(x, \mu^2)$  for quarks, antiquarks of flavour  $i$  and gluons

- Flavor non-singlet combinations

$$q_{ns,ik}^\pm = (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k) \text{ and } q_{ns}^v = \sum_{i=1}^{n_f} (q_i - \bar{q}_i)$$

- splitting functions  $P_{ns}^\pm$  and  $P_{ns}^v = P_{ns}^- + P_{ns}^s$

- Flavor singlet evolution

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \text{ and } q_s = \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

- quark-quark splitting function  $P_{qq} = P_{ns}^+ + P_{ps}$

- Mellin transformation relates to anomalous dimensions  $\gamma_{ik}(N)$  of twist-two operators

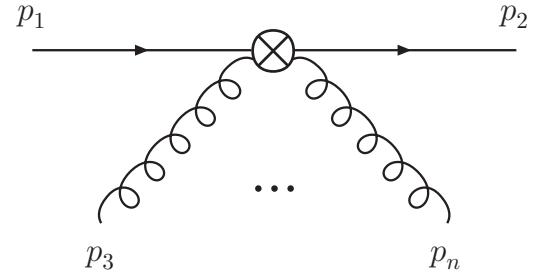
$$\gamma_{ik}^{(n)}(N, \alpha_s) = - \int_0^1 dx \ x^{N-1} P_{ik}^{(n)}(x, \alpha_s)$$

# *Non-singlet*

# Operator matrix elements

- Non-singlet operator of spin- $N$  and twist two

$$O_{\{\mu_1, \dots, \mu_N\}}^{\text{ns}} = \bar{\psi} \lambda^\alpha \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi ,$$
$$\alpha = 3, 8, \dots, (n_f^2 - 1)$$



## Calculation

- Anomalous dimensions  $\gamma(N)$  from ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for  $\gamma_{\text{ns}}^\pm$
  - 1 three- and 29 four-loop meta diagrams for  $\gamma_{\text{ns}}^s$

# Anomalous dimensions

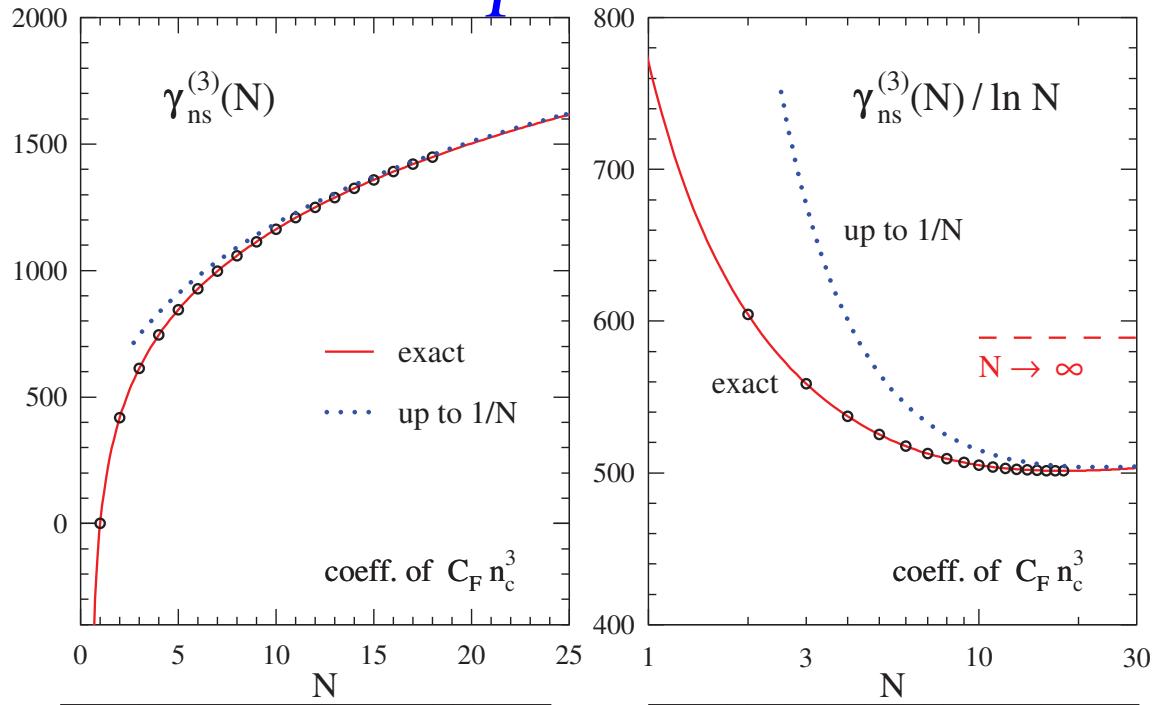
- Anomalous dimensions  $\gamma(N)$  of leading twist non-singlet local operators
  - expressible in harmonic sums up to weight 7
  - $2 \cdot 3^{w-1}$  sums at weight  $w$
- Reciprocity relation  $\gamma(N) = \gamma_u(N + \gamma(N) - \beta(a))$  reduces number of  $2^{w-1}$  sums at weight  $w$  for  $\gamma_u$ 
  - additional denominators with powers  $1/(N+1)$  give  $2^{w+1} - 1$  objects (255 at weight 7)
- Constraints at large- $x$ /small- $x$  ( $N \rightarrow \infty/N \rightarrow 0$ ) give additional 46 conditions

## Upshot

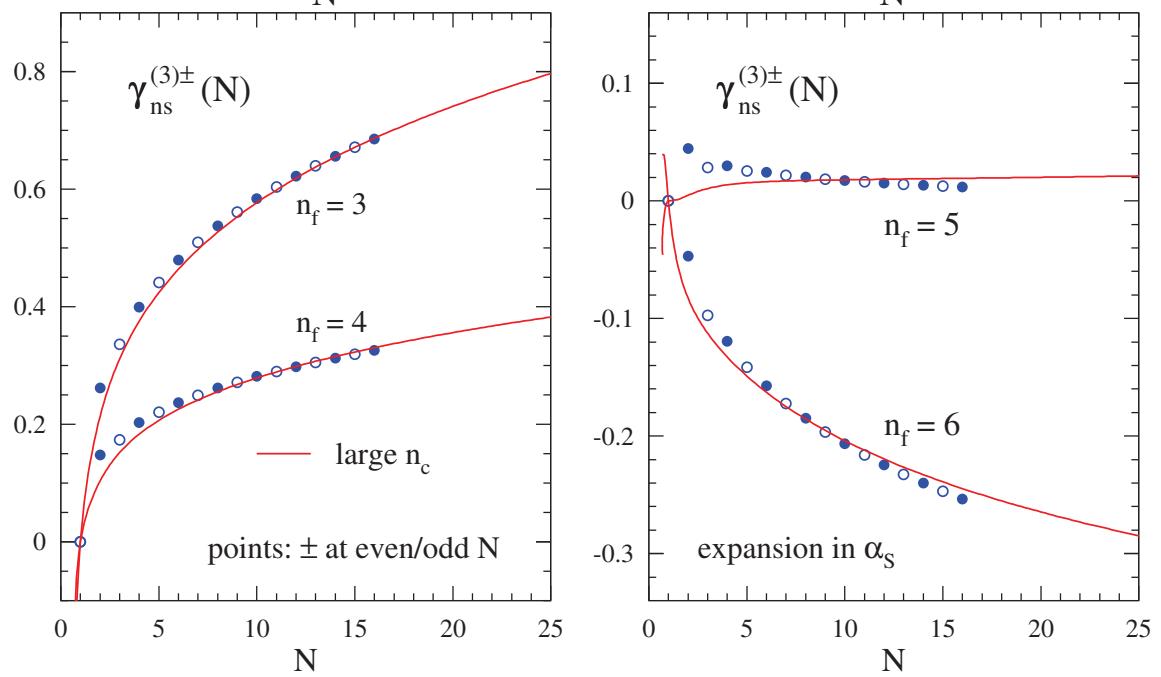
- Computation of Mellin moments up to  $N = 18$  for anomalous dimensions feasible
- Reconstruction of analytic all- $N$  expressions in large- $n_c$  limit from solution of Diophantine equations

# Mellin moments at four loops

- Top:  
 $n_f^0$  part of anomalous dimensions  $\gamma_{ns}^{(3)\pm}(N)$  in large- $n_c$  limit and large- $N$  expansion

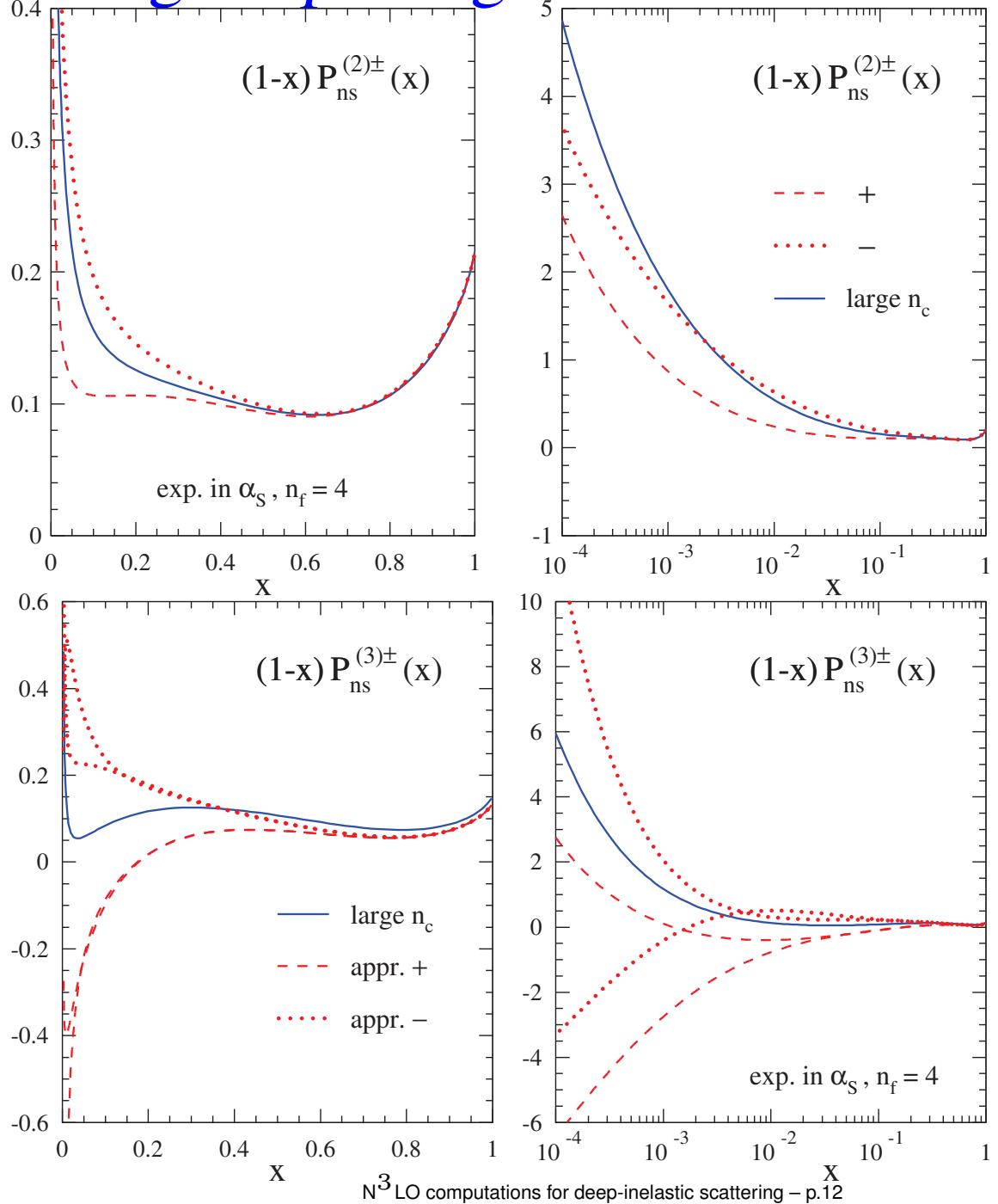


- Bottom: results for even- $N$  ( $\gamma_{ns}^{(3)+}(N)$ ) and odd- $N$  ( $\gamma_{ns}^{(3)-}(N)$ ) in large- $n_c$  limit for  $n_f = 3, \dots, 6$



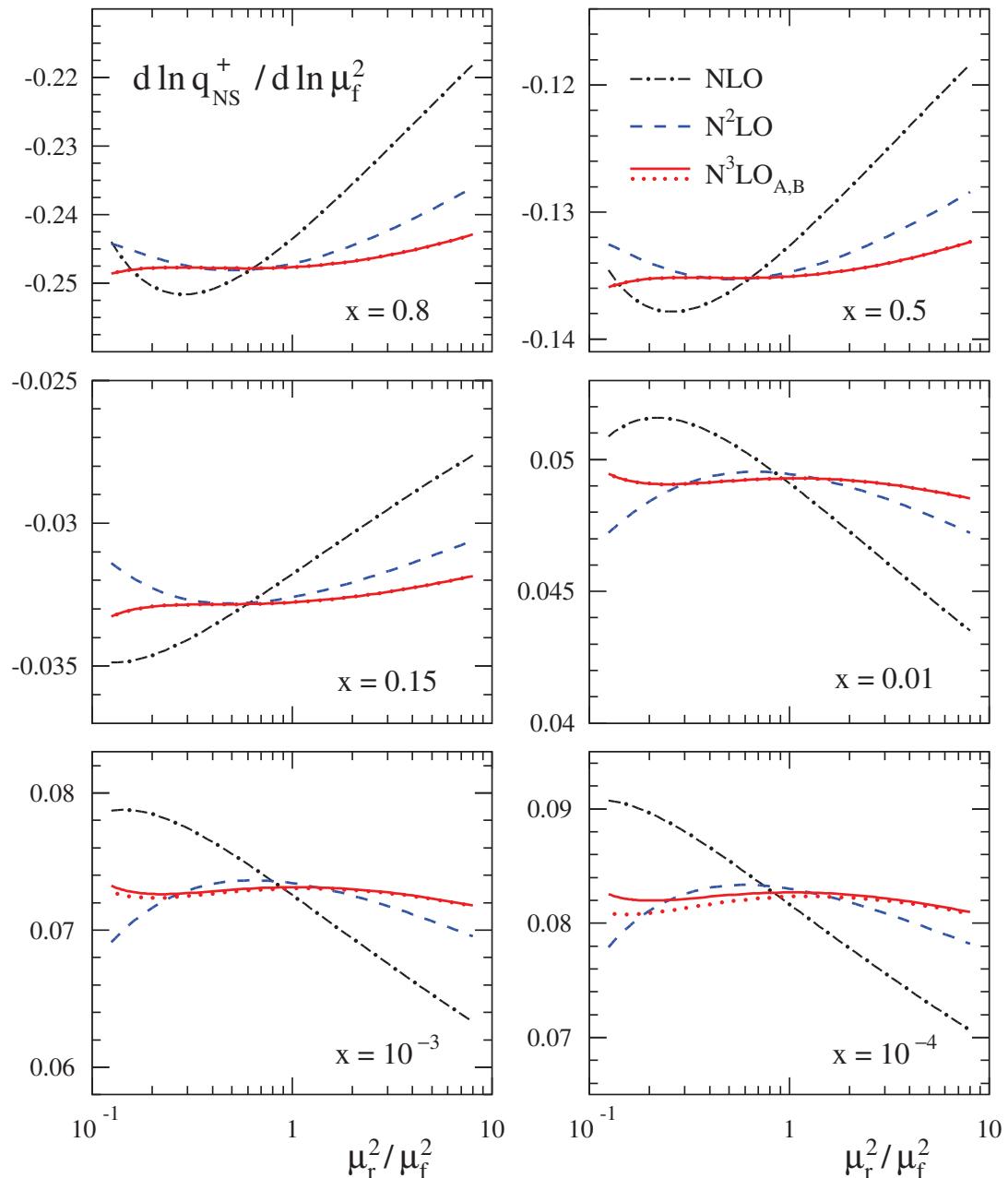
# Four-loop non-singlet splitting functions

- Top:  
three-loop  $P_{\text{ns}}^{(2)\pm}(x)$   
and large- $n_c$  limit  
with  $n_f = 4$
- Bottom:  
four-loop  $P_{\text{ns}}^{(3)\pm}(x)$   
and uncertainty bands  
beyond large- $n_c$  limit  
with  $n_f = 4$



# Scale stability of evolution

- Renormalization scale dependence of evolution kernel  $d \ln q_{\text{ns}}^+ / d \ln \mu_f^2$ 
  - non-singlet shape  
 $xq_{\text{ns}}^+(x, \mu_0^2) = x^{0.5}(1-x)^3$
- NLO, NNLO and N<sup>3</sup>LO predictions
  - remaining uncertainty of four-loop splitting function  $P_{\text{ns}}^{(3)+}$  almost invisible



# Mellin moments at five loops

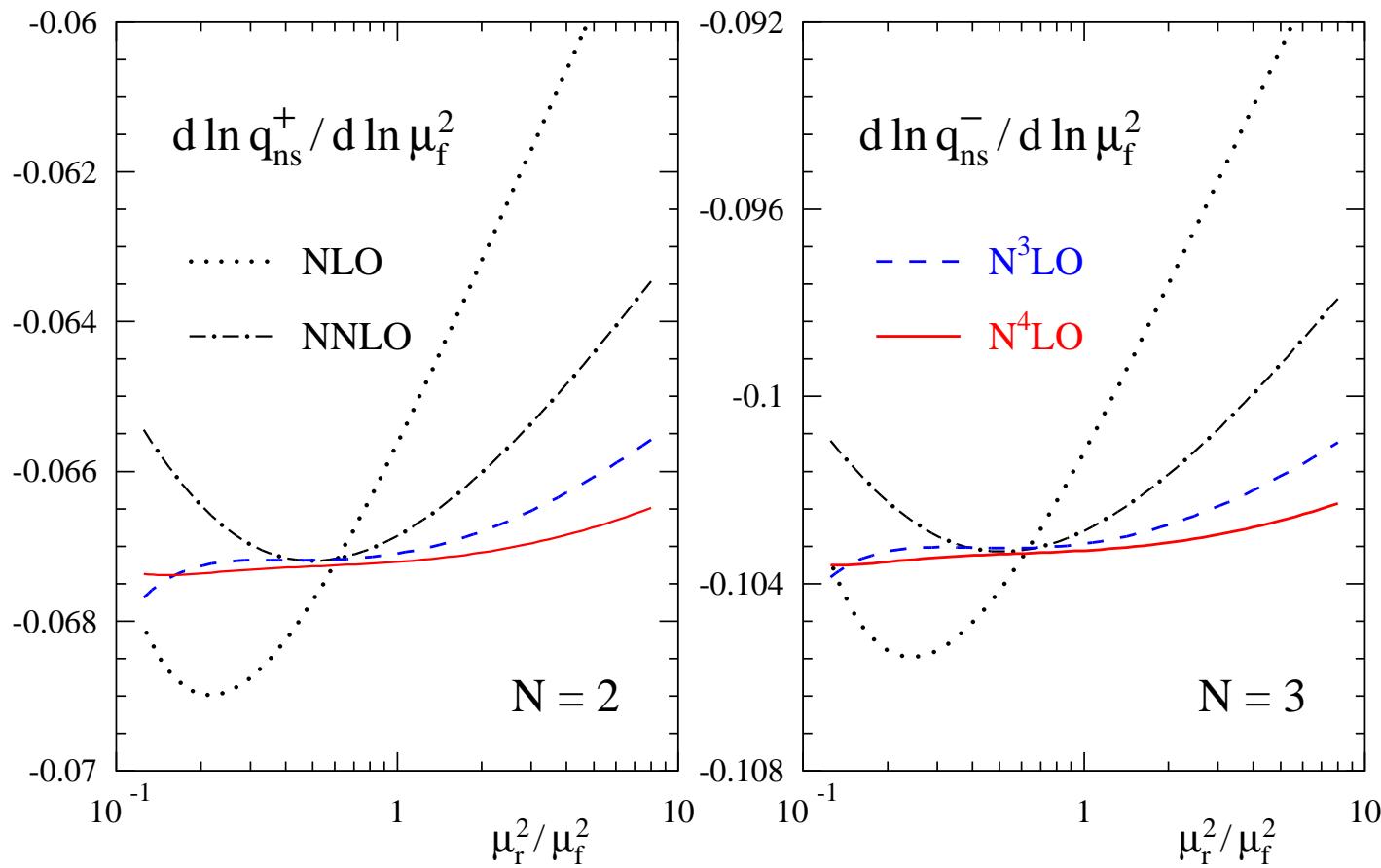
- Moments  $N = 2$  and  $N = 3$  for nonsinglet anomalous dimensions  $\gamma_{\text{ns}}^{\pm}$ 
  - implementation by Herzog, Ruijl '17 of local  $R^*$  operation Chetyrkin, Tkachov '82; Chetyrkin, Smirnov '84 for reduction of five-loop self-energy diagrams to four-loop ones computed with **Forcer** Ruijl, Ueda, Vermaseren '17

$$\begin{aligned} \gamma_{\text{ns}}^{(4)+}(N=2) = & C_F^5 \left[ \frac{9306376}{19683} - \frac{802784}{729} \zeta_3 - \frac{557440}{81} \zeta_5 + \frac{12544}{9} \zeta_3^2 + 8512 \zeta_7 \right] \\ & - C_A C_F^4 \left[ \frac{81862744}{19683} - \frac{1600592}{243} \zeta_3 + \frac{59840}{81} \zeta_4 - \frac{142240}{27} \zeta_5 + 3072 \zeta_3^2 - \frac{35200}{9} \zeta_6 + 19936 \zeta_7 \right] \\ & + C_A^2 C_F^3 \left[ \frac{63340406}{6561} - \frac{1003192}{243} \zeta_3 - \frac{229472}{81} \zeta_4 + \frac{61696}{27} \zeta_5 + \frac{30976}{9} \zeta_3^2 - \frac{35200}{9} \zeta_6 + 15680 \zeta_7 \right] \\ & - C_A^3 C_F^2 \left[ \frac{220224724}{19683} + \frac{4115536}{729} \zeta_3 - \frac{170968}{27} \zeta_4 - \frac{3640624}{243} \zeta_5 + \frac{70400}{27} \zeta_3^2 + \frac{123200}{27} \zeta_6 + \frac{331856}{27} \zeta_7 \right] \\ & + C_A^4 C_F \left[ \frac{266532611}{39366} + \frac{2588144}{729} \zeta_3 - \frac{221920}{81} \zeta_4 - \frac{3102208}{243} \zeta_5 + \frac{74912}{81} \zeta_3^2 + \frac{334400}{81} \zeta_6 + \frac{178976}{27} \zeta_7 \right] \\ & - \frac{d_{AA}^{(4)}}{N_F} C_F \left[ \frac{15344}{81} - \frac{12064}{27} \zeta_3 - \frac{704}{3} \zeta_4 + \frac{58400}{27} \zeta_5 - \frac{6016}{3} \zeta_3^2 - \frac{19040}{9} \zeta_7 \right] \\ & + \frac{d_{FA}^{(4)}}{N_F} C_F \left[ \frac{23968}{81} - \frac{733504}{81} \zeta_3 + \frac{176320}{81} \zeta_5 + \frac{6400}{3} \zeta_3^2 + \frac{77056}{9} \zeta_7 \right] \\ & - \frac{d_{FA}^{(4)}}{N_F} C_A \left[ \frac{82768}{81} - \frac{555520}{81} \zeta_3 + \frac{10912}{9} \zeta_4 - \frac{1292960}{81} \zeta_5 + \frac{84352}{27} \zeta_3^2 + \frac{140800}{27} \zeta_6 + 12768 \zeta_7 \right] \\ & + n_f C_F \left[ \frac{182496}{19683} - \frac{463520}{243} \zeta_3 + \frac{21248}{81} \zeta_4 - \frac{16480}{27} \zeta_5 + \frac{6656}{9} \zeta_3^2 - \frac{6400}{3} \zeta_6 + \frac{8960}{3} \zeta_7 \right] \\ & - n_f C_A C_F^3 \left[ \frac{3375082}{6561} - \frac{420068}{243} \zeta_3 - \frac{48256}{81} \zeta_4 + \frac{458032}{81} \zeta_5 + \frac{3968}{3} \zeta_3^2 - \frac{8000}{3} \zeta_6 + \frac{4480}{9} \zeta_7 \right] \\ & + n_f C_A C_F^2 \left[ \frac{15291499}{13122} + \frac{1561600}{243} \zeta_3 - \frac{114536}{27} \zeta_4 - \frac{252544}{243} \zeta_5 + \frac{24896}{27} \zeta_3^2 + \frac{13600}{27} \zeta_6 + \frac{11200}{27} \zeta_7 \right] \\ & - n_f C_A^2 C_F \left[ \frac{48846580}{19683} + \frac{4314308}{729} \zeta_3 - \frac{274768}{81} \zeta_4 - \frac{1389080}{81} \zeta_5 + \frac{27080}{81} \zeta_3^2 + \frac{184000}{81} \zeta_6 + \frac{39088}{27} \zeta_7 \right] \\ & + n_f \frac{d_{AA}^{(4)}}{N_F} \left[ \frac{22096}{27} + \frac{43712}{81} \zeta_3 - \frac{512}{9} \zeta_4 - \frac{217280}{81} \zeta_5 - \frac{25088}{27} \zeta_3^2 + \frac{25600}{9} \zeta_6 - 24640 \zeta_7 \right] \\ & - n_f C_F \frac{d_{FA}^{(4)}}{N_F} \left[ \frac{170752}{81} - \frac{328832}{81} \zeta_3 + \frac{650240}{81} \zeta_5 - \frac{8192}{9} \zeta_3^2 - \frac{35840}{9} \zeta_7 \right] \\ & + n_f C_A \frac{d_{AA}^{(4)}}{N_F} \left[ \frac{207824}{81} + \frac{251392}{81} \zeta_3 - \frac{5632}{9} \zeta_4 - \frac{522880}{81} \zeta_5 + \frac{15872}{27} \zeta_3^2 + \frac{70400}{27} \zeta_6 - \frac{29120}{9} \zeta_7 \right] \\ & + n_f^2 C_F^3 \left[ \frac{1082297}{6561} - \frac{145792}{243} \zeta_3 + \frac{1072}{81} \zeta_4 + \frac{55552}{81} \zeta_5 + \frac{1792}{9} \zeta_3^2 - \frac{3200}{9} \zeta_6 \right] \\ & + n_f^2 C_A C_F^2 \left[ \frac{332254}{2187} + \frac{85016}{243} \zeta_3 + \frac{20752}{27} \zeta_4 - \frac{28544}{81} \zeta_5 - \frac{13952}{27} \zeta_3^2 + \frac{1600}{27} \zeta_6 \right] \\ & + n_f^2 C_A^2 C_F \left[ \frac{631400}{6561} + \frac{214268}{243} \zeta_3 - \frac{784}{81} \zeta_4 - \frac{53344}{243} \zeta_5 + \frac{25472}{81} \zeta_3^2 + \frac{22400}{81} \zeta_6 \right] \\ & - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{43744}{81} - \frac{35648}{81} \zeta_3 - \frac{1792}{9} \zeta_4 - \frac{52480}{81} \zeta_5 + \frac{2048}{27} \zeta_3^2 + \frac{12800}{27} \zeta_6 \right] \\ & + n_f^3 C_F^2 \left[ \frac{166510}{19683} + \frac{11872}{729} \zeta_3 - \frac{2752}{3} \zeta_4 + \frac{4096}{81} \zeta_5 \right] - n_f^4 C_F \left[ \frac{5504}{19683} + \frac{1024}{729} \zeta_3 - \frac{128}{81} \zeta_4 \right] \end{aligned}$$

$$\begin{aligned} \gamma_{\text{ns}}^{(4)-}(N=3) = & C_F^5 \left[ \frac{99382175}{80621568} + \frac{23328}{1981} \zeta_3 - \frac{3395975}{162} \zeta_5 - \frac{9650}{9} \zeta_3^2 + \frac{34685}{2} \zeta_7 \right] \\ & - C_A C_F^4 \left[ \frac{286028134219}{80621568} - \frac{23916529}{776} \zeta_3 - \frac{4490}{81} \zeta_5 + \frac{134090}{108} \zeta_4 - \frac{2468075}{9} \zeta_6 + \frac{55000}{4} \zeta_7 \right] \\ & + C_A^2 C_F^3 \left[ \frac{154041281}{3359232} - \frac{154041281}{2916} \zeta_3 + \frac{732787}{1296} \zeta_4 + \frac{1972075}{216} \zeta_5 - \frac{63830}{9} \zeta_3^2 - \frac{79750}{9} \zeta_6 + \frac{139895}{4} \zeta_7 \right] \\ & - C_A^3 C_F^2 \left[ \frac{166662991819}{20155392} - \frac{36397493}{2916} \zeta_3 - \frac{103763}{54} \zeta_4 + \frac{30994565}{3888} \zeta_5 - \frac{133990}{27} \zeta_3^2 - \frac{72875}{54} \zeta_6 + \frac{2127335}{108} \zeta_7 \right] \\ & + C_A^4 C_F \left[ \frac{7593279965}{1007769} - \frac{27693563}{23328} \zeta_3 - \frac{1791229}{1296} \zeta_4 - \frac{9417425}{1944} \zeta_5 - \frac{96700}{81} \zeta_3^2 + \frac{163625}{81} \zeta_6 + \frac{199640}{27} \zeta_7 \right] \\ & - \frac{d_{AA}^{(4)}}{N_F} C_F \left[ \frac{81725}{162} - \frac{33505}{18} \zeta_3 - \frac{1100}{3} \zeta_4 + \frac{52025}{18} \zeta_5 - \frac{7000}{3} \zeta_3^2 - \frac{48125}{36} \zeta_7 \right] \\ & - \frac{d_{FA}^{(4)}}{N_F} C_F \left[ \frac{231575}{36} + \frac{6351445}{324} \zeta_3 - \frac{2927225}{162} \zeta_5 + \frac{23210}{3} \zeta_3^2 - \frac{200410}{9} \zeta_7 \right] \\ & + \frac{d_{FA}^{(4)}}{N_F} C_A \left[ \frac{165871}{54} + \frac{1816625}{162} \zeta_3 - \frac{41800}{9} \zeta_4 - \frac{4456145}{162} \zeta_5 + \frac{196880}{27} \zeta_3^2 + \frac{200750}{27} \zeta_6 - \frac{7525}{4} \zeta_7 \right] \\ & + n_f C_F \left[ \frac{40310784}{40310784} - \frac{1776621549}{486} + \frac{1332919}{9} \zeta_3 + \frac{5000}{9} \zeta_5 + \frac{33290}{81} \zeta_4 - \frac{30325}{81} \zeta_5 - \frac{10000}{9} \zeta_6 + \frac{14000}{3} \zeta_7 \right] \\ & - n_f C_A C_F^3 \left[ \frac{3737356319}{3359232} - \frac{2327111}{432} \zeta_3 + \frac{1280}{3} \zeta_5 + \frac{262069}{648} \zeta_4 + \frac{1693715}{162} \zeta_5 - \frac{14000}{3} \zeta_6 + \frac{7000}{3} \zeta_7 \right] \\ & + n_f C_A^2 C_F^2 \left[ \frac{5637513931}{2711207} + \frac{2711207}{27} \zeta_3 - \frac{5020}{9} \zeta_4 - \frac{47499}{108} \zeta_5 + \frac{508820}{243} \zeta_6 - \frac{20375}{27} \zeta_6 + \frac{50155}{108} \zeta_7 \right] \\ & - n_f C_A^3 C_F \left[ \frac{8766012215}{2519424} + \frac{45697231}{5832} \zeta_3 + \frac{1195}{81} \zeta_5 - \frac{2848403}{648} \zeta_4 - \frac{1808870}{243} \zeta_5 + \frac{222250}{81} \zeta_6 + \frac{250915}{108} \zeta_7 \right] \\ & - n_f C_F \frac{d_{FA}^{(4)}}{N_F} \left[ \frac{24385}{81} - \frac{334010}{81} \zeta_3 - \frac{8480}{9} \zeta_5 + \frac{1622600}{81} \zeta_5 - \frac{135380}{81} \zeta_7 \right] \\ & + n_f^2 C_F \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{297889}{162} + \frac{154970}{81} \zeta_3 - \frac{62600}{27} \zeta_5 + \frac{3700}{9} \zeta_4 - \frac{122780}{81} \zeta_5 - \frac{36500}{27} \zeta_6 - \frac{910}{9} \zeta_7 \right] \\ & + n_f^2 C_A \frac{d_{AA}^{(4)}}{N_F} \left[ \frac{241835}{162} + \frac{33487}{81} \zeta_3 + \frac{30560}{27} \zeta_5 - \frac{10780}{9} \zeta_4 - \frac{316900}{81} \zeta_5 + \frac{110000}{27} \zeta_6 - \frac{71960}{9} \zeta_7 \right] \\ & + n_f^2 C_F^3 \left[ \frac{512848319}{1679616} - \frac{57109}{54} \zeta_3 + \frac{2800}{9} \zeta_5^2 + \frac{9118}{81} \zeta_4 + \frac{86440}{81} \zeta_5 - \frac{5000}{9} \zeta_6 \right] \\ & + n_f^2 C_A C_F^2 \left[ \frac{1080083}{5832} - \frac{296729}{972} \zeta_3 - \frac{21800}{27} \zeta_5^2 + \frac{56327}{54} \zeta_4 - \frac{42860}{81} \zeta_5 + \frac{2500}{27} \zeta_6 \right] \\ & + n_f^2 C_A^2 C_F \left[ \frac{61747877}{419904} + \frac{2496811}{1944} \zeta_3 + \frac{39800}{81} \zeta_5^2 - \frac{3503}{3} \zeta_4 - \frac{88990}{243} \zeta_5 + \frac{35000}{81} \zeta_6 \right] \\ & - n_f^2 \frac{d_{FF}^{(4)}}{N_F} \left[ \frac{19435}{27} - \frac{53366}{81} \zeta_3 + \frac{3200}{27} \zeta_5^2 - \frac{3160}{9} \zeta_4 - \frac{70000}{81} \zeta_5 + \frac{20000}{27} \zeta_6 \right] \\ & + n_f^3 C_F^2 \left[ \frac{28758139}{1259712} + \frac{21673}{729} \zeta_3 - \frac{610}{9} \zeta_4 + \frac{800}{27} \zeta_5 \right] \\ & + n_f^3 C_A C_F \left[ \frac{13729181}{1259712} + \frac{14947}{729} \zeta_3 + \frac{4390}{81} \zeta_4 - \frac{6400}{81} \zeta_5 \right] - n_f^4 C_F \left[ \frac{259993}{629856} + \frac{1660}{729} \zeta_3 - \frac{200}{81} \zeta_4 \right] \end{aligned}$$

