

Lessons learned from hadronization tuning with

Genie

version 3

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on behalf of the GENIE Collaboration

Outline

GENIE version 3

Hadronization in neutrino interactions

The AGKY hadronization model

Review of charged averaged multiplicity data

Review of charged averaged multiplicity data

Tuning the AGKY hadronization model against bubble chamber data in GENIE v3

Conclusions

Introduction to GENIE

GENIE Version 3



graphics by

grafiche.test1@gmail.com

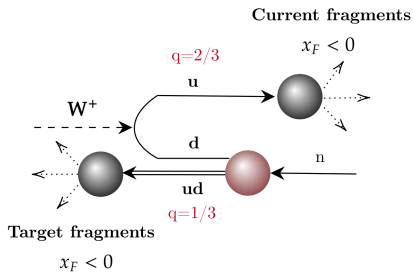
- “Comprehensive Model Configurations” (CMC)
 - Self-consistent collections of primary process models
 - Complete characterisation against public data
 - Possibility to host configurations provided by experiments
- Access to tunes against datasets
 - Same interface
 - Documentation:
 - ⇒ Manual
 - ⇒ Dedicated web page – tunes.genie-mc.org/
 - ⇒ Publications to be released soon

Previously, GENIE released a single ‘Default’ comprehensive model

- Constructed from the most ‘up-to-date’ underlying models
- Some of the parameters were tuned to data
- Missing parameter uncertainty characterization

Hadronization in neutrino interactions - Overview

Hadronization models provide information of the final state hadrons after a DIS interaction



νn neutrino interaction.

It affects:

- E_ν reconstruction
→ Detector response to hadrons
- Efficiency to classify events as NC/CC
→ Shower mismodelling
- Background estimation

**There is no model to describe
exclusive hadronic multiparticle production**

AGKY hadronization model

- The GENIE Collaboration developed the AGKY model [AGKY model, Eur.Phys.J.C63:1-10,2009]
- An effective KNO-based hadronization model was built for low W
- "Integrated" with PYTHIA to cover the full kinematic space
- Wide validity range
- Originally 'tuned' and validated against bubble chamber data
 - Captures several observations on the characteristics of neutrino-induced hadron showers

AGKY main ingredients

- **Low- W empirical model** for SIS/DIS events at $W < 2.3\text{GeV}/c^2$
- **PYTHIA 6** for events with $W > 3\text{GeV}/c^2$
- **Linear transition** from the low- W empirical model to PYTHIA for $2.3 < W < 3\text{GeV}/c^2$

Low-W empirical model implementation

In this talk, we focus on the **description** of the hadronization data and tuning of $\langle n_{ch} \rangle$

The **averaged charged multiplicity** follows an empirical logarithmic law:

$$\langle n_{ch} \rangle = \alpha_{ch} + \beta_{ch} \cdot \ln \left(\frac{W^2}{\text{GeV}^2/c^4} \right)$$

- α_{ch} and β_{ch} are free parameters and depend on the type of interaction
- In GENIE, α_{ch} and β_{ch} are those measured by the FNAL 12FT and BEBC 7FT bubble chambers.

Parameter	νp	νn	$\bar{\nu} p$	$\bar{\nu} n$
α_{ch}	0.40	-0.20	0.02	0.80
β_{ch}	1.42	1.42	1.28	0.95

An insight to the previous AGKY ‘tuning’ approach

- The parameters used consider a linear fit to the multiplicity distributions
 - The experiment extracted those parameters by assuming the linearity over all the $\ln(W^2)$ range
- The FNAL and BEBC data was extracted from neutrino interactions on 2H

Problems with the current approach

- In GENIE, the low- W empirical model is only strictly applied at $W < 2.3\text{GeV}/c^2$
 - Only a small subset of this data satisfies this condition
 - Therefore, the agreement at the PYTHIA region does not improve
- Both experiments released data on H and 2H , but only 2H data is used
- Some of the datasets have been reanalyzed

