Localized Tuning of Particle Accelerator Focusing

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Background

The beta function describes the size of the beam. The sigma of the normal distribution of the beam is proportional to the square root of the beta function.

$$\sigma_{x,y} \propto \sqrt{\beta_{x,y}}$$

Quadrupoles are the magnets that focus the beam. They have a linearly increasing field strength that pushes the particles towards the center.

Transfer matrices are used to see what effect the quadrupole, or series of quadrupoles, had on a particle's trajectory.

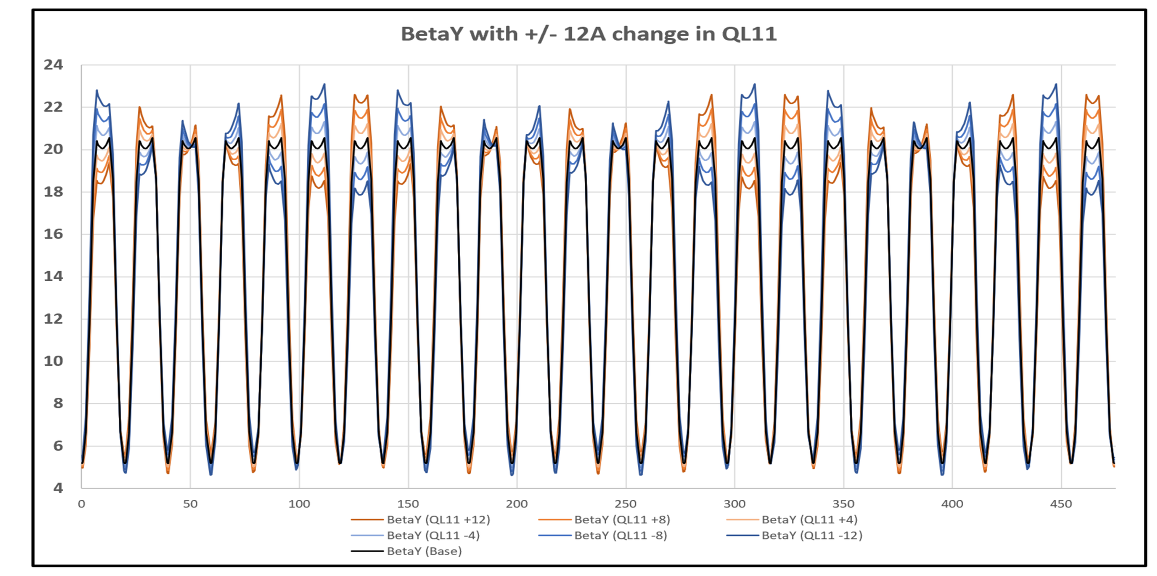
$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

Motivation

The aim of the project was to find a means to locally adjust the beta function in an accelerator.

Existing methods of beta tuning have a non-local effect on the beam. This presents a problem when adjustments to the beta function need to be made only at a specific section of the accelerator, as is the case during injection, extraction, and collimation.

A method for local tuning would be beneficial for the operation of accelerators



Non-local Example – Jeffrey Eldred, 2020

Quadrupole Coefficients

To achieve local tuning, it is necessary to derive coefficients for the quadrupoles that will allow the values of Q to be manipulated without the overall transfer matrix, M, changing.

$$M = Q_5 T_{54} Q_4 T_{43} Q_3 T_{32} Q_2 T_{21} Q_1$$

$$Q_i = \begin{bmatrix} 1 & 0 \\ q_i & 1 \end{bmatrix},$$

$$T_{ji} = \begin{bmatrix} \sqrt{\frac{\beta_j}{\beta_i}}(\cos\phi_{ji} + \alpha_i \sin\phi_{ji}) & \sqrt{\beta_i\beta_j}\sin\phi_{ji} \\ -\frac{1+\alpha_i\alpha_j}{\sqrt{\beta_i\beta_j}}\sin\phi_{ji} + \frac{\alpha_i-\alpha_j}{\sqrt{\beta_i\beta_j}}\cos\phi_{ji} & \sqrt{\frac{\beta_i}{\beta_j}}(\cos\phi_{ji} - \alpha_j\sin\phi_{ji}) \end{bmatrix}$$