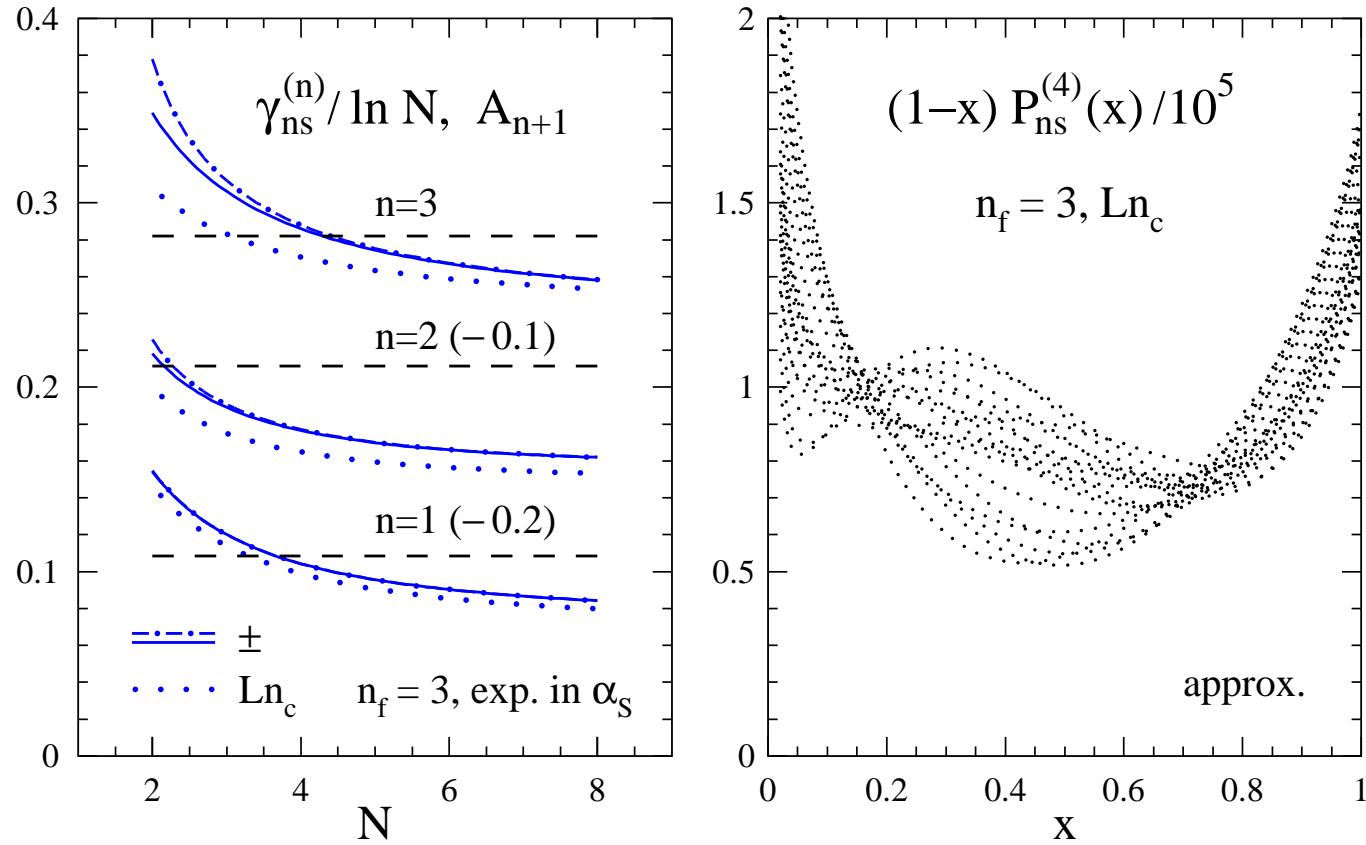
$$\begin{aligned} \gamma_{\text{ns}}^{(4)\nu}(N=3) = & \gamma_{\text{ns}}^{(4)-}(N=3) \\ & + n_f \frac{d_{abc} d^{abc}}{N_F} \left[ C_F^2 \left[ \frac{79906955}{46656} + \frac{246955}{54} \zeta_3 - \frac{504550}{81} \zeta_5 \right] \right. \\ & \left. - C_A C_F \left[ \frac{797321}{3888} - \frac{475655}{54} \zeta_3 + \frac{17600}{9} \zeta_4 + \frac{516950}{81} \zeta_5 - \frac{500}{9} \zeta_3^2 + \frac{2800}{9} \zeta_7 \right] \right] \\ & + C_A^2 \left[ \frac{166535}{486} - \frac{1783913}{324} \zeta_3 + \frac{5555}{9} \zeta_4 + \frac{507515}{81} \zeta_5 - \frac{2035}{27} \zeta_3^2 - \frac{5500}{27} \zeta_6 - \frac{2765}{18} \zeta_7 \right] \\ & + n_f C_A \left[ \frac{285985}{3888} + \frac{41954}{81} \zeta_3 + \frac{160}{27} \zeta_4 - \frac{1010}{9} \zeta_5 - \frac{56480}{81} \zeta_5 + \frac{1000}{27} \zeta_6 \right] \\ & + n_f C_F \left[ \frac{1098323}{3888} - \frac{49720}{81} \zeta_3 + \frac{3200}{9} \zeta_4 \right] - n_f^2 \left[ \frac{21823}{1944} \right] \end{aligned}$$

# Scale stability of evolution



- Renormalization-scale dependence of  $d \ln q_{ns}^\pm / d \ln \mu_f^2$  at  $N=2$  and  $N=3$  using NLO, NNLO, N<sup>3</sup>LO and N<sup>4</sup>LO predictions with  $\alpha_s(\mu_f) = 0.2$  and  $n_f = 4$

# Five-loop splitting function at large- $x$



- Left: Non-singlet anomalous dimensions  $\gamma_{\text{ns}}^{(n) \pm}(N) / \ln N$  for non-even/odd  $2 \leq N \leq 8$  for  $n_f = 3$  compared to their limits for large- $n_c$  and for  $N \rightarrow \infty$  (straight lines)
- Right: 20 trial functions approximating  $P_{\text{ns}}^{(4) \pm}(N)$  in large- $n_c$  limit for  $n_f = 3$  with uncertainty band for five-loop cusp anomalous dimension  $A_5$

# *Singlet*

# Color factors of $SU(n_c)$

- Quadratic Casimir factors  $C_r \delta^{ab} \equiv \text{Tr} ( T_r^a T_r^b )$ 
  - fundamental representation  $C_F = (n_c^2 - 1)/(2n_c)$ ;  
adjoint representation  $C_A = n_c$
- Quartic Casimir invariants occur for the first time at four loops
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$  for representations labels  $x, y$  with generators  $T_r^a$   
$$d_r^{abcd} = \frac{1}{6} \text{Tr} ( T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations} )$$

- $SU(n_c)$  with fermions in fundamental representation

$$d_{AA}^{(4)}/n_A = \frac{1}{24} n_c^2 (n_c^2 + 36) ,$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48} n_c (n_c^2 + 6) ,$$

$$d_{FF}^{(4)}/n_A = \frac{1}{96} (n_c^2 - 6 + 18n_c^{-2})$$

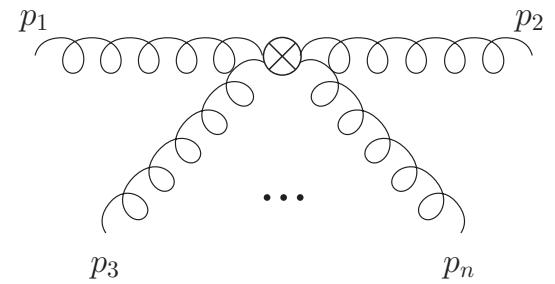
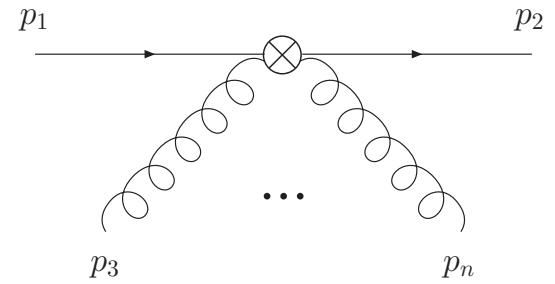
- trace-normalized with  $T_F = \frac{1}{2}$
- dimensions of representations  $n_F = n_c$  and  $n_A = (n_c^2 - 1)$

# Operator matrix elements

- Singlet operators of spin- $N$  and twist two

$$O_{\{\mu_1, \dots, \mu_N\}}^q = \bar{\psi} \gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}} \psi ,$$

$$O_{\{\mu_1, \dots, \mu_N\}}^g = F_{\nu \{\mu_1} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N\}}^\nu$$



- Quartic Casimir terms at four loops  
are effectively ‘leading-order’

- anomalous dimensions fulfil relation for  $\mathcal{N} = 1$  supersymmetry

$$\gamma_{qq}^{(3)}(N) + \gamma_{gq}^{(3)}(N) - \gamma_{qg}^{(3)}(N) - \gamma_{gg}^{(3)}(N) \stackrel{Q}{=} 0$$

- color-factor substitutions for  $n_f = 1$  Majorana fermions (factor  $2n_f$ )

$$(2n_f)^2 \frac{d_{FF}^{(4)}}{n_A} = 2n_f \frac{d_{FA}^{(4)}}{n_A} = 2n_f \frac{d_{FF}^{(4)}}{n_F} = \frac{d_{FA}^{(4)}}{n_F} = \frac{d_{AA}^{(4)}}{n_A}$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums

- quartic Casimir terms fulfil stronger condition Belitsky, Müller, Schäfer ‘99

$$\gamma_{qg}^{(0)}(N) \gamma_{gq}^{(3)}(N) \stackrel{Q}{=} \gamma_{gq}^{(0)}(N) \gamma_{qg}^{(3)}(N)$$

# Calculation and results

- Splitting functions (diagonal) in the large- $x$  limit

$$P_{ii}^{(n-1)}(x) = \frac{A_{n,i}}{(1-x)_+} + B_{n,i} \delta(1-x) + C_{n,i} \ln(1-x) + D_{n,i}$$

- Cusp anomalous dimensions related by Casimir scaling up to three loops

$$A_{n,g} = \frac{C_A}{C_F} A_{n,q} \text{ for } n \leq 3$$

- at four loops Casimir scaling holds in large  $n_c$ -limit

Dixon '17

- Generalized Casimir scaling at four loops for new color factors

S. M., Ruijl, Ueda, Vermaseren, Vogt '18

$$\bullet \quad A_{4,g} \left| \begin{array}{c} d_A^{abcd} d_A^{abcd} \\ \hline n_A \end{array} \right. = A_{4,q} \left| \begin{array}{c} d_F^{abcd} d_A^{abcd} \\ \hline n_F \end{array} \right.$$

$$\bullet \quad A_{4,g} \left| \begin{array}{c} d_F^{abcd} d_A^{abcd} \\ \hline n_A \end{array} \right. = A_{4,q} \left| \begin{array}{c} d_F^{abcd} d_F^{abcd} \\ \hline n_F \end{array} \right.$$

$$\bullet \quad A_{4,g} \left| \begin{array}{c} d_F^{abcd} d_F^{abcd} \\ \hline n_A \end{array} \right. = 0$$

# Quark cusp anomalous dimensions

$$\begin{aligned}
A_{4,q} = & C_F C_A^3 \left( \frac{84278}{81} - \frac{88400}{81} \zeta_2 + \frac{20944}{27} \zeta_3 + 1804 \zeta_4 - \frac{352}{3} \zeta_2 \zeta_3 - \frac{3608}{9} \zeta_5 \right. \\
& \left. - 16 \zeta_3^2 - \frac{2504}{3} \zeta_6 \right) + \frac{d_{FA}^{(4)}}{n_c} \left( -128 \zeta_2 + \frac{128}{3} \zeta_3 + \frac{3520}{3} \zeta_5 - 384 \zeta_3^2 - 992 \zeta_6 \right) \\
& + C_F^3 n_f \left( \frac{572}{9} + \frac{592}{3} \zeta_3 - 320 \zeta_5 \right) \\
& + C_F^2 C_A n_f \left( -\frac{34066}{81} + \frac{440}{3} \zeta_2 + \frac{3712}{9} \zeta_3 - 176 \zeta_4 - 128 \zeta_2 \zeta_3 + 160 \zeta_5 \right) \\
& + C_F C_A^2 n_f \left( -\frac{24137}{81} + \frac{20320}{81} \zeta_2 - \frac{23104}{27} \zeta_3 - \frac{176}{3} \zeta_4 + \frac{448}{3} \zeta_2 \zeta_3 + \frac{2096}{9} \zeta_5 \right) \\
& + n_f \frac{d_{FF}^{(4)}}{n_c} \left( 256 \zeta_2 - \frac{256}{3} \zeta_3 - \frac{1280}{3} \zeta_5 \right) + C_F^2 n_f^2 \left( \frac{2392}{81} - \frac{640}{9} \zeta_3 + 32 \zeta_4 \right) \\
& + C_F C_A n_f^2 \left( \frac{923}{81} - \frac{608}{81} \zeta_2 + \frac{2240}{27} \zeta_3 - \frac{112}{3} \zeta_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27} \zeta_3 \right)
\end{aligned}$$

**Large- $n_c$**  (Henn, Lee, Smirnov, Smirnov, Steinhauser '16; S. M., Ruijl, Ueda, Vermaseren, Vogt '17);  
 **$n_f$  terms** (Grozin '18; Henn, Peraro, Stahlhofen, Wasser '19);  **$n_f^2$  terms** (Davies, Ruijl, Ueda, Vermaseren, Vogt '16; Lee, Smirnov, Smirnov, Steinhauser '17);  **$n_f^3$  terms** (Gracey '94; Beneke, Braun, '95);  
**quartic colour factors** (Lee, Smirnov, Smirnov, Steinhauser '19; Henn, Korchemsky, Mistlberger '19)

# Coefficients of $\delta(1 - x)$

S. M., Ruijl, Ueda, Vermaseren, Vogt '18 (and updated)

$C_F^4$	$C_F^3 C_A$	$C_F^2 C_A^2$	$C_F C_A^3$	$d_{FA}^{(4)}/n_F$
$196.5 \pm 1.$	$-687.5 \pm 1.5$	$1219.5 \pm 2.$	$295.7 \pm 0.5$	$-998.0 \pm 0.2$
$n_f C_F^3$	$n_f C_F^2 C_A$	$n_f C_F C_A^2$	$n_f d_{FF}^{(4)}/n_F$	
$80.780 \pm 0.005$	$-455.247 \pm 0.005$	$-274.466 \pm 0.01$	$-143.6 \pm 0.2$	
$C_F n_c^3$		$n_f C_F n_c^2$		
716.5577		-484.8864		
$n_f^2 C_F^2$	$n_f^2 C_F C_A$		$n_f^3 C_F$	
-5.775288	51.03056		2.261237	

- Numerical values for color coefficients of  $\delta(1 - x)$  part  $B_4^q$  in quark splitting function
  - exact values rounded to seven digits
  - errors correlated due to known exact results in large- $n_c$  limit

## *Coefficient functions*

# Threshold resummation

- Coefficient function in large  $x$ -limit have large logarithms at  $n^{\text{th}}$ -order

$$\alpha_s^n \frac{\ln^{2n-1}(1-x)}{(1-x)_+} \longleftrightarrow \alpha_s^n \ln^{2n}(N)$$

- Threshold resummation in Mellin space

$$C^N = (1 + \alpha_s g_{01} + \alpha_s^2 g_{02} + \dots) \cdot \exp(G^N) + \mathcal{O}(N^{-1} \ln^n N)$$

- Control over logarithms  $\ln(N)$  with  $\lambda = \beta_0 \alpha_s \ln(N)$  to  $N^k \text{LL}$  accuracy

$$G^N = \ln(N)g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \alpha_s^2 g_4(\lambda) + \alpha_s^3 g_5(\lambda) + \dots$$

- $g_1(\lambda)$ : LL Sterman '87; Appell, Mackenzie, Sterman '88
- $g_2(\lambda)$ : NLL Catani Trenatdue '89
- $g_3(\lambda)$ : NNLL or  $N^2\text{LL}$  Vogt '00; Catani, Grazzini, de Florian, Nason '03
- $g_4(\lambda)$ :  $N^3\text{LL}$  S.M., Vermaseren, Vogt '05
- $g_5(\lambda)$ :  $N^4\text{LL}$  Das, S.M., Vogt '19
- Resummed  $G^N$  predicts fixed orders in perturbation theory
  - generating functional for towers of large logarithms

# *Resummation exponent*

- Factorization in soft and collinear limit  $\longrightarrow$  product of radiative factors

$$G^N = \ln \Delta_q + \ln J_q + \Delta^{\text{int}}$$

- Renormalization group equations for radiative factors  $\Delta_q$ ,  $J_q$  and  $\Delta^{\text{int}}$

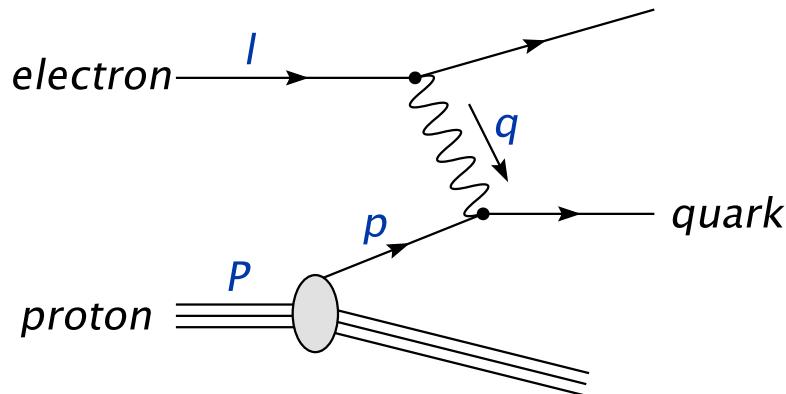
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$$\ln \Delta_p = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \int_{\mu_f^2}^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A^p(\alpha_s(q^2))$$



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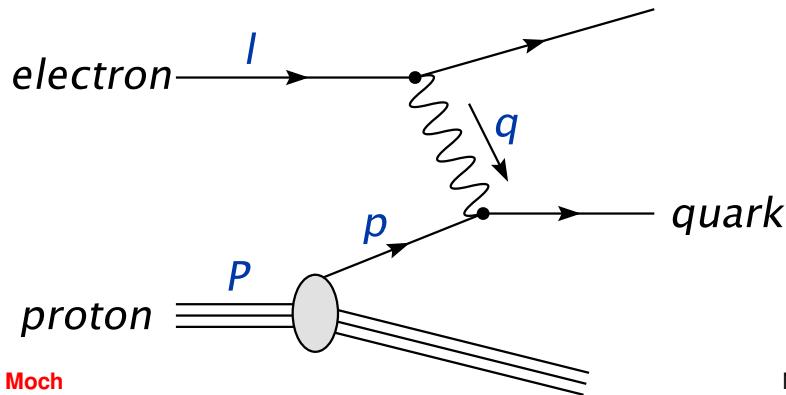
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- $J_p$ : collinear emission from “unobserved” final state parton  $p$

$$\ln J_p = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left[ \int_{(1-z)^2 Q^2}^{(1-z)Q^2} \frac{dq^2}{q^2} A^p(\alpha_s(q^2)) + B^J(\alpha_s([1-z]Q^2)) \right]$$



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 $\Delta^{\text{int}} = 0$  in DIS to all orders Forte, Ridolfi '02; Gardi, Roberts '02

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## Challenge

- Determination of evolution kernel  $B^J$  for jet function  $J_p$  at four loops

# Virtual corrections and real emissions

## Soft and collinear factorization in $D = 4 - 2\epsilon$ -dimensions

- Bare (partonic) structure function  $\mathcal{T}_n$  in  $D = 4 - 2\epsilon$ -dimensions

- $\mathcal{T}_n$  combines

- virtual corrections  $\mathcal{F}_n$  (dependent on  $\delta(1-x)$ )

- pure real-emission contributions  $\mathcal{S}_n$

(dependent on  $D$ -dimensional +-distributions  $f_{k,\epsilon}$ )

$$f_{k,\epsilon}(x) = \epsilon[(1-x)^{-1-k\epsilon}]_+ = -\frac{1}{k} \delta(1-x) + \epsilon \sum_{i=0} \frac{(-k\epsilon)^i}{i!} \frac{\ln^i(1-x)}{(1-x)_+}$$

- Laurent-series for  $\mathcal{T}_n$  at  $n^{\text{th}}$ -order

- mass-factorization predicts  $\frac{1}{\epsilon^n}$

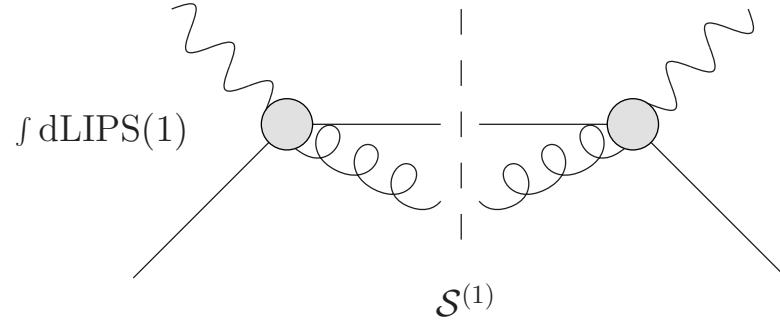
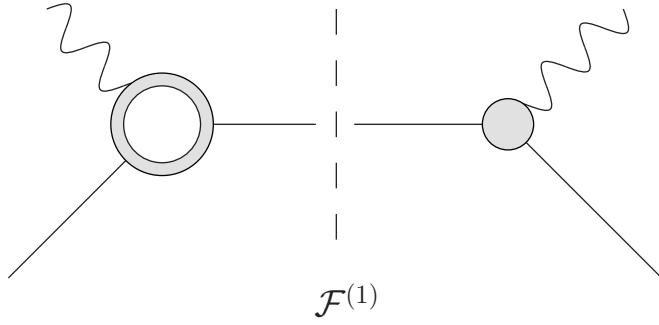
- soft and collinear singularities in  $\mathcal{F}_n$  and  $\mathcal{S}_n$  behave as  $\frac{1}{\epsilon^{2n}}$

- Infrared finiteness implies cancellation of poles between  $\mathcal{F}_n$  and  $\mathcal{S}_n$

Kinoshita '62; Lee, Nauenberg '64

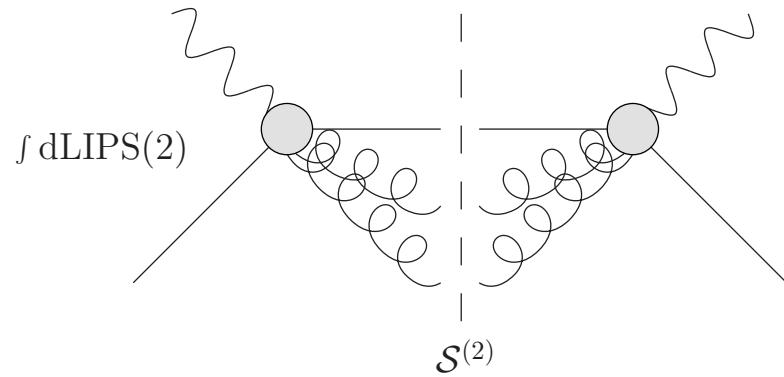
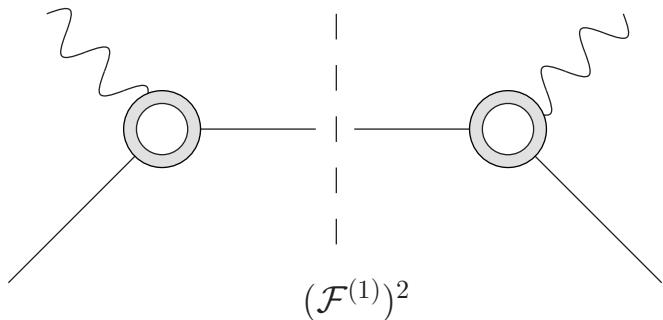
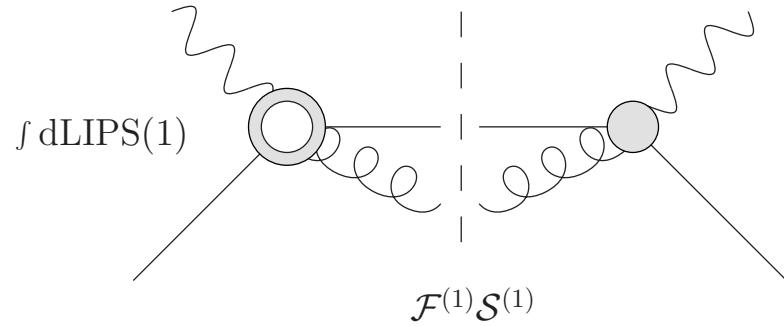
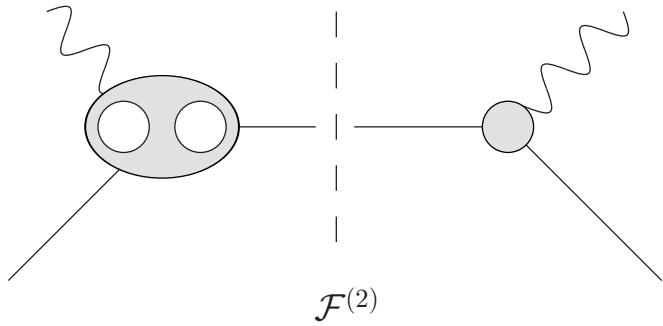
- Constructive approach to  $\mathcal{F}_n$  and  $\mathcal{S}_n$