There is no clear agreement between each of the analysis

An insight to the previous AGKY ‘tuning’ approach

Experiment	W^2 [GeV ² /c ⁴]	Target	α_{ch}	β_{ch}
$\nu_\mu + p \rightarrow \mu^- X^{++}$				
FNAL (1976)	[1.5,150]	H	1.09 ± 0.38	1.09 ± 0.03
BEBC (1983)	[12,112]	H	-0.05 ± 0.11	1.43 ± 0.04
FNAL (1983)	[1.5,160]	² H	0.05 ± 0.07	1.42 ± 0.03
BEBC (1990)	[6,150]	H	0.911 ± 0.224	1.131 ± 0.086
BEBC (1992)	[12,144]	H	0.40 ± 0.13	1.25 ± 0.04
$\nu_\mu + n \rightarrow \mu^- X^+$				
BEBC (1984)	[6, 112]	² H	1.75 ± 0.12	1.31 ± 0.04
FNAL (1983)	[1.5,160]	² H	-0.20 ± 0.07	1.42 ± 0.03
$\bar{\nu}_\mu + p \rightarrow \mu^+ X^0$				
FNAL (1982)	[1.7,74]	H	-0.44 ± 0.13	1.48 ± 0.06
BEBC (1982)	[5,75]	² H	0.02 ± 0.20	1.28 ± 0.08
BEBC (1983)	[12,96]	H	-0.56 ± 0.25	1.42 ± 0.08
BEBC (1990)	[6,144]	H	$0.222 \pm 0.0.362$	1.117 ± 0.100
BEBC (1992)	[12,144]	H	-0.44 ± 0.20	1.30 ± 0.06
$\bar{\nu}_\mu + n \rightarrow \mu^+ X^-$				
BEBC (1982)	[1.5,56]	² H	0.80 ± 0.09	0.95 ± 0.04

An insight to the previous AGKY ‘tuning’ approach

Solutions to the current approach

- In GENIE, the low- W empirical model is only strictly applied at $W^2 < 2.3\text{GeV}/c^2$
 - The AGKY model needs to be tuned as a whole
- Both experiments released data on H and 2H , but only neutrino hadroproduction on 2H data is used
 - Other studies already show existing tensions between H and 2H datasets
 - Need to review the cuts applied to the data to remove possible data-MC mismodelling
 - Partial fits may help to understand the tensions
 - Both H and 2H data will be considered in the tune
- Some of the datasets have been reanalyzed
 - Include in the fit the most updated analysis

There is no clear agreement between each of the analysis

PYTHIA 6 tuning

- The PYTHIA MC is based on the **Lund string fragmentation framework**
- The default PYTHIA parameters are tuned to high energy pp experiments ($\sqrt{s} \sim 35$ GeV)
- NOMAD (NUX) PYTHIA6 tuning was adopted in 2007.
- Some PYTHIA6 defaults were restored in later GENIE re-tune (2010).

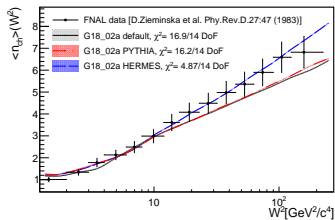
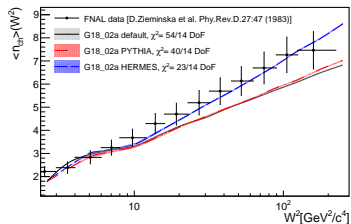
	PYTHIA default	NUX 2001	GENIE 2010 re-tune
$s\bar{s}$ production suppression	0.30	0.21	0.30
$\langle p_T^2 \rangle$ (GeV^2)	0.36	0.44	0.44
Non-gaussian p_T tail parameterization	0.01	0.01	0.01
Fragmentation cut-off energy (GeV)	0.80	0.20	0.20

In the GENIE PYTHIA study,

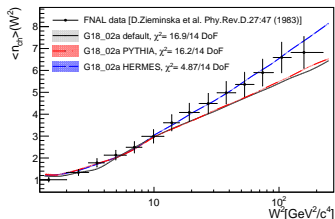
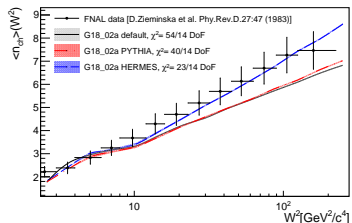
- Not enough knobs were found to influence predictions
- Parameter uncertainties were not quantified

PYTHIA 6 tuning

- Another study [arXiv:1412.4301] proved that GENIE shows better agreement against $\langle n_{ch} \rangle$ data when using the parameters extracted by the HERMES experiment
 - HERMES is a $e^- e^+$ experiment operating at 27.6 GeV
 - The improvement was mainly due to the change of Lund a and Lund b
- However, **PYTHIA was not directly tuned** against the available neutrino hadroproduction bubble chamber data.