To simplify the calculations, the expression was converted to normalized coordinates. (Jeff Eldred, Beams 8341-V1)

$$Q_i = B_i \tilde{Q}_i B_i^{-1}, \quad T_{ji} = B_j R_{ji} B_i^{-1},$$

$$B_i = \begin{bmatrix} \sqrt{\beta_i} & 0 \\ \frac{-\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{bmatrix}, \quad B_i^{-1} = \begin{bmatrix} \frac{1}{\sqrt{\beta_i}} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \sqrt{\beta_i} \end{bmatrix},$$

$$\tilde{Q}_i = \begin{bmatrix} 1 & 0 \\ k_i & 1 \end{bmatrix}, \quad R_{ji} = \begin{bmatrix} \cos\phi_{ji} & \sin\phi_{ji} \\ -\sin\phi_{ji} & \cos\phi_{ji} \end{bmatrix},$$

$$\tilde{M} = B_3^{-1} M B_1^{-1} = \tilde{Q}_3 R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1.$$

$$k_i = \beta_i q_i$$

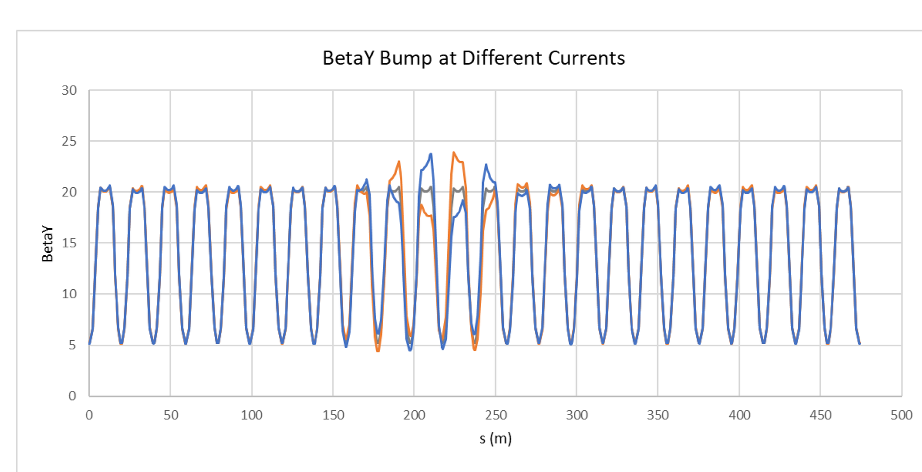
From that, the following system of equations is derived. The aim was to solve each coefficient in terms of K3 and test to determine if the derived coefficients produced a local beta bump.

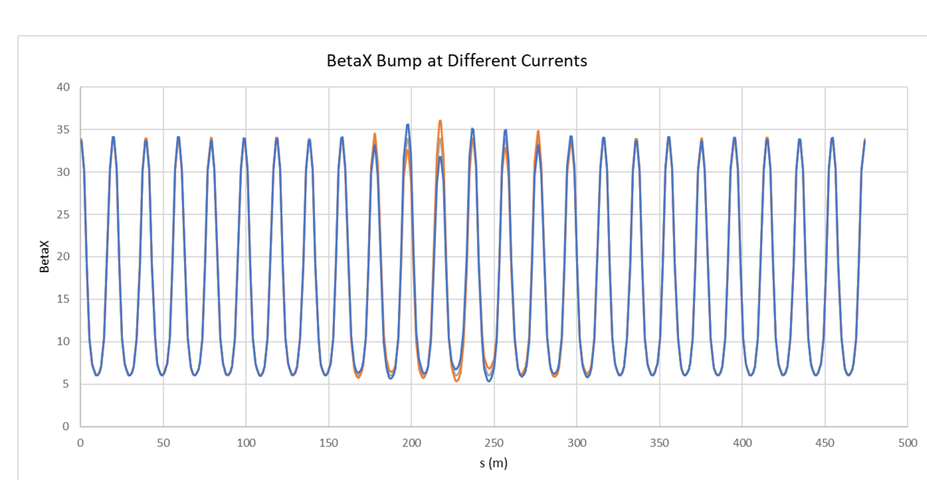
$$0 = k_4 s_{54} c_{41} + k_3 s_{53} c_{31} + k_2 s_{52} c_{21} + k_1 s_{51}$$
$$0 = k_4 s_{54} s_{41} + k_3 s_{53} s_{31} + k_2 s_{52} s_{21}$$
$$0 = k_5 c_{51} + k_4 c_{54} c_{41} + k_3 c_{53} c_{31} + k_2 c_{52} c_{21} + k_1 c_{51}$$
$$0 = k_5 s_{51} + k_4 c_{54} s_{41} + k_3 c_{53} s_{31} + k_2 c_{52} s_{21}$$

Quadrupole Coefficients Results

Below are the resulting coefficients and plots.

- $k1=(((c_{21}*(c_{31}*(c_{52}*s_{53}-c_{53}*s_{52})*s_{41}*s_{51}*s_{54}-(c_{41}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{51}*s_{53}-c_{51}*(c_{53}*s_{54}-c_{54}*s_{53})*s_{41}*s_{52})*s_{31})+(c_{31}*(c_{41}*(c_{53}*s_{54}-c_{54}*s_{53})*s_{51}*s_{52}-c_{51}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41}*s_{53})+c_{41}*c_{51}*(c_{52}*s_{53}-c_{53}*s_{52})*s_{31}*s_{54})*s_{21})*k3)/((c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*s_{41}*s_{54}-(c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*s_{52}-c_{51}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41})*s_{21})*s_{51}))$
- $k2=((-(c_{31}*(c_{51}*s_{53}-c_{53}*s_{51})*s_{41}*s_{54}-(c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*s_{53}-c_{51}*(c_{53}*s_{54}-c_{54}*s_{53})*s_{41})*s_{31})*k3)/(c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*s_{41}*s_{54}-(c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*s_{52}-c_{51}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41})*s_{21}))$
- $k4=((-(c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*s_{31}*s_{53}-(c_{31}*(c_{51}*s_{53}-c_{53}*s_{51})*s_{52}-c_{51}*(c_{52}*s_{53}-c_{53}*s_{52})*s_{31})*s_{21})*k3)/(c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*s_{41}*s_{54}-(c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*s_{52}-c_{51}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41})*s_{21}))$
- $k5=((-(c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*(c_{53}*s_{54}-c_{54}*s_{53})*s_{31}*s_{41}-(c_{31}*(c_{51}*s_{53}-c_{53}*s_{51})*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41}-c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*(c_{52}*s_{53}-c_{53}*s_{52})*s_{31})*s_{21})*k3)/((c_{21}*(c_{51}*s_{52}-c_{52}*s_{51})*s_{41}*s_{54}-(c_{41}*(c_{51}*s_{54}-c_{54}*s_{51})*s_{52}-c_{51}*(c_{52}*s_{54}-c_{54}*s_{52})*s_{41})*s_{21})*s_{51}))$





Beta Functions

The aim of the second project was to find the beta funtions that would describe the effect the quad kicks had on the beta values. To find the beta functions, the transfer matrices had to be found

$$M(s_3|s_1) = B_3 \tilde{Q}_3^{1/2} R_{32} \tilde{Q}_2 R_{21} \tilde{Q}_1^{1/2} B_1^{-1}$$

$$\beta_j = \begin{bmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \end{bmatrix} \begin{bmatrix} \beta_i \\ 0 \\ \beta_i^{-1} \end{bmatrix}$$

$$\beta_j = M_{11}^2 \beta_i + M_{12}^2 \beta_i^{-1}$$

Running the calculations in Sage resulted in these expressions:

$$beta2h = C21 \cdot S21 \cdot beta2 \cdot k1 + beta2$$

$$beta3h = C31 \cdot S31 \cdot beta3 \cdot k1 + 2 \cdot C32 \cdot S32 \cdot beta3 \cdot k2 + beta3$$

$$beta4h = C41 \cdot S41 \cdot beta4 \cdot k1 + 2 \cdot C42 \cdot S42 \cdot beta4 \cdot k2 + 2 \cdot C43 \cdot C43 \cdot beta4 \cdot k3 + beta4$$

Summary

The goal of this project was to find a means to locally tune beta functions in an accelerator. The first half of the project was aimed at deriving the coefficients needed to produce the local bump, and the second half of the project sought to derive the resulting beta functions.

The project was successful on both counts; a local bump was produced, and the beta functions were found. This result will potentially have implications for the operation of accelerators going forward. Specifically, it may aid in beam injection, extraction, and collimation.