# Factorization of result (1 loop)



$$\mathcal{T}_1^b = 2 \operatorname{Re} \mathcal{F}_1 \delta(1-x) + \mathcal{S}_1$$

# Factorization of result (2 loops and higher)

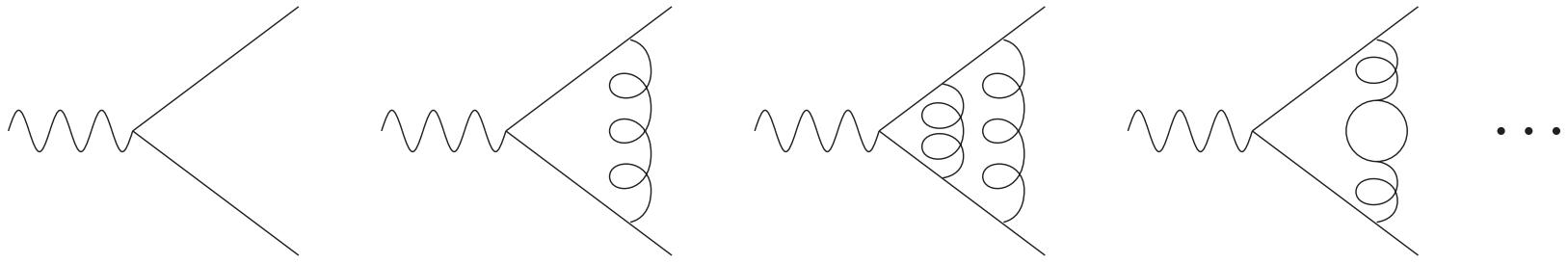


$$\mathcal{T}_2^b = (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \delta(1-x) + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{S}_1 + \mathcal{S}_2$$

$$\mathcal{T}_3^b = (2 \operatorname{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \delta(1-x) + (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \mathcal{S}_1 + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{S}_2 + \mathcal{S}_3$$

$$\begin{aligned} \mathcal{T}_4^b = & (2 \operatorname{Re} \mathcal{F}_4 + |\mathcal{F}_2|^2 + 2 |\mathcal{F}_1 \mathcal{F}_3|) \delta(1-x) \\ & + (2 \operatorname{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \mathcal{S}_1 + (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \mathcal{S}_2 + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{S}_3 + \mathcal{S}_4 \end{aligned}$$

# Quark form factor in QCD



- QCD corrections to vertex  $\gamma^* q \bar{q}$ , i.e.  $\Gamma_\mu = ie_q (\bar{u} \gamma_\mu u) \mathcal{F}_q(Q^2, \alpha_s)$ 
  - gauge invariant quantity
  - infrared divergent (dimensional regularization  $D = 4 - 2\epsilon$ )
- Form factor  $\mathcal{F}(Q^2, \alpha_s)$  exponentiates [Collins '80; Sen '81; Korchemsky '88; Magnea, Sterman '90; Contopanagos, Laenen, Sterman '97; Magnea '00](#) (long history)

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}(Q^2, \alpha_s, \epsilon) = \frac{1}{2} K(\alpha_s, \epsilon) + \frac{1}{2} G\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right).$$

- Renormalization group equations for functions  $G$  and  $K$ 
  - all  $Q^2$ -scale dependence in  $G$  (finite in  $\epsilon$ )
  - pure counter term function  $K$  (contains poles in  $\frac{1}{\epsilon}$ )
- Cusp anomalous dimension  $A$  governs evolution for  $G$  and  $K$

# Exponentiation of form factor

- Solution for  $\ln \mathcal{F}(Q^2, \alpha_s, \epsilon)$  in  $D$ -dimensions

- boundary condition  $\mathcal{F}(0, \alpha_s, \epsilon) = 1$

$$\ln \mathcal{F}\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) =$$

$$\frac{1}{2} \int_0^{Q^2/\mu^2} \frac{d\xi}{\xi} \left( K(\alpha_s, \epsilon) + G(1, \bar{a}(\xi\mu^2, \alpha_s, \epsilon), \epsilon) + \int_{\xi}^1 \frac{d\lambda}{\lambda} A(\bar{a}(\lambda\mu^2, \epsilon)) \right)$$

- use running coupling in  $D$ -dimensions from

$$\lambda \frac{\partial}{\partial \lambda} \bar{a}(\lambda, \alpha_s, \epsilon) = -\epsilon \bar{a}(\lambda, \alpha_s, \epsilon) - \beta_0 \bar{a}^2(\lambda, \alpha_s, \epsilon) - \dots$$

- boundary condition  $\bar{a}(1, \alpha_s, \epsilon) = \alpha_s$

## Upshot

- Generating functional for Laurent-series in  $\epsilon$  to all orders

# Solution for form factor

## Result

- Result up to four loops in terms of expansion coefficients of  $A$  and  $G$

$$\mathcal{F}_1 = -\frac{1}{2} \frac{1}{\epsilon^2} A_1 - \frac{1}{2} \frac{1}{\epsilon} G_1$$

$$\mathcal{F}_2 = \frac{1}{8} \frac{1}{\epsilon^4} A_1^2 + \frac{1}{8} \frac{1}{\epsilon^3} A_1 (2G_1 - \beta_0) + \frac{1}{8} \frac{1}{\epsilon^2} (G_1^2 - A_2 - 2\beta_0 G_1) - \frac{1}{4} \frac{1}{\epsilon} G_2$$

$$\begin{aligned} \mathcal{F}_3 = & -\frac{1}{48} \frac{1}{\epsilon^6} A_1^3 - \frac{1}{16} \frac{1}{\epsilon^5} A_1^2 (G_1 - \beta_0) - \frac{1}{144} \frac{1}{\epsilon^4} A_1 (9G_1^2 - 9A_2 - 27\beta_0 G_1 + 8\beta_0^2) \\ & - \frac{1}{144} \frac{1}{\epsilon^3} (3G_1^3 - 9A_2 G_1 - 18A_1 G_2 + 4\beta_1 A_1 - 18\beta_0 G_1^2 + 16\beta_0 A_2 + 24\beta_0^2 G_1) \\ & + \frac{1}{72} \frac{1}{\epsilon^2} (9G_1 G_2 - 4A_3 - 6\beta_1 G_1 - 24\beta_0 G_2) - \frac{1}{6} \frac{1}{\epsilon} G_3 \end{aligned}$$

$$\mathcal{F}_4 = \dots$$

- Expansion in terms of bare coupling  $a_s^b = \alpha_s^b / (4\pi)$

$$\mathcal{F}(Q^2, \alpha_s^b) = 1 + \sum_{l=1} \left( a_s^b \right)^l \left( \frac{Q^2}{\mu^2} \right)^{-l\epsilon} \mathcal{F}_l$$

# Solution for form factor

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$$\mathcal{F}_4 = \dots$$

$\mathcal{F}_2$ : Hamberg, van Neerven, Matsuura '88; Harlander '00; Gehrmann, Huber, Maitre '05; S.M. Vermaseren, Vogt '05

$\mathcal{F}_3$ : S.M. Vermaseren, Vogt '05; Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser '09; Gehrmann, Glover, Huber, Ikizlerli, Studerus '10

$\mathcal{F}_4$ : Henn, Smirnov, Smirnov, Steinhauser, Lee '16; Lee, Smirnov, Smirnov, Steinhauser '17 & '19; von Manteuffel, Schabinger '19

# Universality of subleading infrared poles

- Universal subleading infrared poles in function  $G$  Dixon, Magnea, Sterman '08
- Coefficients  $G_n$  at  $n$ -loops are composed of:
  - twice the  $\delta(1 - x)$  part  $B^q$  in parton splitting function
  - single-logarithmic anomalous dimension of *eikonal* form factor
  - terms associated with QCD beta function

$$G_1 = 2B_1^q + f_1^q + \varepsilon f_{01}^q ,$$

$$G_2 = 2B_2^q + (f_2^q + \beta_0 f_{01}^q) + \varepsilon f_{02}^q ,$$

$$G_3 = 2B_3^q + (f_3^q + \beta_1 f_{01}^q + \beta_0 f_{02}^q) + \varepsilon f_{03}^q ,$$

$$G_4 = 2B_4^q + (f_4^q + \beta_2 f_{01}^q + \beta_1 f_{02}^q + \beta_0 f_{03}^q) + \varepsilon f_{04}^q$$

- $f$ -function shares maximal non-Abelian property and Casimir scaling with cusp anomalous dimensions

$$f_1^q = 0 ,$$

$$f_2^q = C_F \left\{ C_A \left( \frac{808}{27} - \frac{22}{3} \zeta_2 - 28 \zeta_3 \right) + n_f \left( -\frac{112}{27} + \frac{4}{3} \zeta_2 \right) \right\} ,$$

$$f_3^q = \dots$$

$$\begin{aligned}
f_4^q &= C_F C_A^3 \left( \frac{9364079}{6561} - \frac{1186735}{729} \zeta_2 - \frac{837988}{243} \zeta_3 + \frac{115801}{27} \zeta_4 + \frac{11896}{9} \zeta_2 \zeta_3 + 3952 \zeta_5 \right. \\
&\quad \left. - \frac{4796}{9} \zeta_3^2 - \frac{129547}{54} \zeta_6 - 416 \zeta_2 \zeta_5 - 720 \zeta_3 \zeta_4 - 1700 \zeta_7 - \frac{1}{24} f_{4, d_{FA}^{(4)}}^q \right) \\
&\quad + \frac{d_{FA}^{(4)}}{n_c} f_{4, d_{FA}^{(4)}}^q + C_F^3 n_f f_{4, n_f}^q C_F^3 + C_F^2 C_A n_f f_{4, n_f}^q C_F^2 C_A + C_F C_A^2 n_f \left( -\frac{243859}{432} \right. \\
&\quad \left. + \frac{389228}{729} \zeta_2 + \frac{105193}{243} \zeta_3 - \frac{22667}{18} \zeta_4 - \frac{848}{9} \zeta_2 \zeta_3 - \frac{860}{27} \zeta_5 + \frac{2740}{9} \zeta_3^2 + \frac{5179}{9} \zeta_6 \right. \\
&\quad \left. + \frac{1}{24} b_{4, d_{FF}^{(4)}}^q - \frac{1}{2} f_{4, n_f}^q C_F^2 C_A - \frac{1}{4} f_{4, n_f}^q C_F^3 \right) + n_f \frac{d_{FF}^{(4)}}{n_c} \left( -384 + \frac{4544}{3} \zeta_2 \right. \\
&\quad \left. - \frac{5312}{9} \zeta_3 - \frac{800}{3} \zeta_4 + 128 \zeta_2 \zeta_3 - \frac{21760}{9} \zeta_5 + \frac{1216}{3} \zeta_3^2 + \frac{1184}{9} \zeta_6 - 2 b_{4, d_{FF}^{(4)}}^q \right) \\
&\quad + C_F^2 n_f^2 \left( \frac{16733}{486} - \frac{172}{9} \zeta_2 - \frac{4568}{81} \zeta_3 + \frac{64}{9} \zeta_4 + \frac{32}{3} \zeta_2 \zeta_3 + \frac{304}{9} \zeta_5 \right) \\
&\quad + C_F C_A n_f^2 \left( \frac{27875}{17496} - \frac{15481}{729} \zeta_2 + \frac{32152}{243} \zeta_3 + \frac{388}{9} \zeta_4 - \frac{224}{9} \zeta_2 \zeta_3 - 112 \zeta_5 \right) \\
&\quad + C_F n_f^3 \left( -\frac{16160}{6561} - \frac{16}{81} \zeta_2 - \frac{400}{243} \zeta_3 + \frac{128}{27} \zeta_4 \right)
\end{aligned}$$

- Numerical values for unknown coefficients

$$f_{4, d_{FA}^{(4)}}^q, f_{4, d_{FA}^{(4)}}^q, f_{4, n_f}^q C_F^3, f_{4, n_f}^q C_F^2 C_A$$

available from Mellin moments for DIS coefficient function at four loops

Davies, Ruijl, Ueda, Vermaseren, Vogt '16; S. M., Ruijl, Ueda, Vermaseren, Vogt (to appear)

- prediction for complete structure of  $\epsilon$ -poles of quark form factor in QCD at four loops (all terms  $\epsilon^{-8} \dots \epsilon^{-1}$ )

$$\begin{aligned} \mathcal{F}_4 \Big|_{1/\epsilon} = & C_F^4 (-2212.8 \pm 0.3) + C_F^3 C_A (-1601.9 \pm 0.5) + C_F^2 C_A^2 (19661.7 \pm 0.5) \\ & + C_F C_A^3 (-13274.1 \pm 1.0) + \frac{d_{FA}^{(4)}}{n_c} (262.3 \pm 12.5) + C_F^3 n_f (2140. \pm 750.) \\ & + C_F^2 C_A n_f (-12800. \pm 750.) + C_F C_A^2 n_f (10320. \mp 560.) + n_f \frac{d_{FF}^{(4)}}{n_c} (53.12744) \\ & + C_F^2 n_f^2 (1604.851) + C_F C_A n_f^2 (-2304.682) + C_F n_f^3 (158.0655) \end{aligned}$$

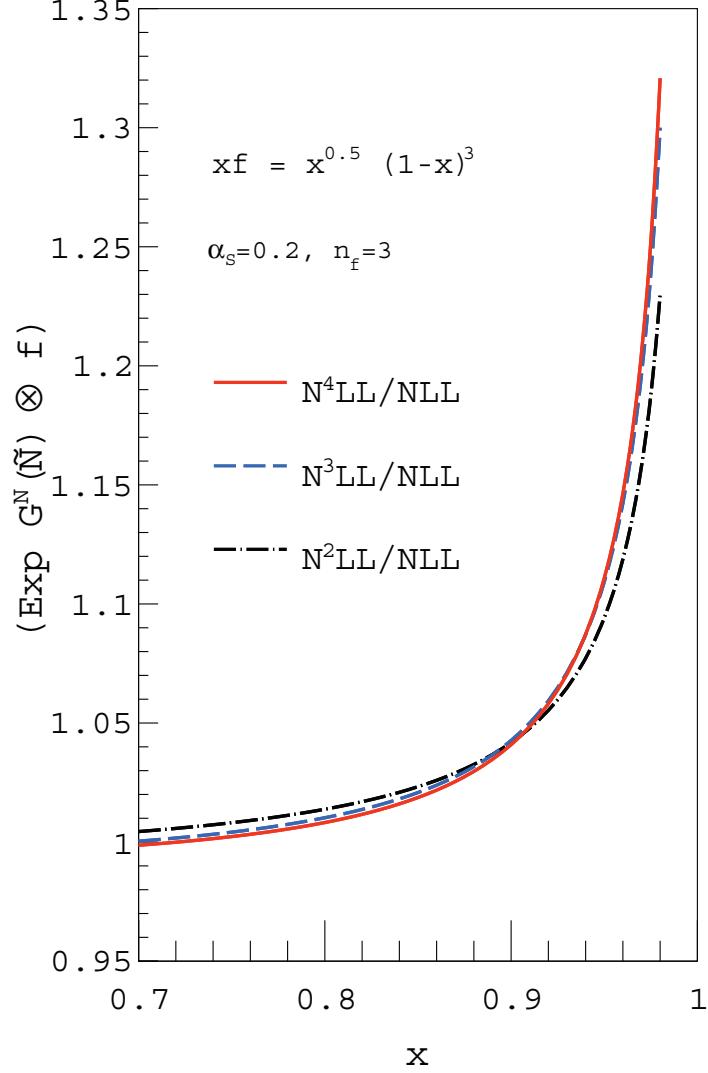
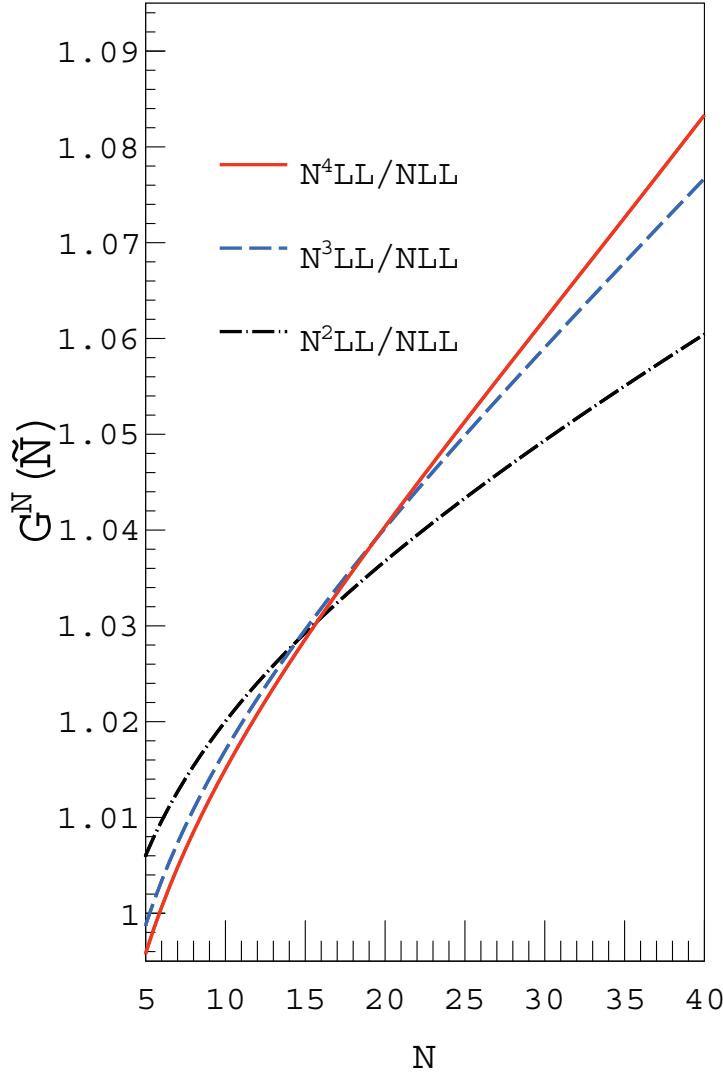
# *DIS coefficient functions at four loops*

## *Result*

- Four-loop coefficient function  $c_{2,q}^{(4)}$  known  $\frac{\ln^7(1-x)}{(1-x)_+}, \dots, \frac{1}{(1-x)_+}$
- New result for  $\frac{1}{(1-x)_+}$  term
  - best estimate (using partial large- $n_c$  information)

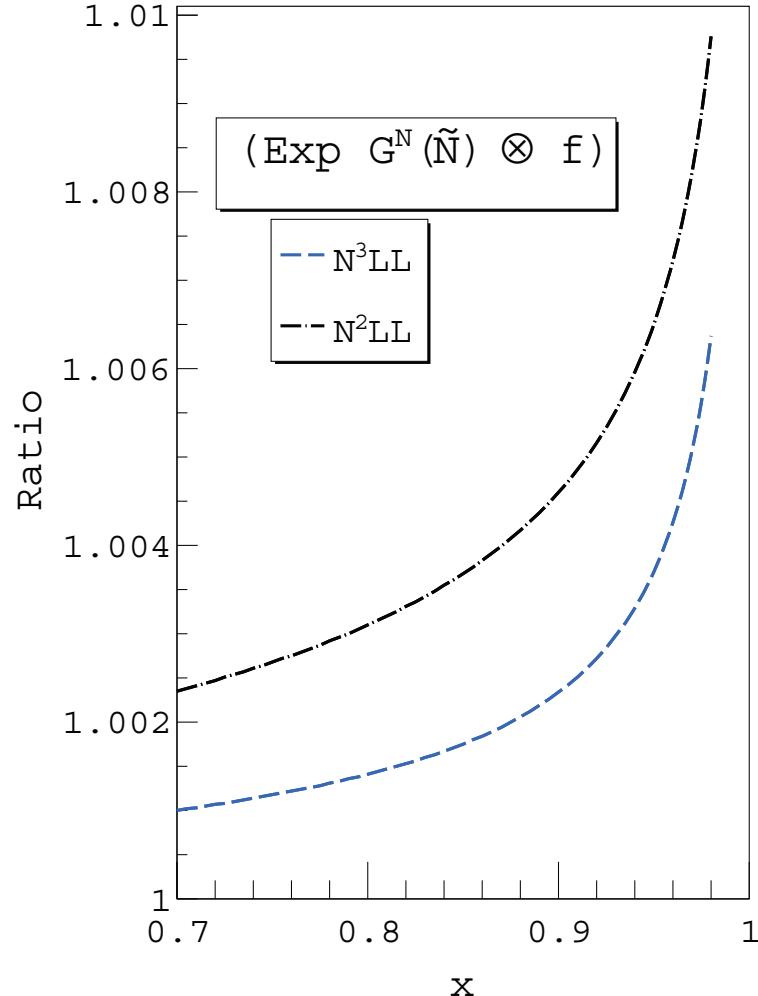
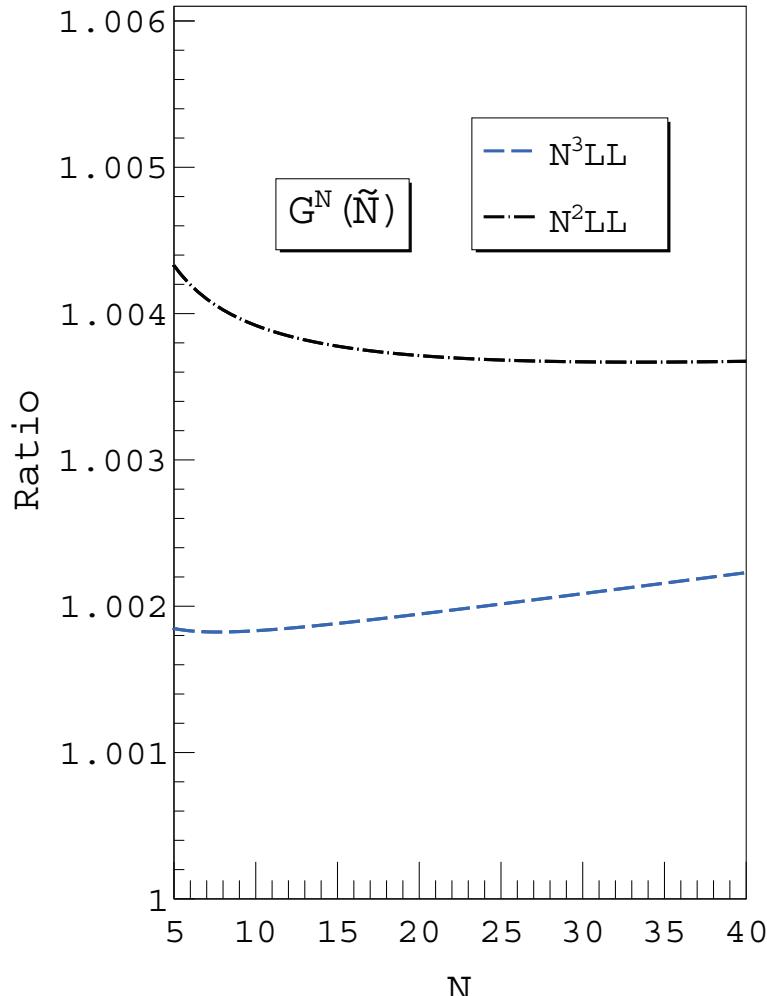
$$c_{2,q}^{(4)} \Big|_{\frac{1}{(1-x)_+}, \text{best}} = (3.874 \pm 0.010) \cdot 10^4 + (-3.494 \pm 0.032) \cdot 10^4 n_f + 2062.715 n_f^2 \\ - 12.08488 n_f^3 + 47.55183 n_f f_{l11}$$

# Numerical results for DIS (I)



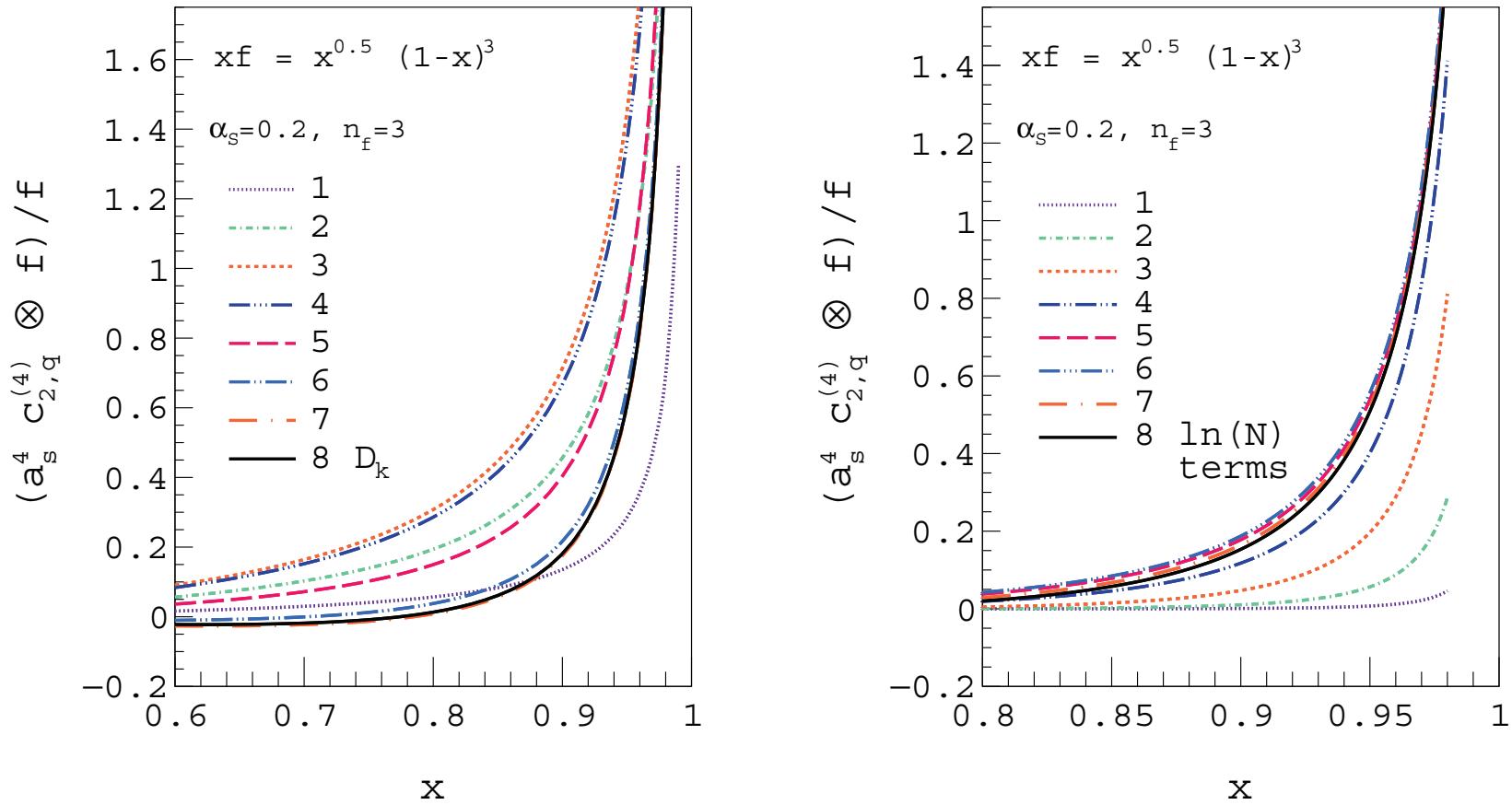
- Left: Resummed exponent  $G^N$  normalized to NLL for DIS plotted successively up to  $N^4\text{LL}$  for  $\alpha_s = 0.2$  and  $n_f = 3$
- Right: Resummed series convoluted with typical shape for a quark distribution  $xf = x^{0.5}(1-x)^3$  up to  $N^4\text{LL}$

# Numerical results for DIS (II)



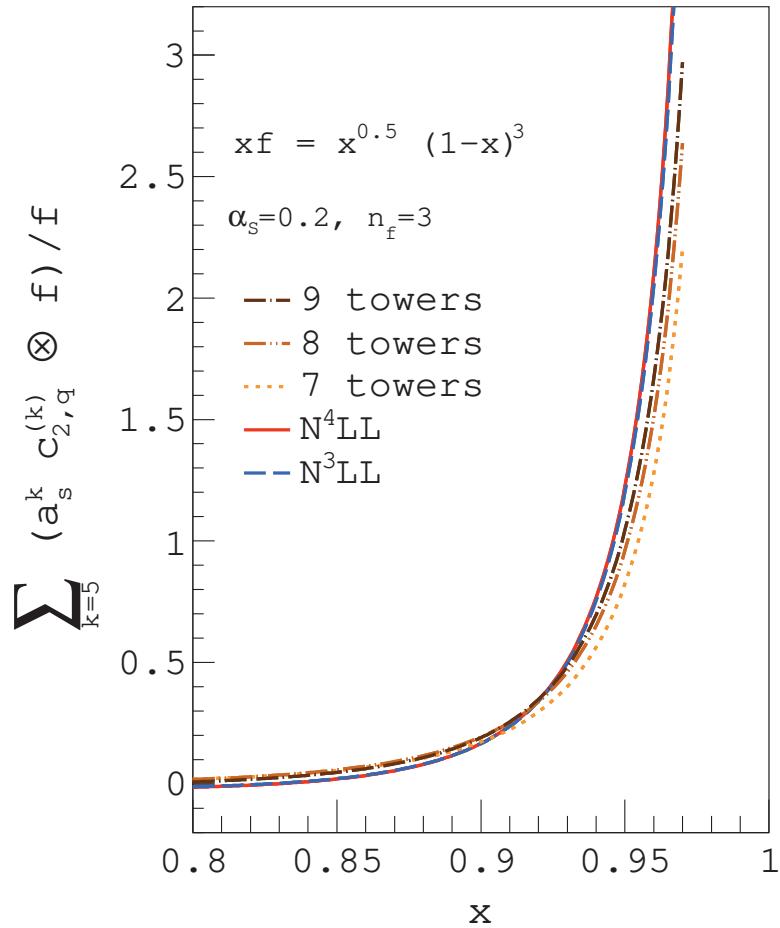
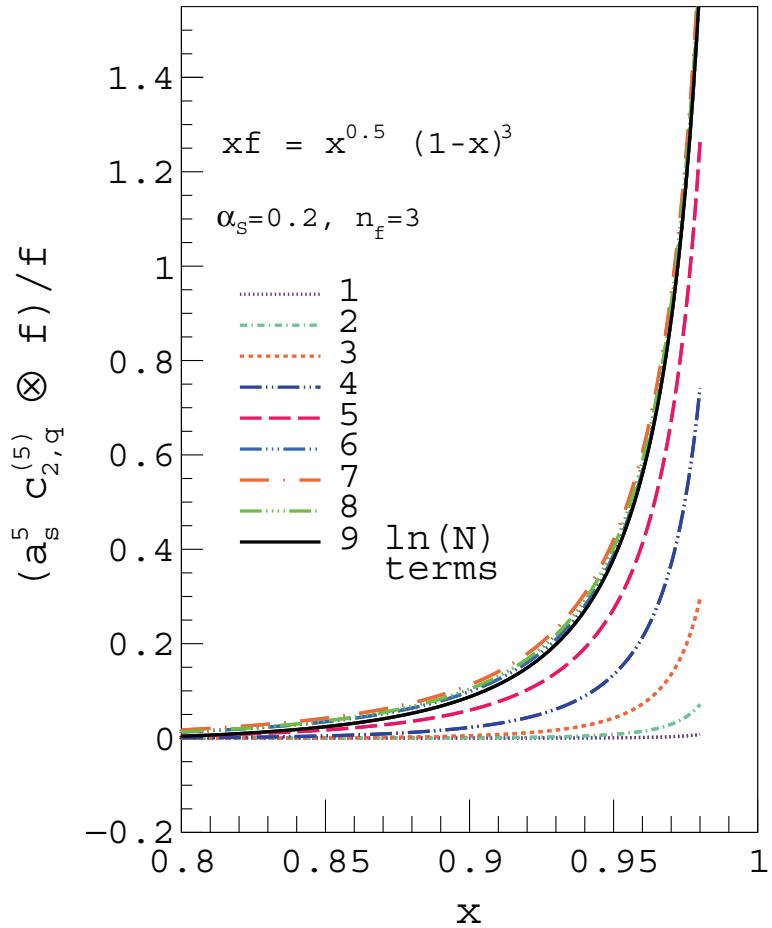
- Comparison between large- $n_c$  approximation and exact result at each resummed order for  $\alpha_s = 0.2$  and  $n_f = 3$  light flavors
- Left: Ratio for DIS resummed exponent  $G^N$  as function of Mellin- $N$
- Right: Ratio for resummed series convoluted with typical input shape  $xf = x^{0.5}(1-x)^3$  plotted against  $x$

# Numerical results for DIS (III)



- Left: DIS Wilson coefficient  $c_{2,q}^{(4)}$  convoluted with input shape  $xf$  with successive addition of plus-distributions  $\mathcal{D}_k = \ln^k(1-x)/(1-x)_+$  starting from highest term
- Right: Same with the successive addition of the DIS  $N$ -space logarithms

# Numerical results for DIS (IV)



- Left: Successive approximations of the five-loop coefficient function  $c_{2,q}^{(5)}$  by large- $N$  terms illustrated by convolution with input shape  $xf$
- Right: Corresponding results for effect of higher terms beyond  $\alpha_s^4$  obtained from tower expansion up to nine towers and from exponentiation up to  $N^4$ LL accuracy

# Summary

- Determination of strong coupling  $\alpha_s$  at 1% precision requires QCD radiative corrections to evolution equations at  $N^3\text{LO}$
- Matrix elements of local operators of twist two
  - non-singlet anomalous dimensions  $\gamma_{\text{ns}}^{(3),\pm,\text{v}}(N)$  (fixed Mellin moments and exact results for large- $n_c$ ) at  $N^3\text{LO}$
  - quartic Casimir contributions to singlet anomalous dimension  $\gamma_{ij}^{(3)}(N)$  at  $N^3\text{LO}$
- Quark and gluon cusp anomalous dimensions
  - generalization of the lower-order ‘Casimir scaling’
- Phenomenology for DIS
  - new estimate for four-loop coefficient function  $c_{2,\text{q}}^{(4)}$  down to  $\frac{1}{(1-x)_+}$
  - resummation to  $N^4\text{LL}$  accuracy