'Default' GENIE agreement with bubble chamber data



- Clearly, at the PYTHIA region, the charged average multiplicity is under predicted.
- Though at the low- W region the prediction seems reasonable, there is no information about the uncertainty

To improve this, we tuned the AGKY model against the average multiplicity data

- The GENIE database has been updated
- All the cuts applied have been revisited

Neutrino production experiments on H

Both BEBC and FNAL released data on H for

- $\nu_{\mu}p \rightarrow \mu^{-}X^{++}$
- $\bar{\nu}_{\mu}p \rightarrow \mu^{+}X^0$

The analysis procedure is similar in both cases:

Neutrino production analysis on νH interactions

- For ν_{μ} interactions on H,
⇒ Scan for events with three (or more) charged secondaries
- For $\bar{\nu}_{\mu}$ interactions on H,
⇒ Scan for events with two¹ (or more) charged secondaries

¹Events with only one secondary can occur. However, these events are removed due to 1) low scanning efficiency and 2) bad $E_{\bar{\nu}}$ reconstruction

Neutrino production experiments on H

Similar cuts are considered in both experiments:

- Muons are detected with **external muon identifiers**
 - ⇒ Muon momenta cuts can be applied for good μ -ID
 - ⇒ The FNAL_15FT experiment also requires $p_{TR}^{\mu} \geq 1 \text{ GeV}/c$
- Tracks are identified by using bubble chamber density, energy loss, range in hydrogen, break point probability and kinematic fits.
 - ⇒ If left unidentified, **the remaining charged hadrons are assumed to be pions**
- **QE events are removed** by applying a cut on either W or/and Q^2
- See backup slides for detailed information about the cuts applied to each dataset.

Neutrino production experiments on H

In bubble chamber experiments on H or 2H , the identification of neutral particles is difficult due to the low density of the medium

- The transverse momentum balance method is used to estimate the neutrino energy
- This allows to correct for missing neutral particles in the event

$$E_{\nu}^{reco} = p_L^{\mu} + p_L^c \left(1 + \frac{|\vec{p}_T^{\mu} + \vec{p}_T^c|}{\sum_{i=1}^{n_{ch}} |\vec{p}_{Ti}|} \right)$$

$L, T \equiv$ longitudinal and transverse components of the momentum, for the muon (μ) and the charged-hadron system (c) relative to the neutrino direction

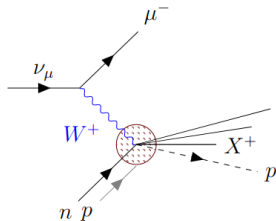
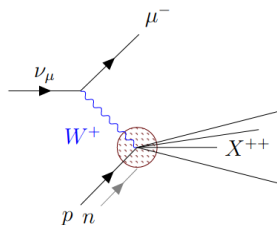
p_{Ti} is the transverse momentum of the i th charged hadron

- BEBC estimated $\Delta E_{\nu} \sim 10\text{-}15\%$ [PhysRevD.27.47]
- In some analysis, cuts on E_{ν}^{reco} are applied

Neutrino production experiments on ${}^2\text{H}$

The analysis is different as both $\nu_\mu p$ and $\nu_\mu n$ are possible

- ν_μ on p :
 - Odd number of prongs
- ν_μ on n :
 - Even number of prongs + no visible spectator
 - Odd number of prongs + visible proton
- Requirements for a visible proton:
 - Backward-going particle ($\vec{p}_p^T < 0$)
 - Forward-going particle with $p < p_{max}^p$ (analysis dependent)
- For FNAL, $p_p > 200\text{MeV}/c$ to be detected as a prong.



Correcting for rescattering effects on 2H samples

The rescattering is produced through primary hadrons interaction with the spectator nucleon in the deuterium (p or n)

- Rescattering events have a clear signature on the **event energy balance**:

$$\varepsilon = \sum_i (E_i - p_{li}) - M_n$$

being E_i and p_{li} the i th secondary particle energy and longitudinal momentum component respectively.

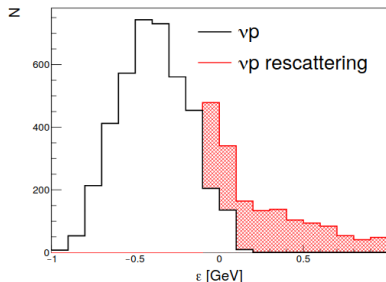
- **No rescattering case**, the mass of the target nucleon, $M_n = m_p$ or m_n
 - In an ideal detector, $\varepsilon \equiv 0$
 - In bubble chambers (missing neutral particles), $\varepsilon < 0$
- **For rescattering events**, $M_i = M_{{}^2H} \Rightarrow \varepsilon > 0$ with a maximum value of $M_d - M_n$

Correcting for rescattering effects on 2H samples

BEBC_12FT: cut on the energy balance

[Z.Phys.C Particles and Fields 24,119-131 (1984)]

- $\varepsilon > 0.1$ GeV
- $\varepsilon > -0.1$ and $(E_T^{miss})^2 > 0.075$ (GeV/c) 2



(b) $\nu_{\mu}p$ events under the even prong topology assumption. Rescattering events are highlighted in red.

Main problems with the data releases

- The **migration between samples is not quantified** (e.g. from $\nu_\mu p$ to $\nu_\mu n$)
- The data releases contain **only statistical errors**.
- Information about **systematic errors** is
 - hardly included in the release
 - If not included it is quoted as a %
 - **No correlations** available between
 - Datasets coming from the same release
 - Different hadronization observables (e.g $\langle n_{ch} \rangle$ and D_{ch})
- Different experiments apply **different corrections** which may affect the physics interpretation
 - ⇒ Rescattering correction for BEBC data only
- Datasets are often **re-analysed**:
 - Some information is lost in between the different papers
 - The W^2 range may differ significantly between them
 - ⇒ This may reduce the available data to fit important modelling aspects, such as the low-W AGKY empirical model

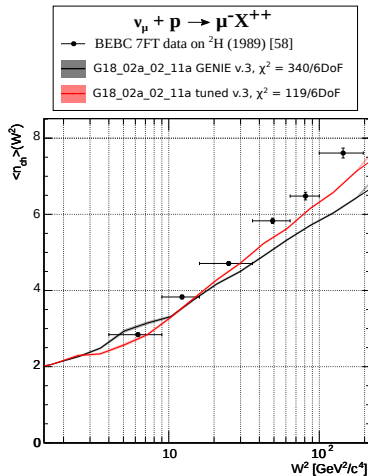
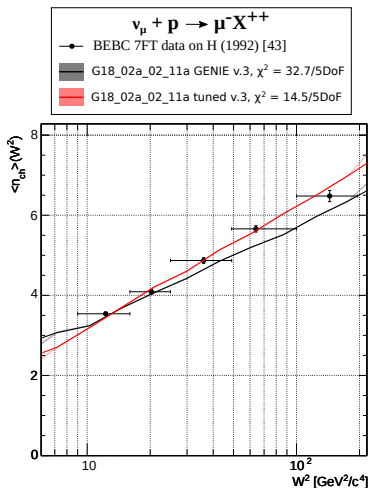
Tuning as a tool to expose tensions between datasets

Few details on the averaged charged multiplicity tune:

- Simulated GENIE events for H and 2H + G18_02a tune as base model
 - Same data analysis is applied to each prediction
 - The latest releases are used for a global tune on $H + {}^2H$
- ⇒ Better agreement + parameter uncertainties
- **Tensions between H and 2H samples are re-encountered**
 - And clear discrepancies between data arise

Tuning as a tool to expose tensions between datasets

Tensions between H and 2H samples are re-encountered



Conclusions

- Neutrino MC generators are a great tool to **expose tensions between data**
- The GENIE Collaboration is moving towards the development and validation of comprehensive model configuration
 - Carefully tuned against neutrino data
- **Old datasets are still very valuable!**
- It is crucial to **understand the analysis procedure** of each of the datasets in order to
 - Use them for tuning purposes
 - Underpin possible model weakness and tensions between data⇒ **Release the analysis code**
- Historical bubble chamber data lack of proper systematic error studies, which is also missing for many nuclear datasets (**correlations between samples**)
- Some of the problems encountered with historical data are present in nuclear datasets

Backup slides

Core GENIE mission - from GENIE by-law

- Framework "... provide a state-of-the-art neutrino MC generator for the world experimental neutrino community ..."
- Universality "... simulate all processes for all neutrino species and nuclear targets, from MeV to PeV energy scales ..."
- Global fit "... perform global fits to neutrino, charged-lepton and hadron scattering data and provide global neutrino interaction model tunes ..."

Neutrino MC generators allow to

- Compare data and models
- Compare dataset against dataset
 - Data quality and data sources are increasing \Rightarrow Tensions
 - \Rightarrow Comparing results from different experiments

GENIE global analysis of neutrino data

Released a single 'Default' comprehensive model

- Constructed from the most 'up-to-date' underlying models
- Some of the parameters were tuned to data
- Missing parameter uncertainty characterization

Other alternative modelling elements were available to users, facilitating

- Physics and logical inconsistencies
- Double counting issues
- Loss of agreement with data

Required improvements in the GENIE Software

- Development of the GENIE Comparison Software
- Deep understanding and maintenance of the data archive
- Interface GENIE to the Professor tool
[<https://professor.hepforge.org/>]

GENIE global analysis of neutrino data

For each CMC, several tunes can be available by

1. Incorporating different experimental data
2. Considering different modelling degrees of freedom
3. Adding parameter priors and/or nuisance parameters

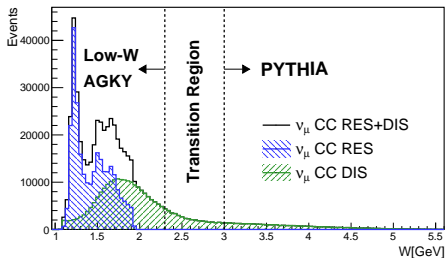
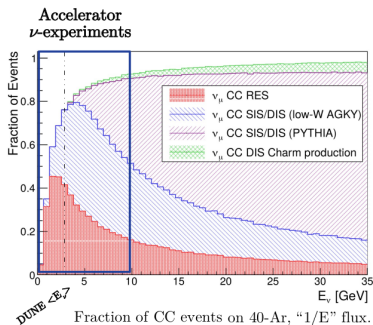
Main GENIE tuning effort

- Re-tune of the bare-nucleon cross-section model - To be published soon
- **Hadronization tune** - main focus of this talk
- Nuclear tunes

AGKY hadronization model

- **Low-W empirical model** for SIS/DIS events at $W < 2.3\text{GeV}/c^2$
- **PYTHIA 6** for events with $W > 3\text{GeV}/c^2$
- **Linear transition** from the low-W empirical model to PYTHIA for $2.3 < W < 3\text{GeV}/c^2$

The Low-W empirical model plays a central role on accelerator neutrino experiments



Low-W empirical model implementation

General workflow

1. Estimate the **averaged charged multiplicity**, $\langle n_{ch} \rangle$,
2. Calculate the **total averaged multiplicity**, $\langle n_{tot} \rangle$,
3. Go from average to **actual multiplicities** on an event-by-event basis, n_{tot} ,
 - The particles are **not independently produced**
 - ⇒ The total multiplicity is estimated with the **KNO scaling law**
4. Generate particle momenta

Data on hadron shower characteristics

- **Average charged and neutral particle multiplicities**
- Multiplicity dispersion and correlations between different hadrons,
- Fragmentation functions (z distributions), ...

See C.Andreopoulos talk @ NuSTEC Workshop on Shallow-and Deep-Inelastic Scattering (11-13 October 2018) for more information

PYTHIA 6 ($W > 3\text{GeV}/c^2$)

- Based on the **Lund string fragmentation framework**
 - Describes the dynamics of one-dimensional relativistic strings that are stretched between colored partons
 - The hadronization is described as break-ups in a string producing $q\bar{q}$ pairs
- The **fragmentation function** gives the probability to produce a hadronic system with a given z

$$f(z) \propto z^{-1}(1-z)^a \cdot \exp\left(\frac{-bm_{\perp}^2}{z}\right)$$

where m_{\perp}^2 is the transverse mass of the hadron and $z = E/\nu$

- Lund a (a) and Lund b (b) are tunable parameters that are responsible to distribute the available energy to the produced hadron
- The default PYTHIA parameters are tuned to high energy pp experiments ($\sqrt{s} \sim 35 \text{ GeV}$)

Information about datasets used in the tune

Experiment	N_p	W^2 [GeV ² /c ⁴]	Target	Cuts	Syst.	In Fit
$\nu_\mu + p \rightarrow \mu^- X^{++}$						
FNAL (1976)	25	[1.5,150]	H	$Q^2 \geq 1$ GeV $E_\nu \geq 15$ GeV $p_{Ch}^L \geq 10$ GeV/c $p_T^\mu \geq 5$ GeV/c $p_\mu \geq 1$ GeV/c $p_\mu \geq 3$ GeV/c	included	✓
BEBC (1983)	11	[12,112]	H	$x_B \geq 0.1$ $W^2 \geq 9$ GeV ² /c ⁴ $E_{visible} \geq 5$ GeV	5-3%	✗
BEBC (1989)	6	[6,144]	² H	ϵ_{cut} $p_\mu \geq 4$ GeV/c $p_L \leq 300$ GeV/c $p_T^\mu \geq 5$ GeV/c $p_\mu^L \geq 1$ GeV/c	included	✓
FNAL (1983)	14	[1.5,160]	² H	$p_{Ch}^L \geq 5$ GeV/c $p_p \leq 340$ MeV/c ² $W \geq 1.5$ GeV/c ² $E_\nu \geq 10$ GeV	10%	✓
BEBC (1990)	6	[6,150]	H	$Q^2 \geq 1$ (GeV/c) ² $p_\mu \geq 3$ GeV/c $W^2 \geq 4$ GeV ² /c ⁴	~ stat	$W^2 < 9$ GeV ²
BEBC (1992)	5	[12,144]	H	$p_\mu \geq 3$ GeV/c	included	✓

Information about datasets used in the tune

Experiment	N_p	W^2 [GeV ² /c ⁴]	Target	Cuts	Syst.	In Fit
$\bar{\nu}_\mu + p \rightarrow \mu^+ X^0$						
FNAL (1981)	10	[1.7,74]	H	$p_{Ch} \geq 5$ GeV/c $p_{FW}^{Tot} \geq 2$ GeV/c $y_B \geq 0.1$ $y_B \leq 0.8$	~ stat	✓
BEBC (1982)	8	[5,75]	² H	$E_{\bar{p}} \geq 5$ GeV $p_\mu \geq 4$ GeV/c $p_p \leq 300$ MeV/c $p_\mu \geq 3$ GeV/c	~ stat	✗
BEBC (1983)	10	[12,96]	H	$x_B \geq 0.1$ $W^2 \geq 9$ GeV ² /c ⁴ $E_{visible} \geq 5$ GeV	5-3%	$W^2 < 10$ GeV ²
BEBC (1989)	6	[6,144]	² H	ϵ_{cut} $p_\mu \geq 4$ GeV/c $p_p \leq 300$ MeV/c $Q^2 \geq 0.1$ GeV ²	included	✓
BEBC (1990)	6	[6,144]	H	$p_\mu \geq 3$ GeV/c $W^2 \geq 4$ GeV ² /c ⁴	~ stat	
BEBC (1992)	5	[12,144]	H	$p_\mu \geq 3$ GeV/c	included	✓

Information about datasets used in the tune

$\nu_\mu + n \rightarrow \mu^- X^+$						
BEBC (1984)	8	[6, 112]	^2H	ε_{cut} $p_\mu \geq 4 \text{ GeV}/c$ $Q^2 \geq 1 \text{ (GeV}/c)^2$ $W^2 \geq 5 \text{ GeV}^2/c^4$ $p_p \leq 300 \text{ MeV}/c^2$	$\sim \text{stat}$	✗
BEBC (1989)	6	[6,144]	^2H	ε_{cut} $p_\mu \geq 4 \text{ GeV}/c$ $p_p \leq 300 \text{ MeV}/c$ $W \geq 5 \text{ GeV}/c^2$	included	✓
FNAL (1983)	14	[1.5,155]	^2H	$p_\mu^T \geq 1 \text{ GeV}/c$ $p_{Ch}^L \geq 5 \text{ GeV}/c$ $E_\nu \geq 10 \text{ GeV}$ $p_p \leq 340 \text{ MeV}/c^2$	10%	✓
$\bar{\nu}_\mu + n \rightarrow \mu^+ X^-$						
BEBC (1982)	8	[1.5,56]	^2H	$p_\mu \geq 4$ $p_p \leq 300 \text{ MeV}/c$	$\sim \text{stat}$	$W^2 < 10 \text{ GeV}^2$
BEBC (1989)	6	[6,144]	^2H	ε_{cut} $p_\mu \geq 4$ $p_p \leq 300 \text{ MeV}/c$	included	✓

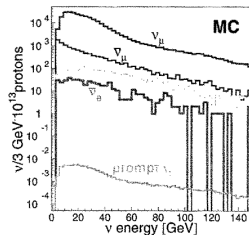
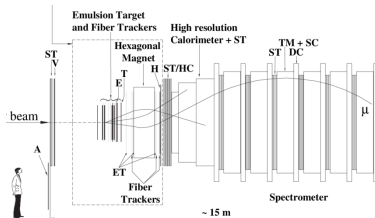
Relevance of nuclear datasets - the CHORUS experiment

The CHORUS² experiment was designed to study $\nu_\mu \rightarrow \nu_\tau$ oscillations

Valuable information on neutrino hadron production was released³

Relevant information for the GENIE event generation:

- Fuji ET-7B nuclear emulsions target \rightarrow The exact mixture of elements in the target is used
- $\langle E_{\nu_\mu} \rangle = 26.9 \text{ GeV} \rightarrow$ Using the CHORUS flux



²[Nuclear Instruments and Methods in Physics Research A 401 (1917) 7-44]

³[arXiv:0707.1586]

Relevance of nuclear datasets - the CHORUS analysis

Tracks are classified as:

Shower particles

Heavily relativistic
charged particles

$$\eta \geq 1$$

Grey particles

Recoil nucleons from
internuclear cascade

$$\eta < 1$$

Black particles

Low energy fragments
emitted from the excited
target nucleus (protons)

- Stop within one emulsion plate
- Removed with the $\beta > 0.25$ cut

where, the pseudorapidity variable is defined as

$$\eta = -\ln \left(\tan \frac{\theta}{2} \right)$$

They released data for averaged charged multiplicity and grey multiplicity as a function of W^2 :

- Define the averaged charge multiplicity data as $\langle n_{ch} \rangle = \langle n_{shower} \rangle - 1$
- Grey multiplicity data as $\langle n_g \rangle$

The CHORUS analysis in GENIE

1. The same cuts are applied to GENIE predictions
2. Black tracks are always removed from the event sample
3. Effectively, the $\langle n_{ch} \rangle$ sample provides information about hadronization
4. The $\langle n_g \rangle$ sample is related with FSI processes after hadronization

So far we are trying to understand the CHORUS analysis in depth so we can use the datasets for tuning purposes

The CHORUS analysis in GENIE

Main problems we are facing:

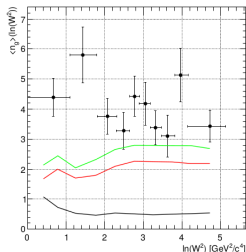
- The need to **understand migrations** between shower, grey and black samples
- The black tracks are removed with $\beta < 0.2\text{GeV}$ applied in other emulsion experiments
- In CHORUS this cut was not applied as black cuts were stopped by the detector
- **Exact method used to reconstruct W^2 :**
- Missing information on the exact methodology to calculate E_{hadro}
- Changes on W^2 affect both samples

Consequences of misunderstanding the analysis

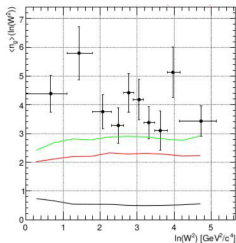
Misunderstandings on the analysis affect the prediction

- In order to use the data for a tune we need to understand **the exact analysis** procedure followed by the experiment
- An example is the reconstruction of W^2 from final state hadrons

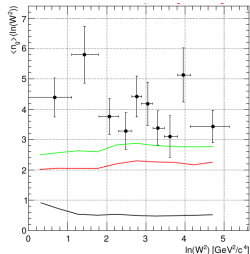
Are neutral pions included in E_{visible} ?



W^2 calculated from
MC-truth information



W_{reco}^2 calculated with
charged particles only



W_{reco}^2 includes π^0

The CHORUS analysis in GENIE

1. Reconstruct the neutrino energy from the muon and visible final state hadrons

$$E_\nu = E_\mu + E_{had}$$

where E_{had} is the energy transfer to the hadronic system,

$$E_{hadro} = KE_{nucl.fragments} + E_{visible}$$

Are neutral pions included in $E_{visible}$?

2. Reconstruct the momentum transferred squared

$$Q_\nu^2 = 2E_\nu(E_\mu - p_\mu \cos\theta_\mu) - m_\mu^2$$

3. Reconstruct the invariant mass squared

$$W_\nu^2 = 2m_n(E_\nu - E_\mu) + m_n^2 - Q_\nu^2$$

where m_n is the nucleon mass