Mathematics of Multi-Scale Loop Integrals

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- Precision physics requires quantum corrections, hence Feynman integrals.

- Standard method for the calculation of Feynman integrals:
  Differential equations and integration-by-parts identities.

- Solving a differential equation requires boundary data.
  This is a simpler problem (less scales).

The method of differential equations

- "Calculation" of Feynman integrals: Express a Feynman integral as a linear combination of standarised iterated integrals (multiple polylogarithms, elliptic multiple polylogarithms, iterated integrals of modular forms, ...).

- This is easy if differential equation is of the form

$$\left[d + \varepsilon A(x)\right] I(\varepsilon, x) \quad = \quad 0,$$

  e.g. $A(x)$ independent of ε and has only simple poles.

- In order to get to this form, we may
  - transform the basis of master integrals $I' = UI$
    (this is like a gauge transformation)
  - transform the kinematic variables $x \to x'$
    (this is like a general coordinate transformation)

Outlook

- All we need to know to compute a Feynman integral are the right transformations $I' = UI$ and $x \to x'$.

- Sounds easy, but it isn't.

- Mathematics comes to our help: We are dealing with a problem from algebraic geometry:
  - Multiple polylogarithms: Moduli space $\mathcal{M}_{0,n}$ of a punctured Riemann sphere.
  - Elliptic multiple polylogarithms: Moduli space $\mathcal{M}_{1,n}$ of a punctured torus.

Be cautious

- At one-loop we only have $\mathrm{Li}_1(x)$ and $\mathrm{Li}_2(x)$. The naive generalisation is

  $$\mathrm{Li}_n(x).$$

  This is not enough. We need at genus zero

  $$\mathrm{Li}_{n_1,\dots,n_k}(x_1,\dots,x_k).$$

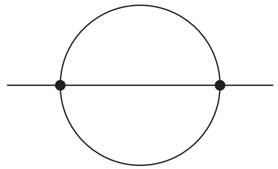
- At the current state-of-the-art we have iterated integrals related to the moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$. The naive generalisation is

  $$\mathcal{M}_{g,n}.$$

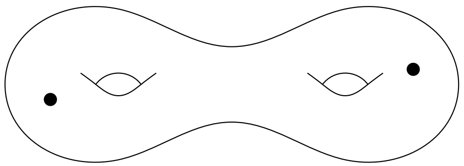
  We expect this not to be enough. We might need to go beyond moduli spaces of curves to moduli spaces of surfaces and beyond.

Cartoons

- QFT:



- Strings:



- Mathematics: