light exotic hadrons on the lattice

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studying (exotic) hadrons with lattice QCD

not concerned with precision, or even **direct** comparison with experiment **at this stage**

answer a simpler question: **"what kinds of hadrons does QCD admit? with what properties?"**

but, intention is to be **as rigorous as possible**

where possible obtain definitive statements about **the QCD spectrum**

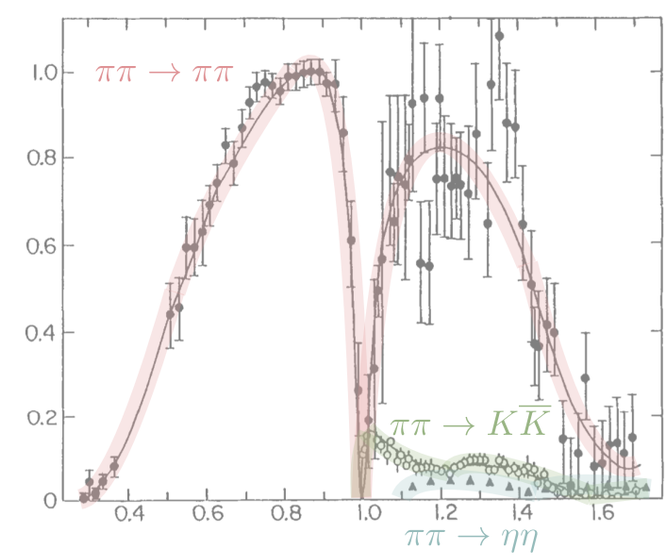
really "a" QCD spectrum,
because we can freely vary the quark masses
and in practice in many cases we will **want** to compute with heavier

"as rigorous as possible"?

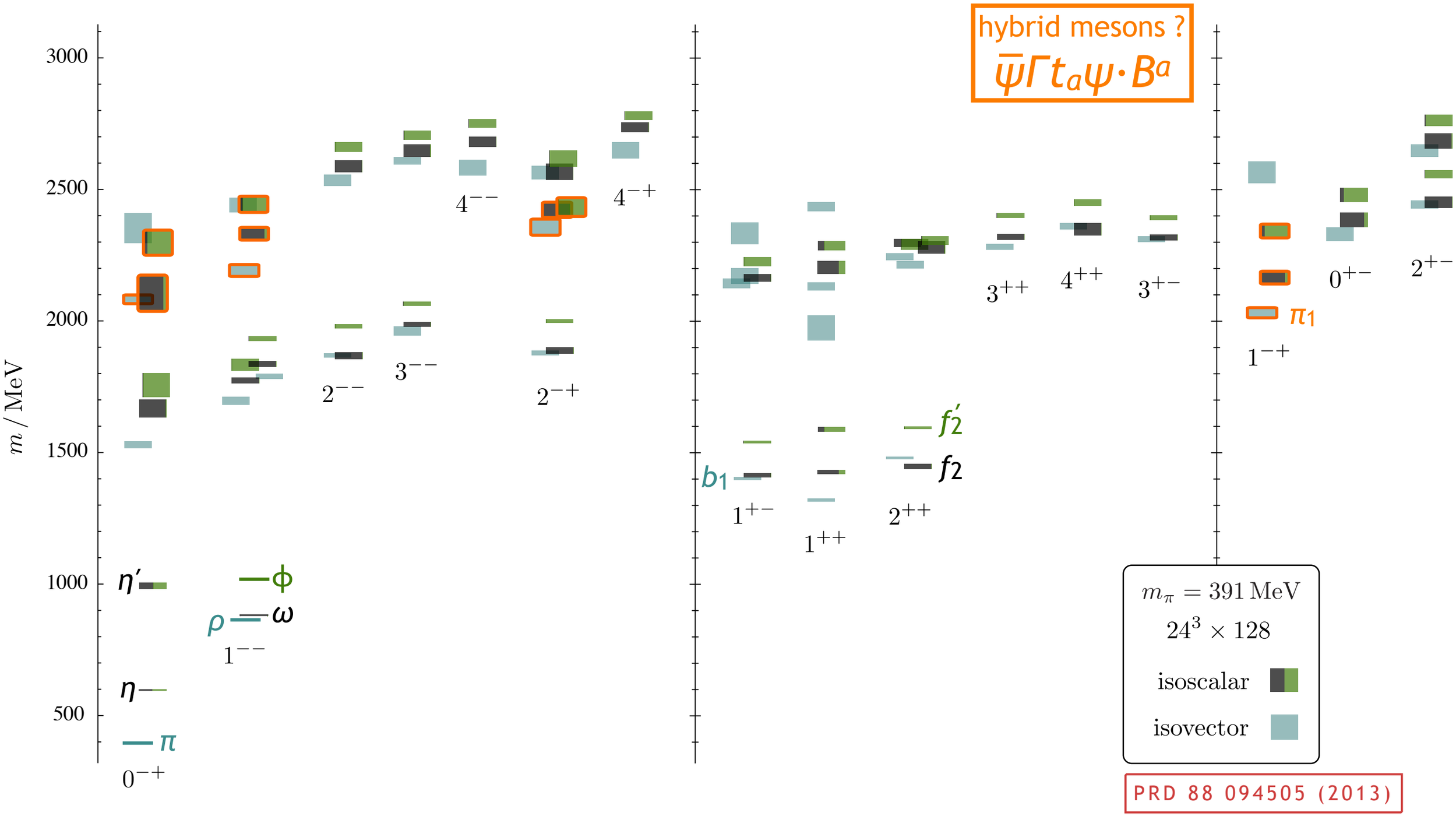
yes, excited states should appear as they actually are, as **unstable resonances**

⇒ we'd like to study scattering amplitudes

so you know in advance that
it is going to get complicated:

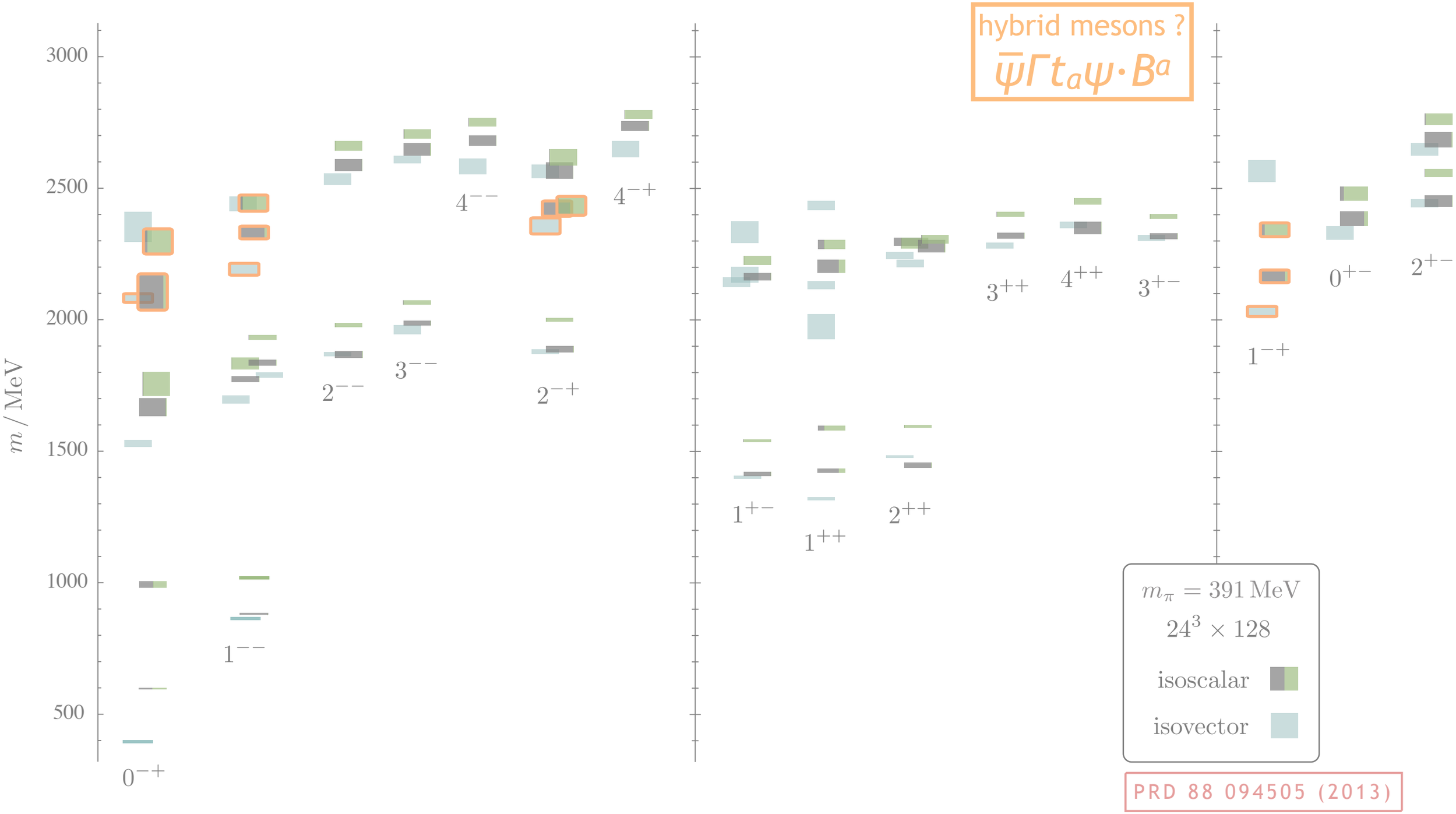


“simpler” calculations with $\bar{\psi}\Gamma D \ldots D\psi$ operator basis



PRD 88 094505 (2013)

calculations with $\bar{\psi}\Gamma D \ldots D\psi$ operator basis



but this doesn't meet my criterion: **as rigorous as possible**

most of these states are actually **resonances**

excited states as unstable resonances

the **discrete spectrum** in a finite-volume can be rigorously related to the **scattering matrix**

a.k.a. "Lüscher method"

need to compute the **complete** spectrum of eigenstates in the relevant energy regions

practically: supplement $q\bar{q}$-like operators with **meson-meson-like operators**

(relatively) easy case: **elastic scattering**

ππ I=1 $J^P=1^-$

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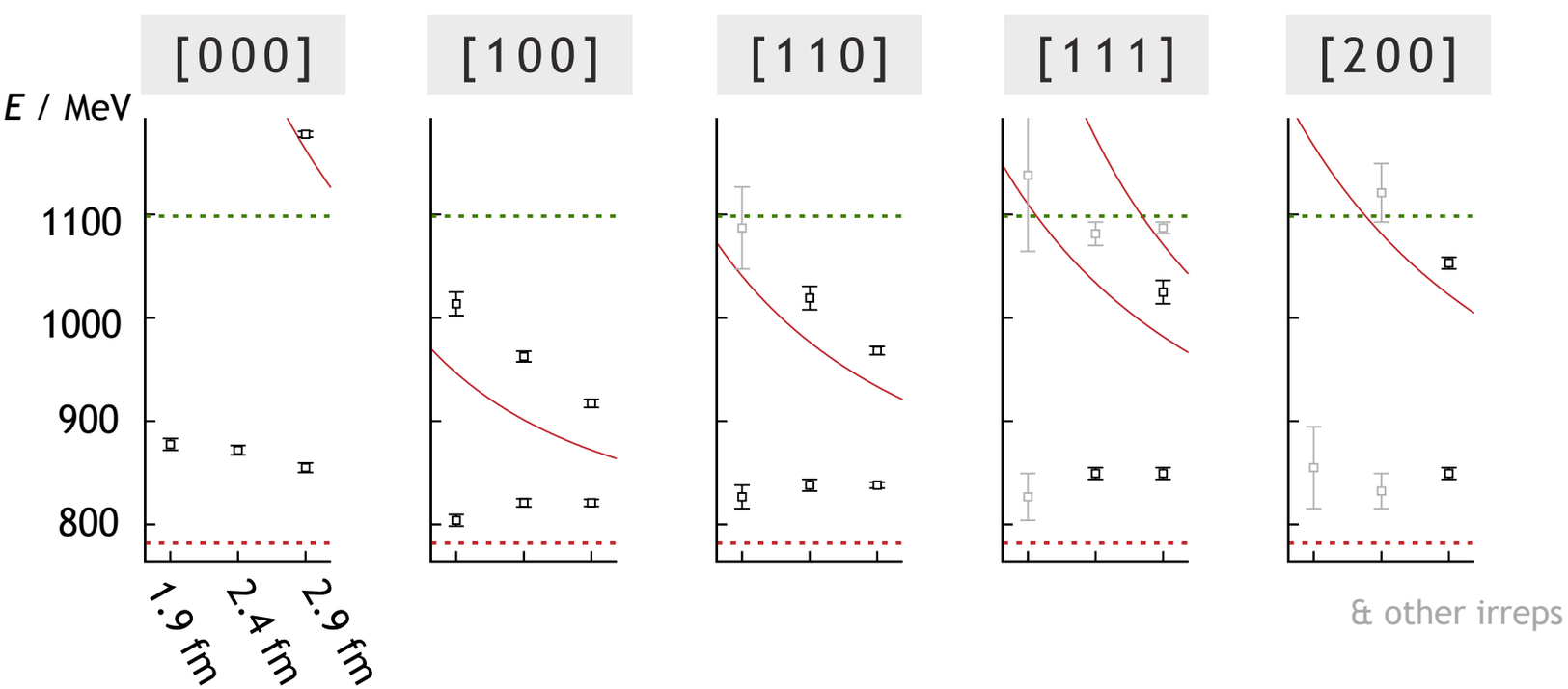
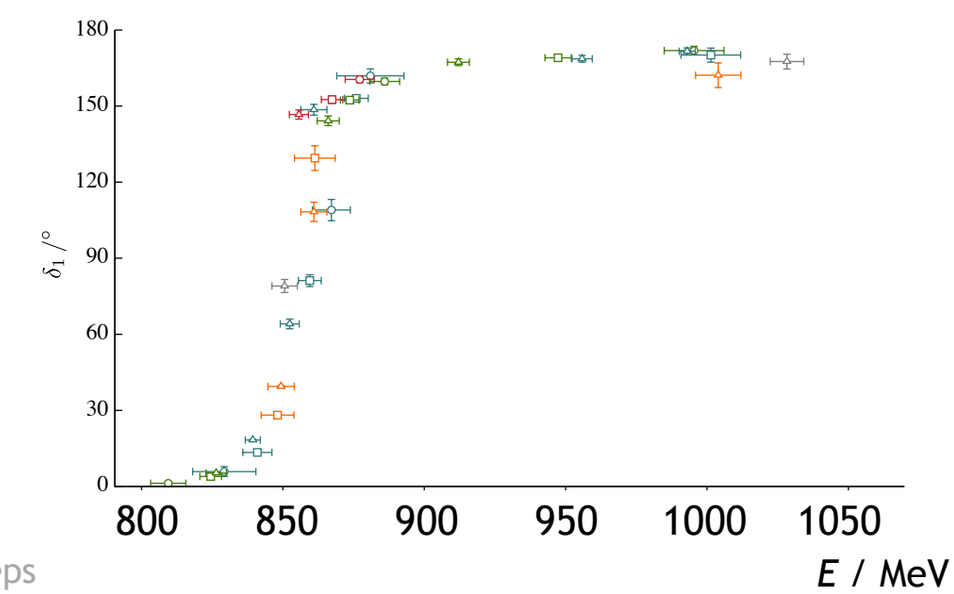
m_π ~ 391 MeV



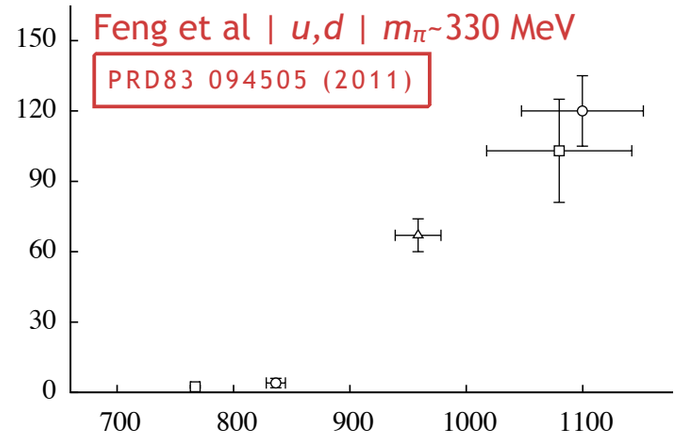
& other irreps

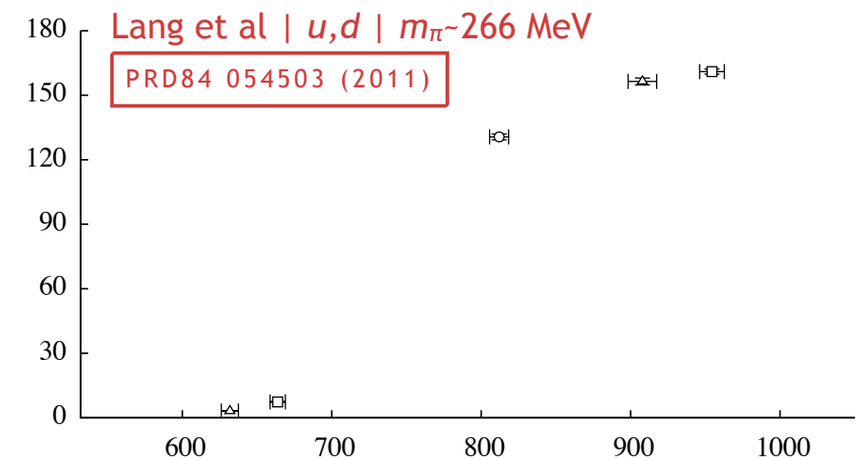
scattering phase-shift

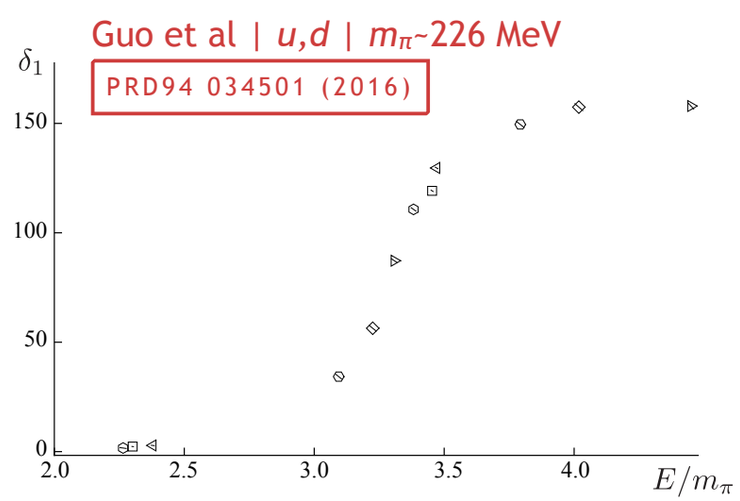


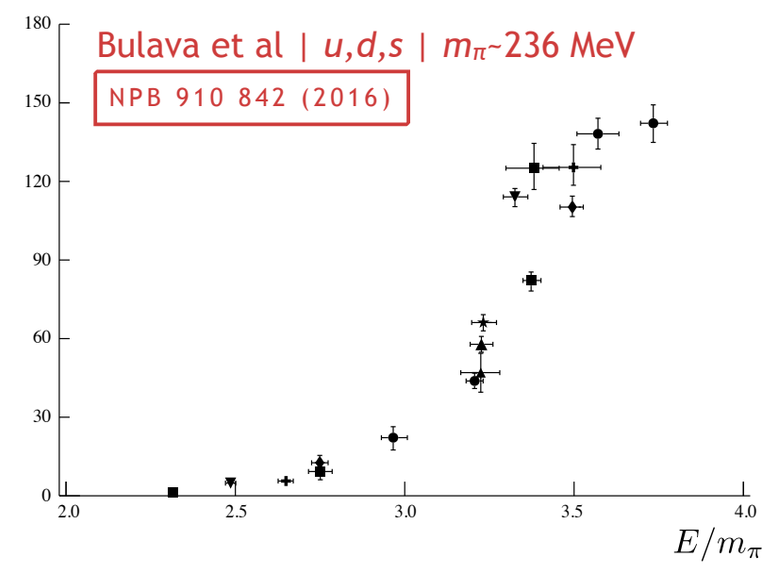
an elastic resonance — the ρ in $\pi\pi$ (isospin=1)

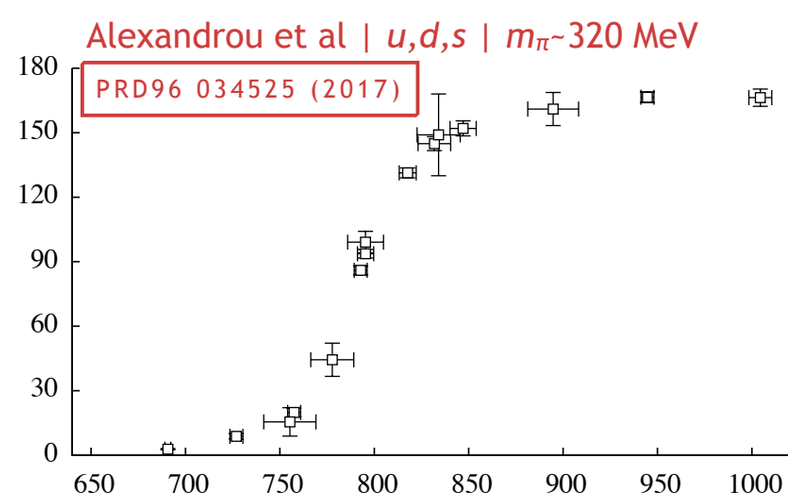
has become a common quantity to compute in lattice QCD …

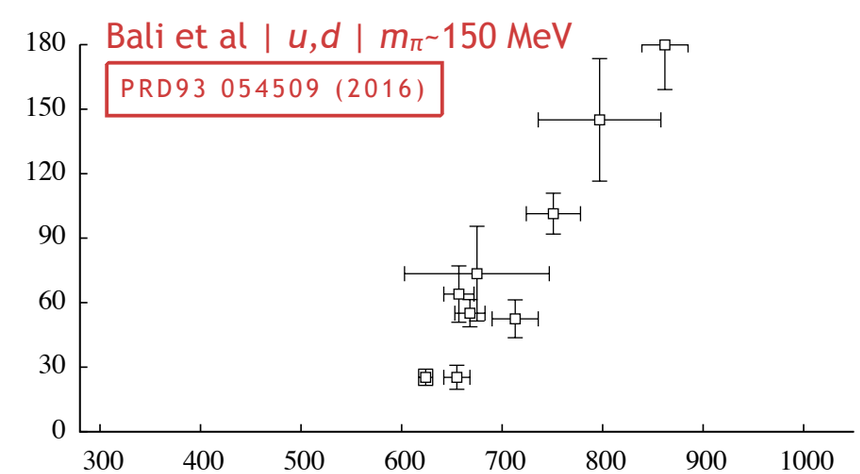










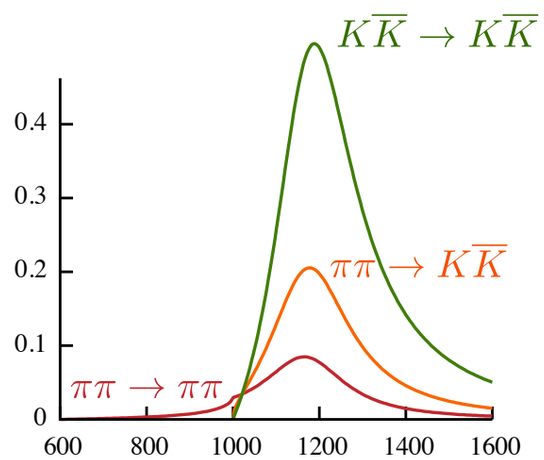


excited states as unstable resonances

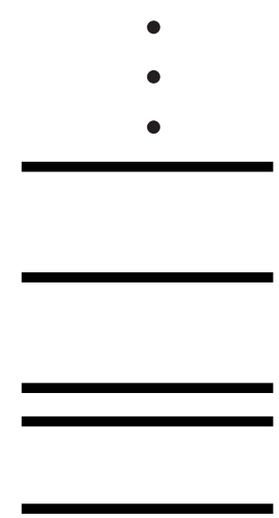
in the case of **coupled-channel scattering** it's more challenging …

e.g. some energy region where $\pi\pi$, $K\bar{K}$ accessible

$$\mathbf{S}(E) = \begin{bmatrix} S_{\pi\pi,\pi\pi}(E) & S_{\pi\pi,K\bar{K}}(E) \\ S_{\pi\pi,K\bar{K}}(E) & S_{K\bar{K},K\bar{K}}(E) \end{bmatrix} \xrightarrow{L\times L\times L} E_n(L)$$



you want to know this



lattice QCD gives you this

?

an approach:

parameterize the energy dependence of the scattering matrix

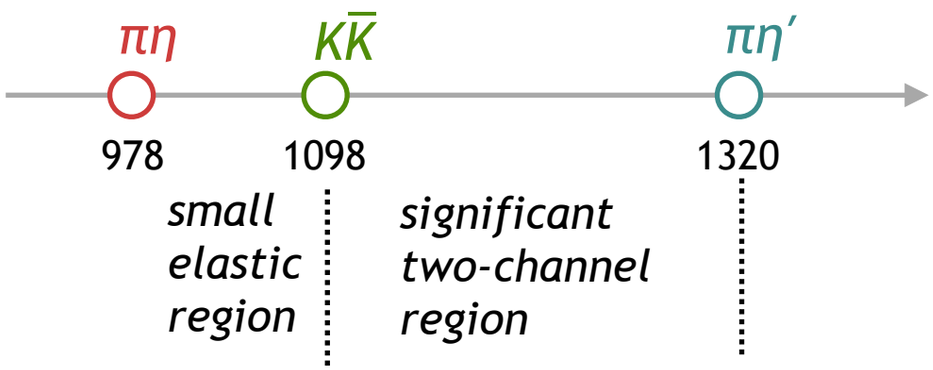
an important observation
— not like experiment
— can't study 'channel-by-channel'
— all channels contribute, have to solve the 'whole problem'

coupled-channel scalar mesons

scalars easiest ‘technically’ (not easiest physics) - 0^- 0^- in *S*-wave

m_π~391 MeV

isospin=1 $\pi\eta$, $K\bar{K}$



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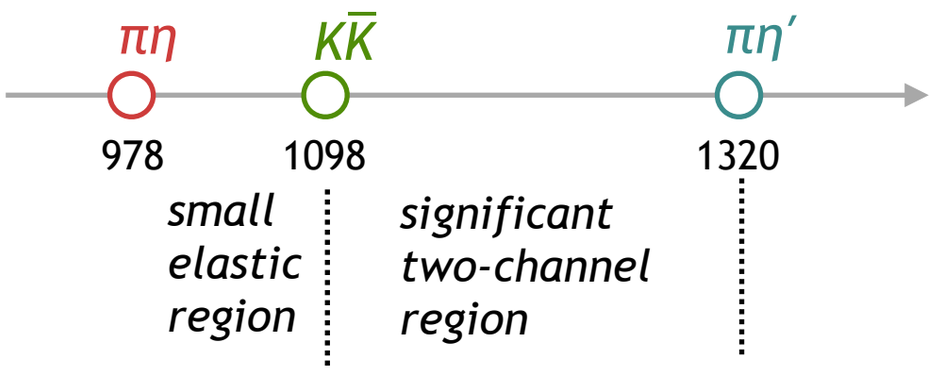

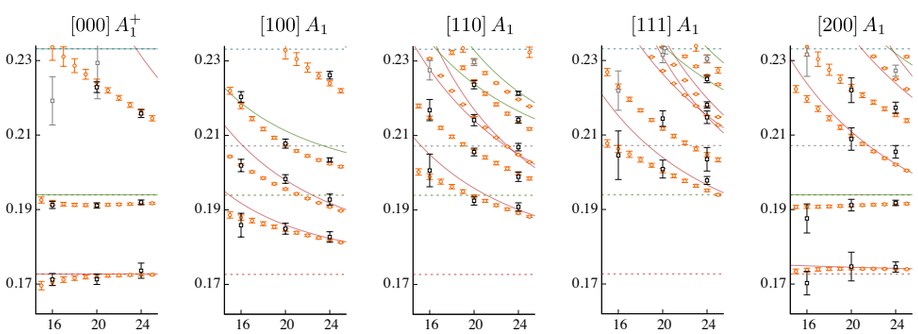

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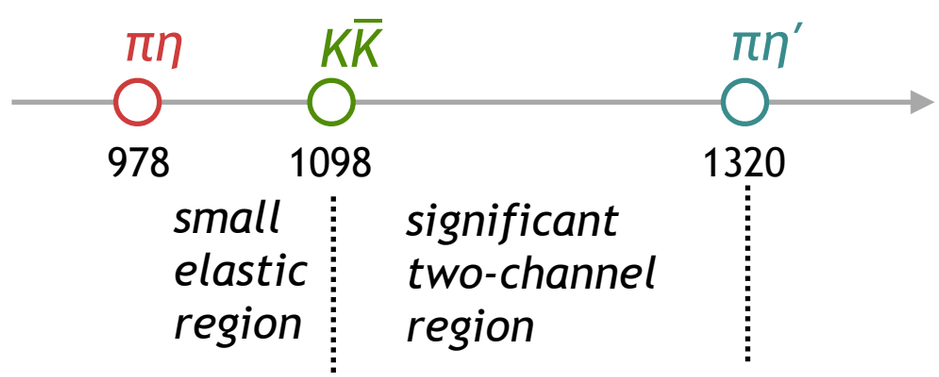

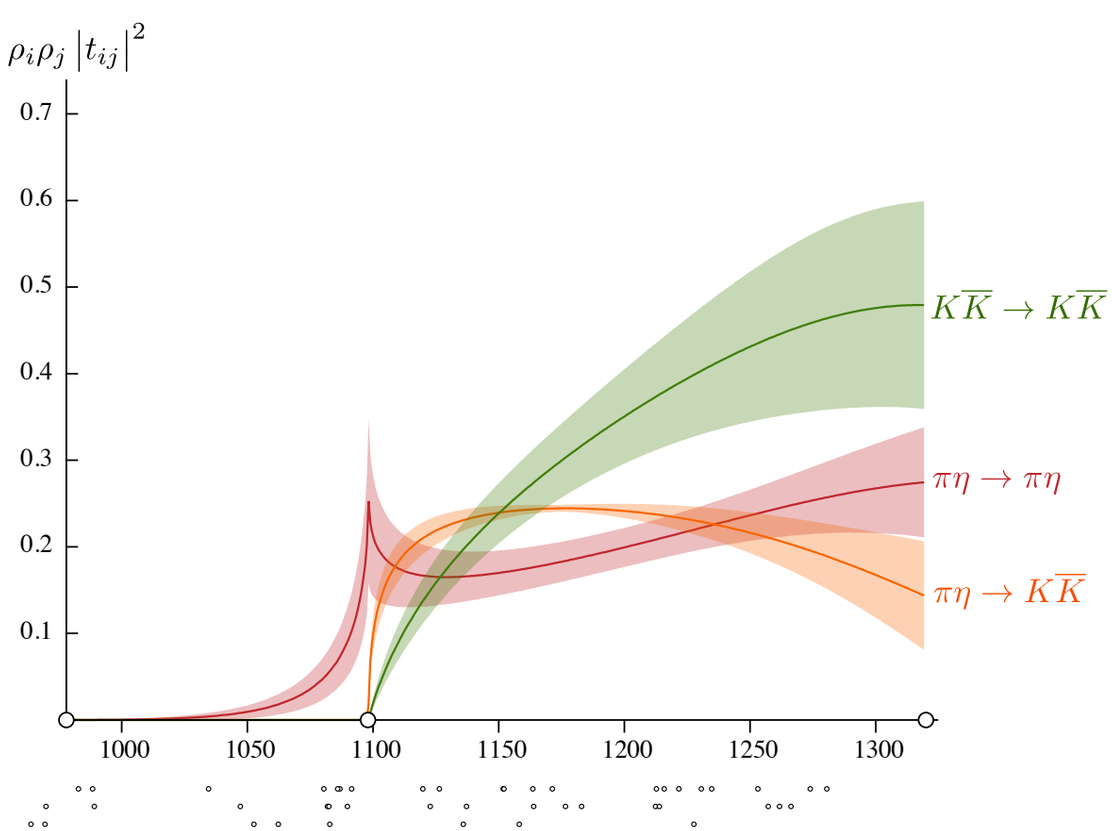

PRD93 094506 (2016)

coupled-channel scalar mesons

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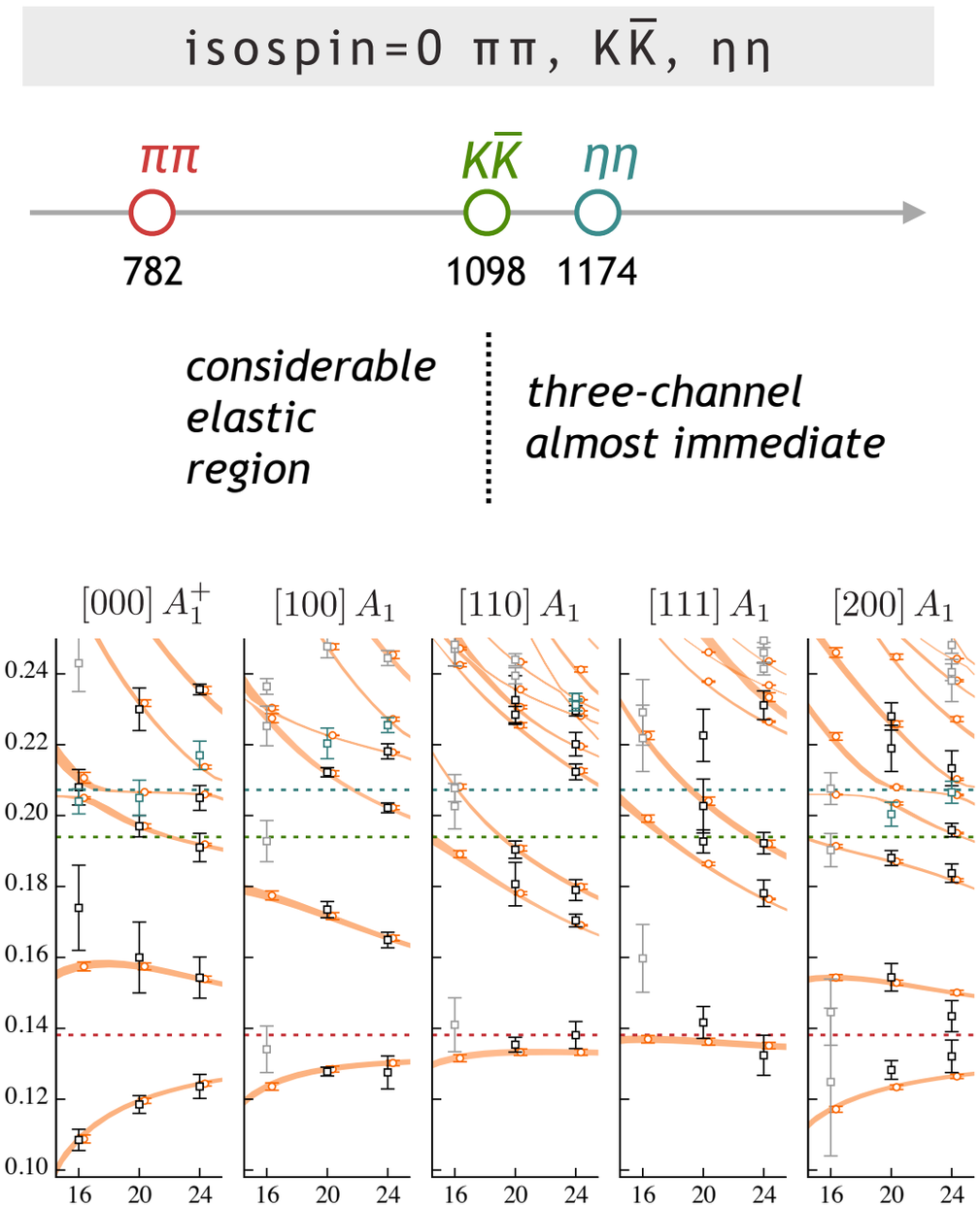
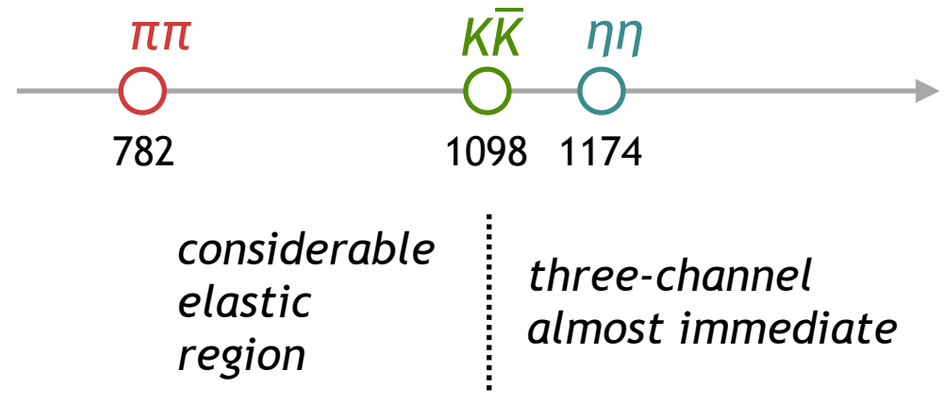


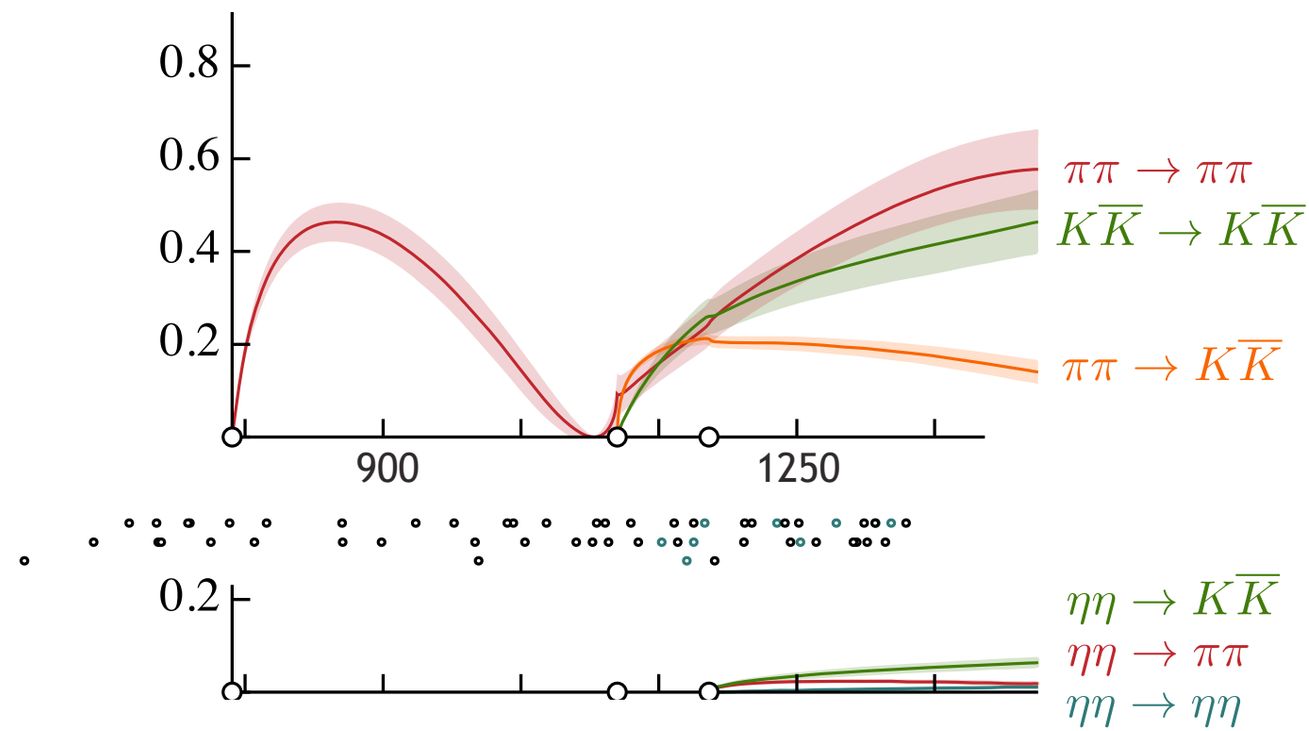
coupled-channel scalar mesons

scalars easiest ‘technically’ (not easiest physics) - 0^- 0^- in *S*-wave

m_π~391 MeV

isospin=0 $\pi\pi$, $K\bar{K}$, $\eta\eta$





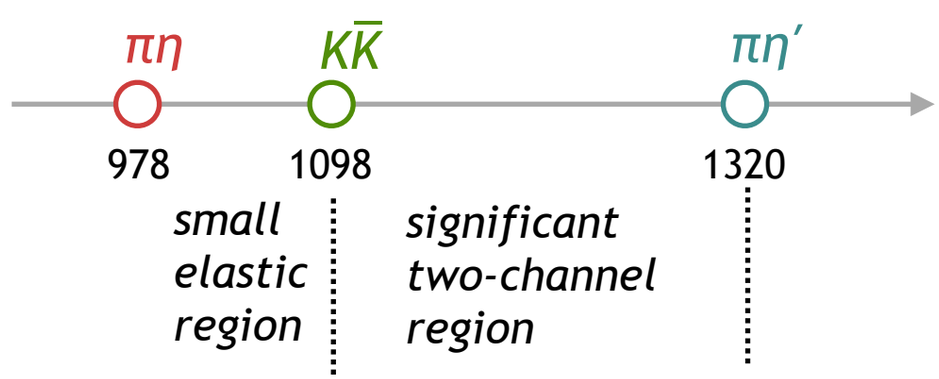
PRD97 054513 (2018)

coupled-channel scalar mesons

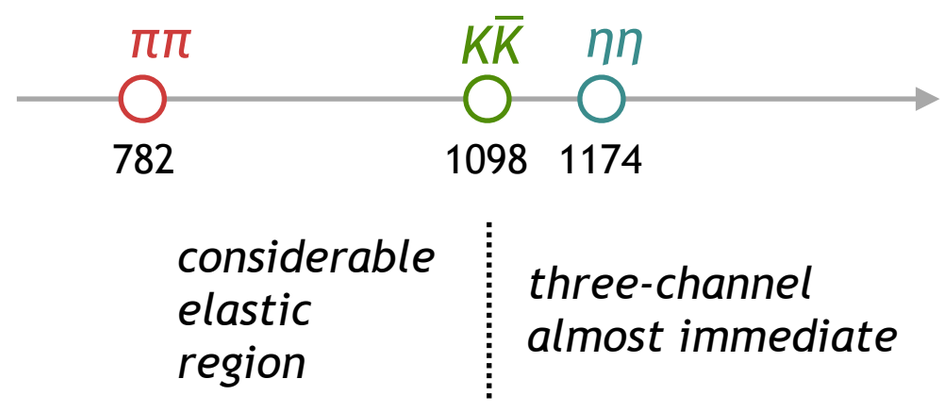
scalars easiest 'technically' (not easiest physics) - 0^- 0^- in S-wave

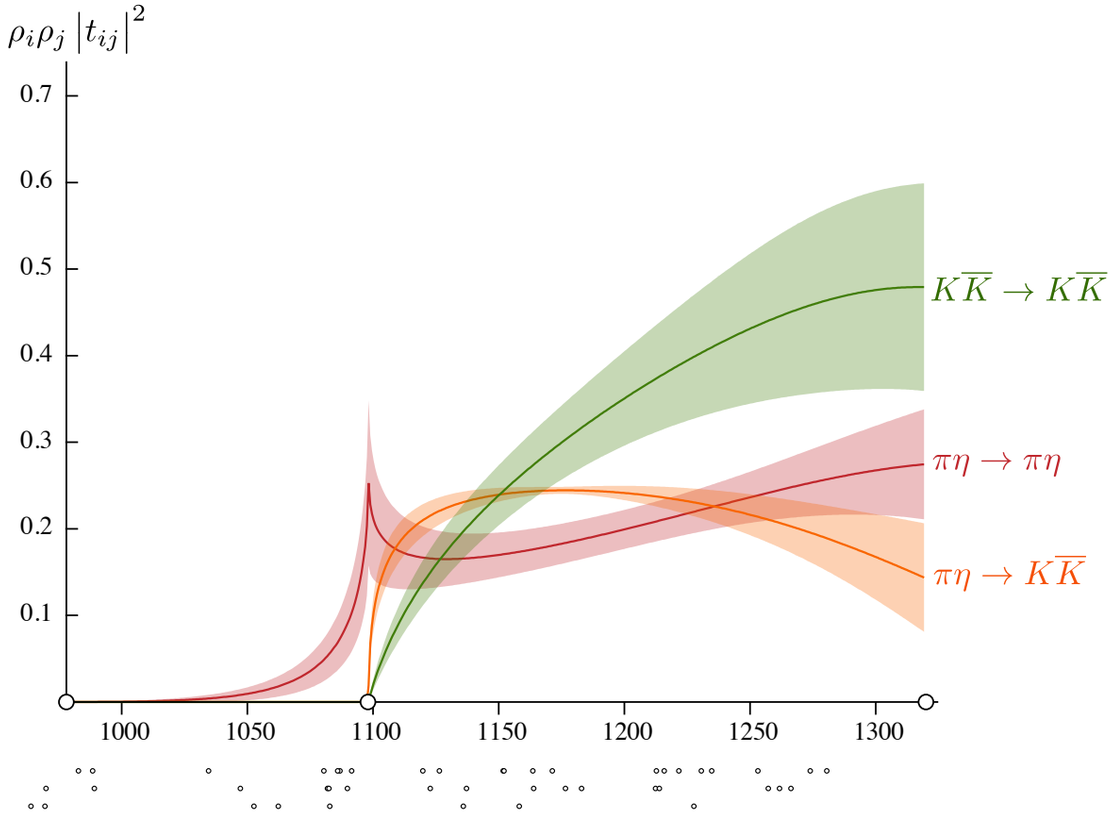
m_π~391 MeV

isospin=1 $\pi\eta$, $K\bar{K}$



isospin=0 $\pi\pi$, $K\bar{K}$, $\eta\eta$





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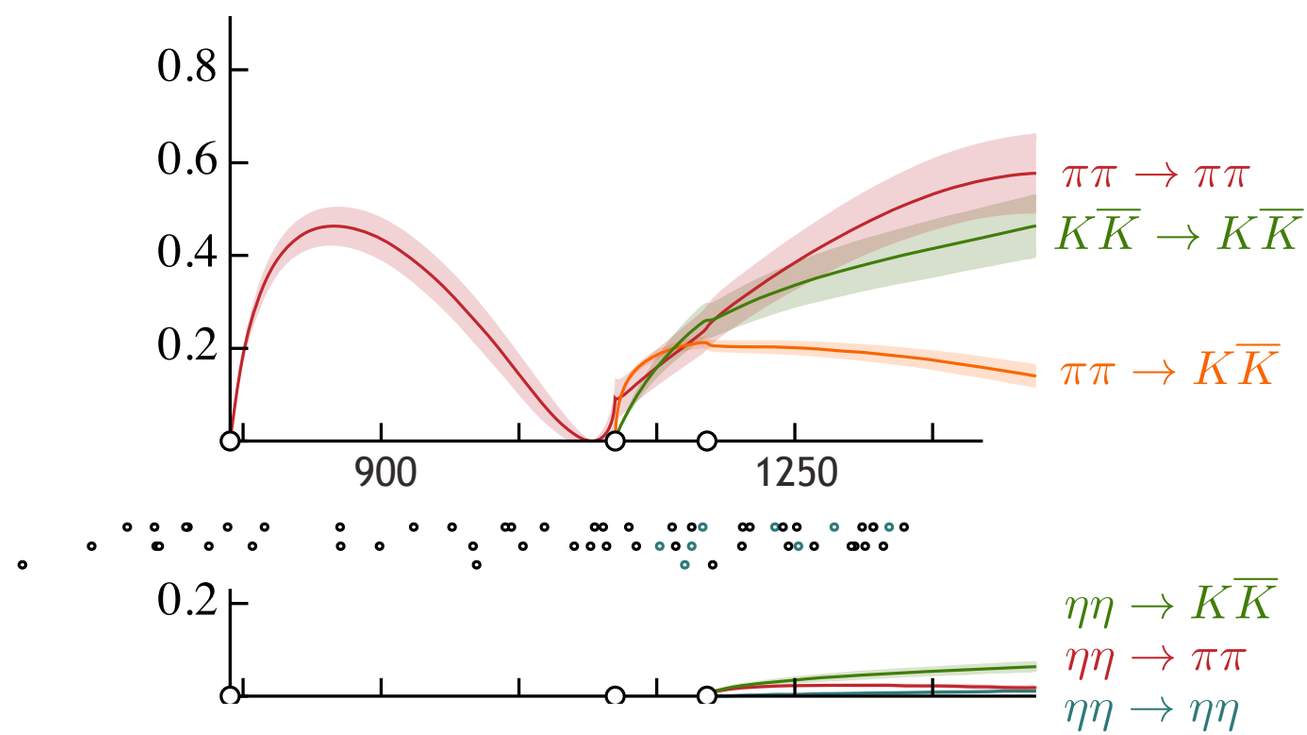


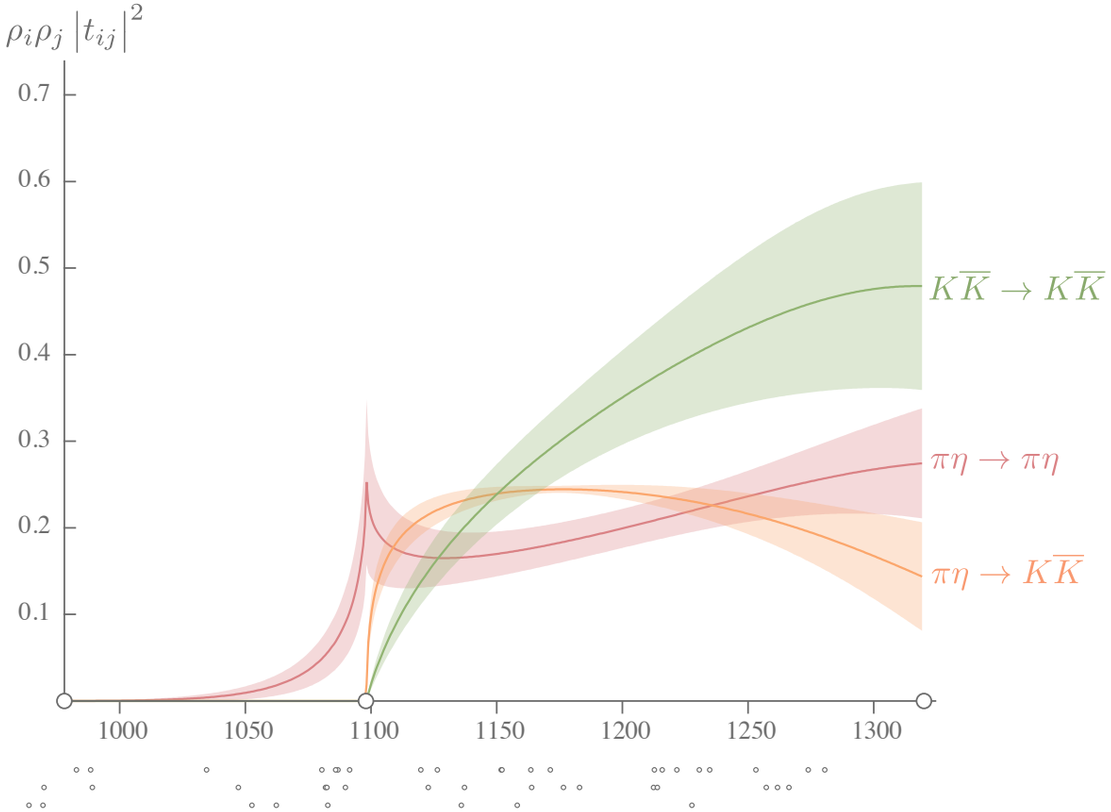
PRD97 054513 (2018)

coupled-channel scalar mesons

… very nice, but where are the **resonances** ?

we need to look for **pole singularities** of $\boldsymbol{t}(s)$

(recall the mantra: **"as rigorous as possible"**)



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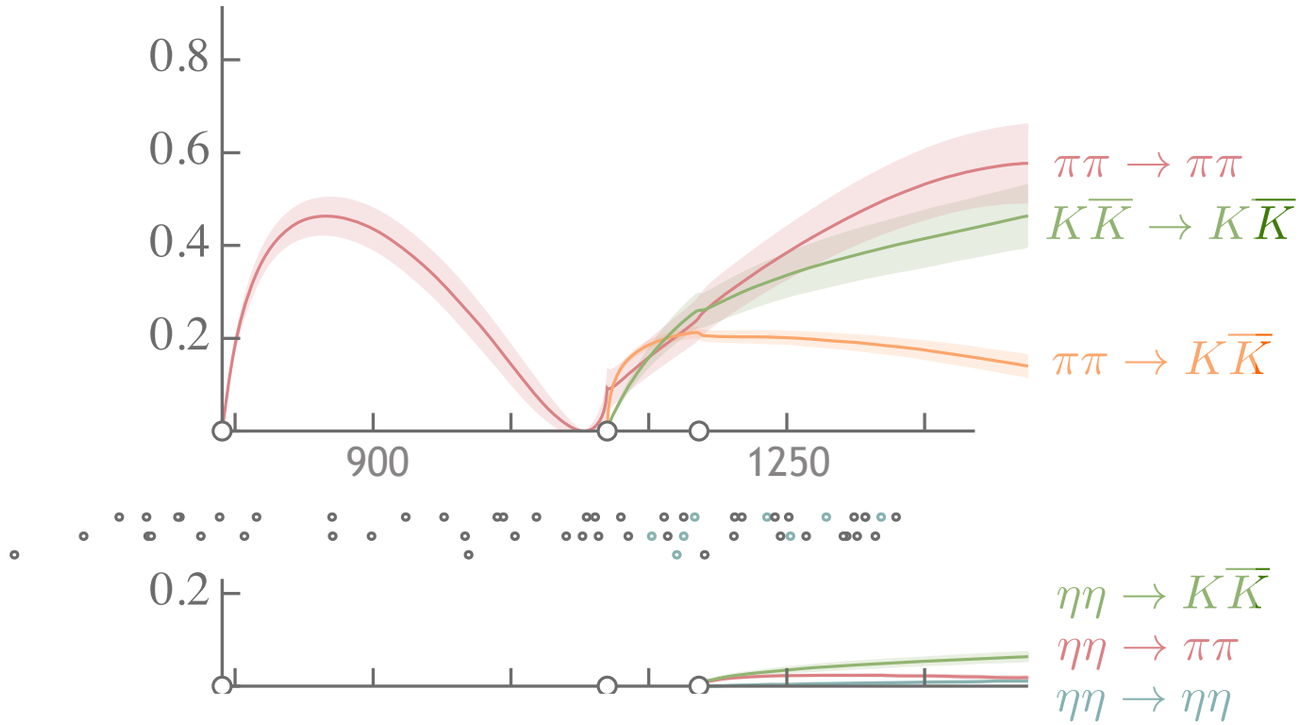
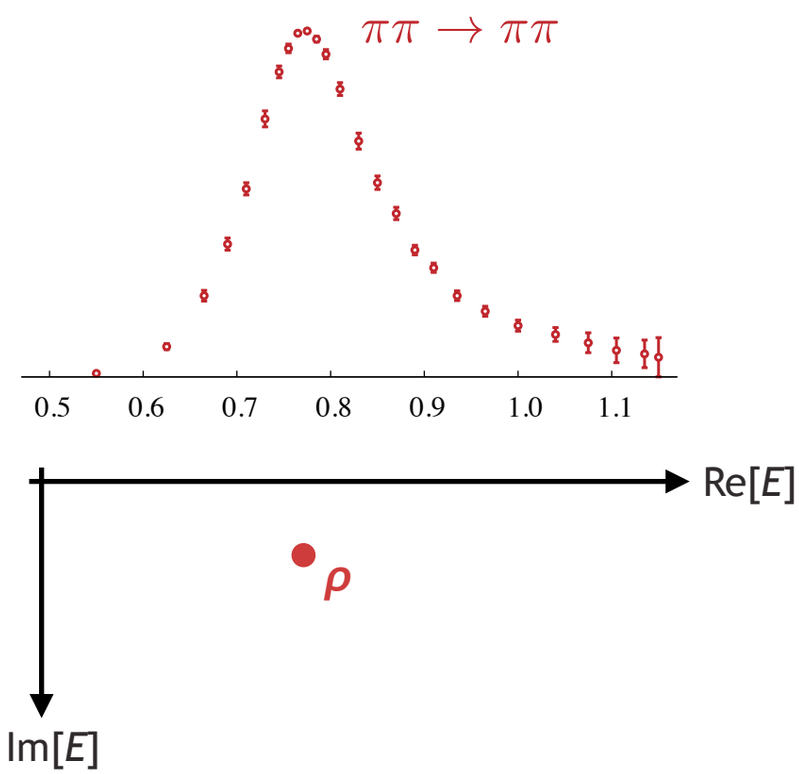


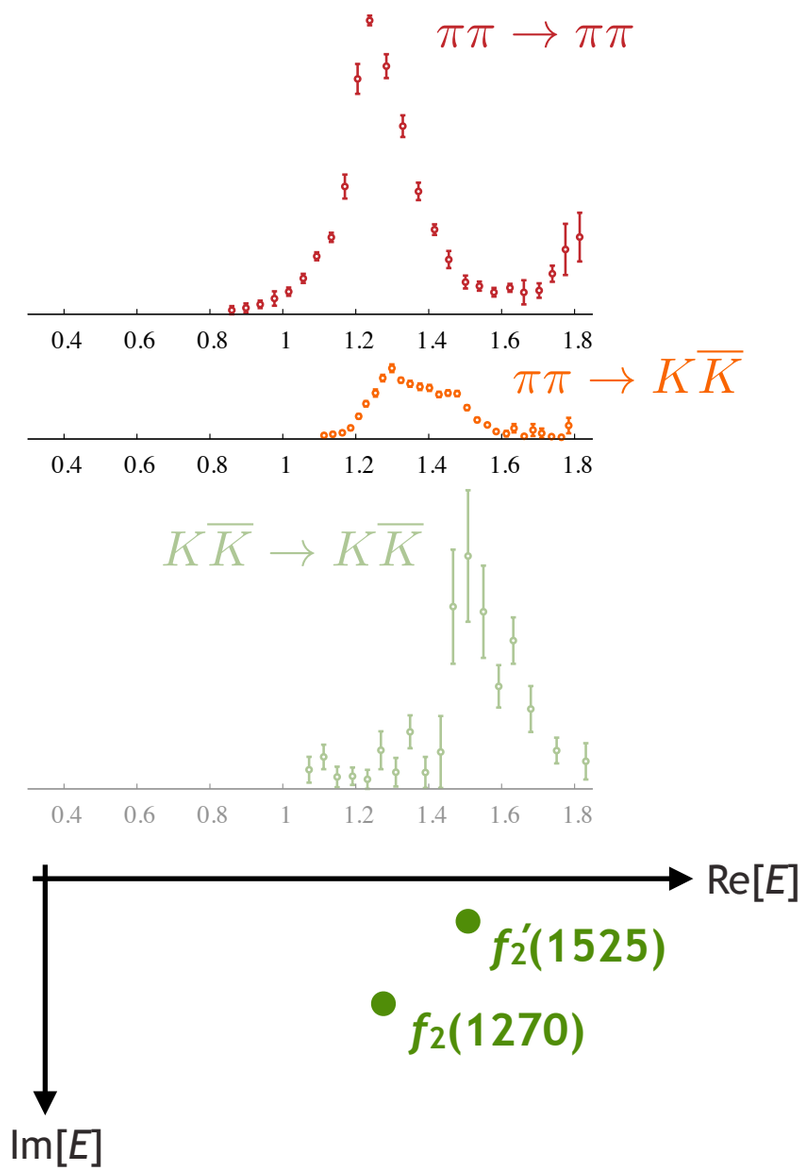
PRD97 054513 (2018)

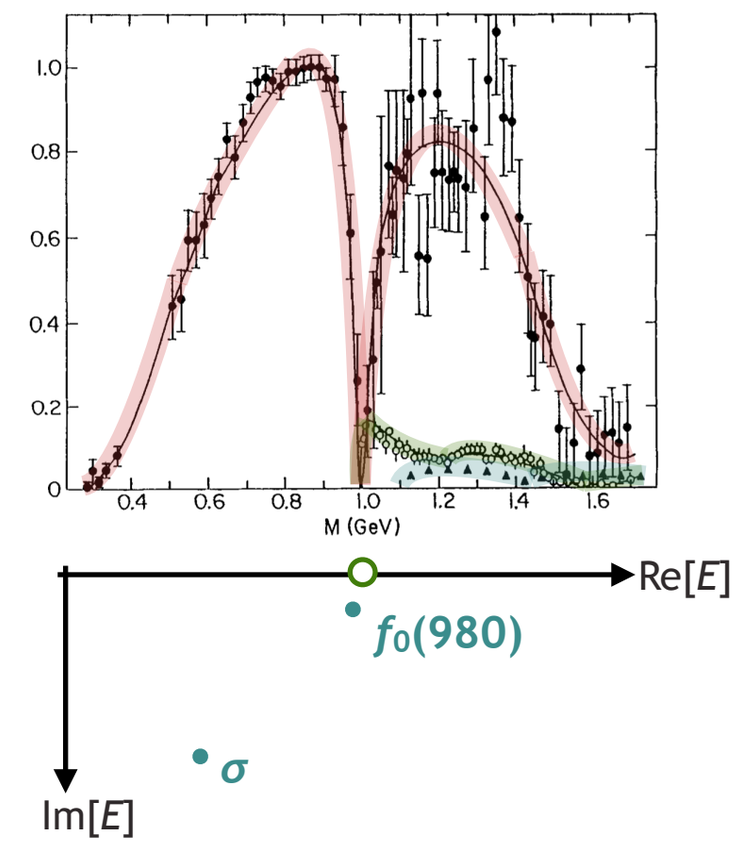
resonances are poles at complex energy

$s = E_{\text{cm}}^2$

$$t_{ab}(s) \sim \frac{c_a c_b}{s_0 - s} \qquad \sqrt{s_0} = m_R \pm i\tfrac{1}{2}\Gamma_R$$







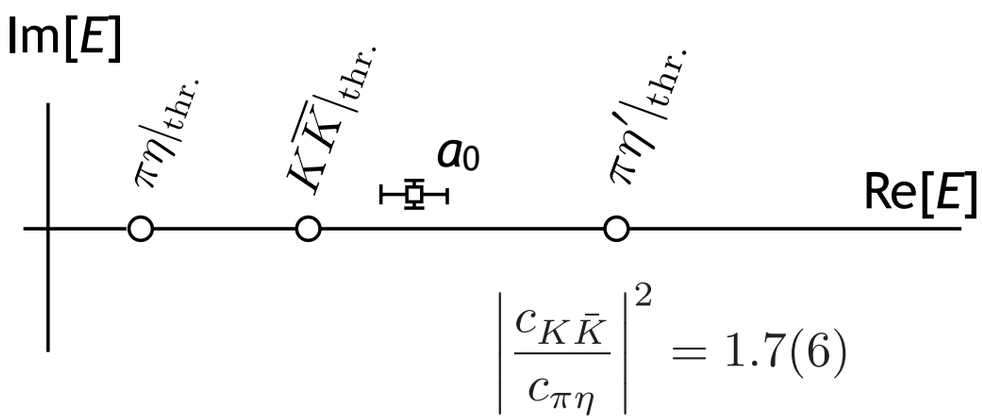
… in practice more common to “produce” resonances …
(i.e. more complicated initial state)

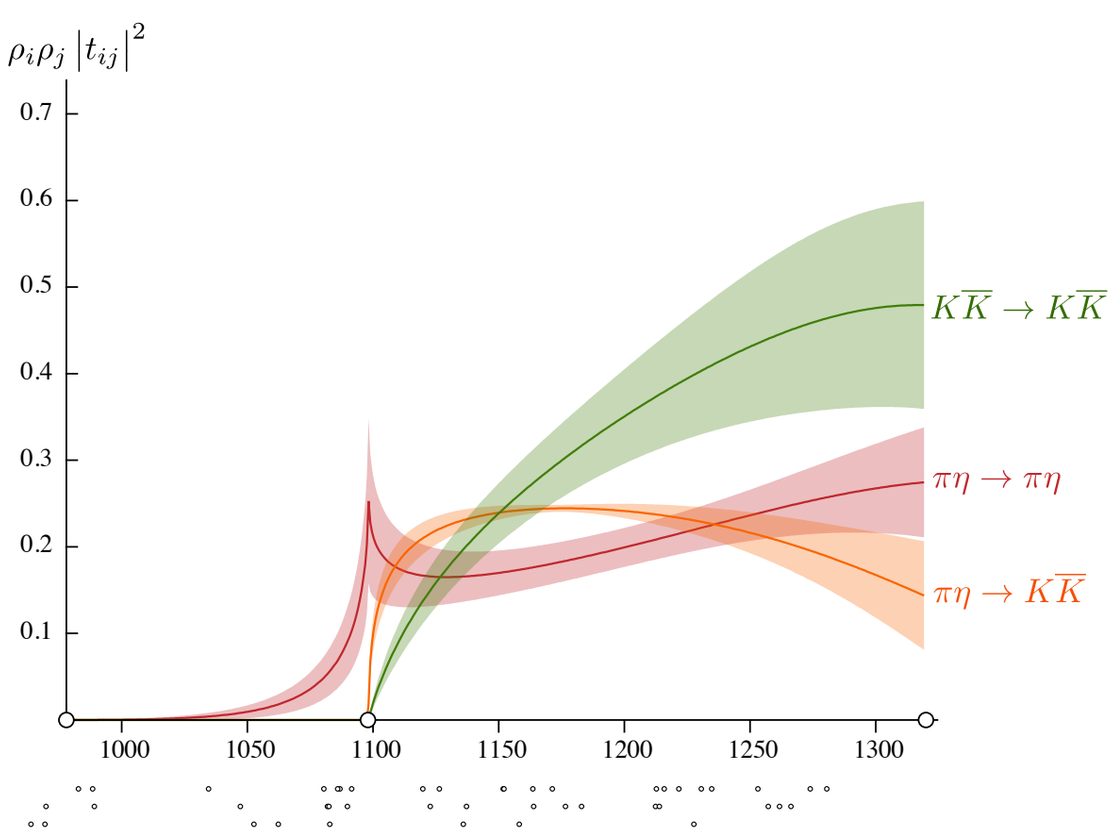
coupled-channel scalar mesons

resonances as pole singularities

m_π~391 MeV

isospin=1 πη, K$\bar{K}$





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coupled-channel scalar mesons

resonances as pole singularities

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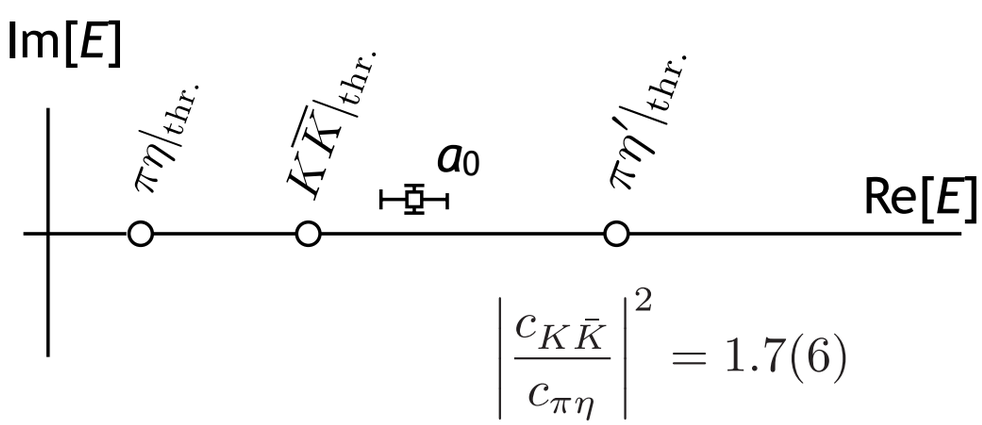

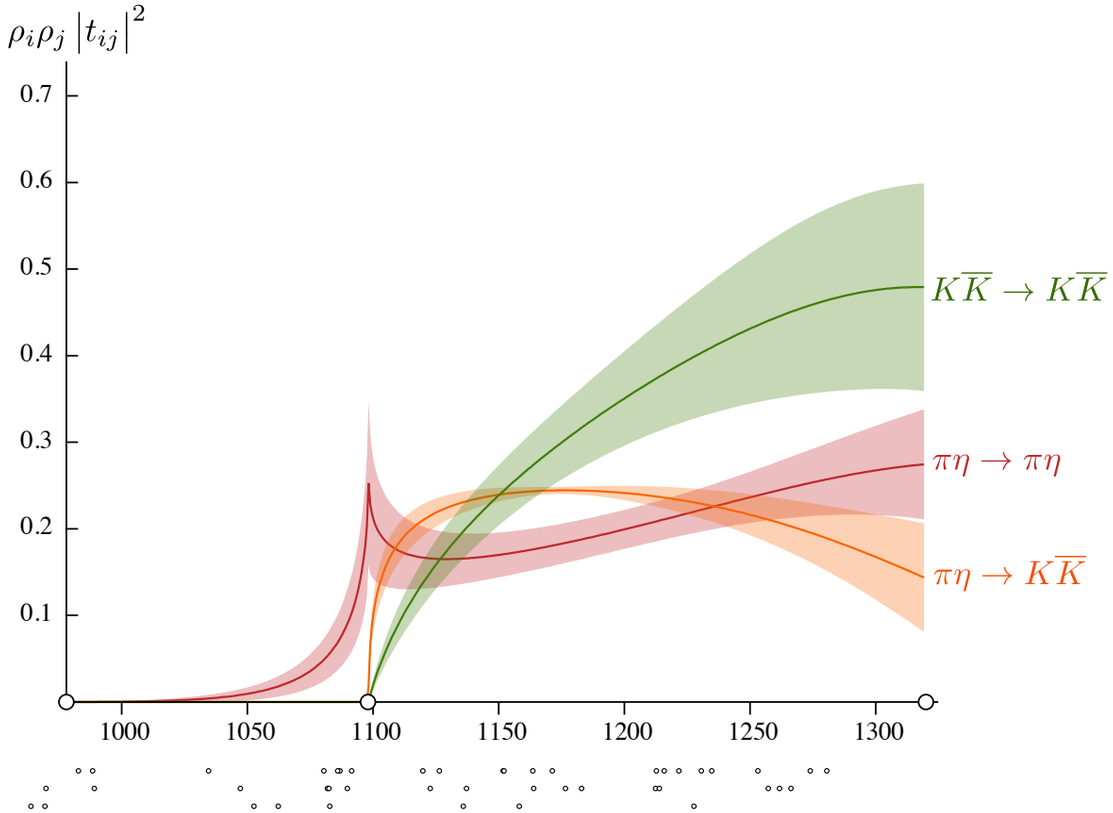

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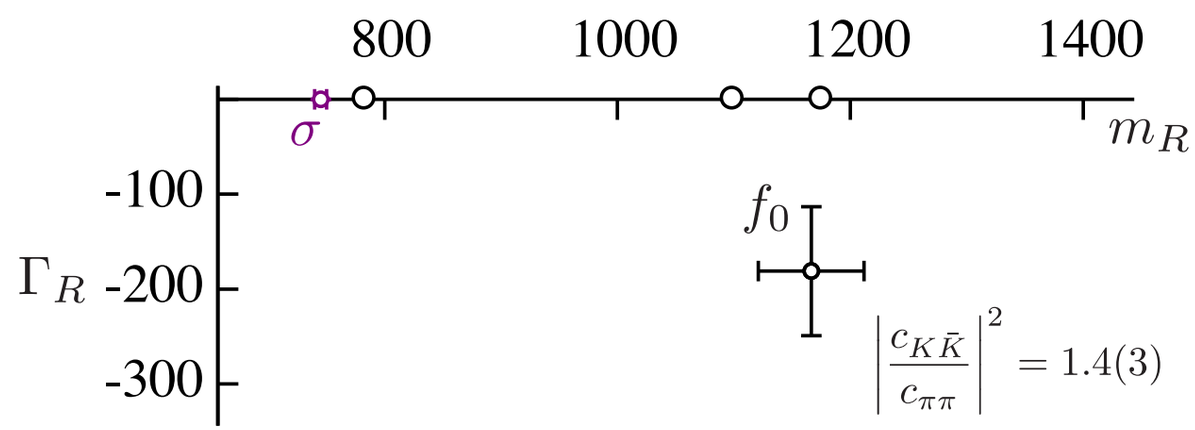
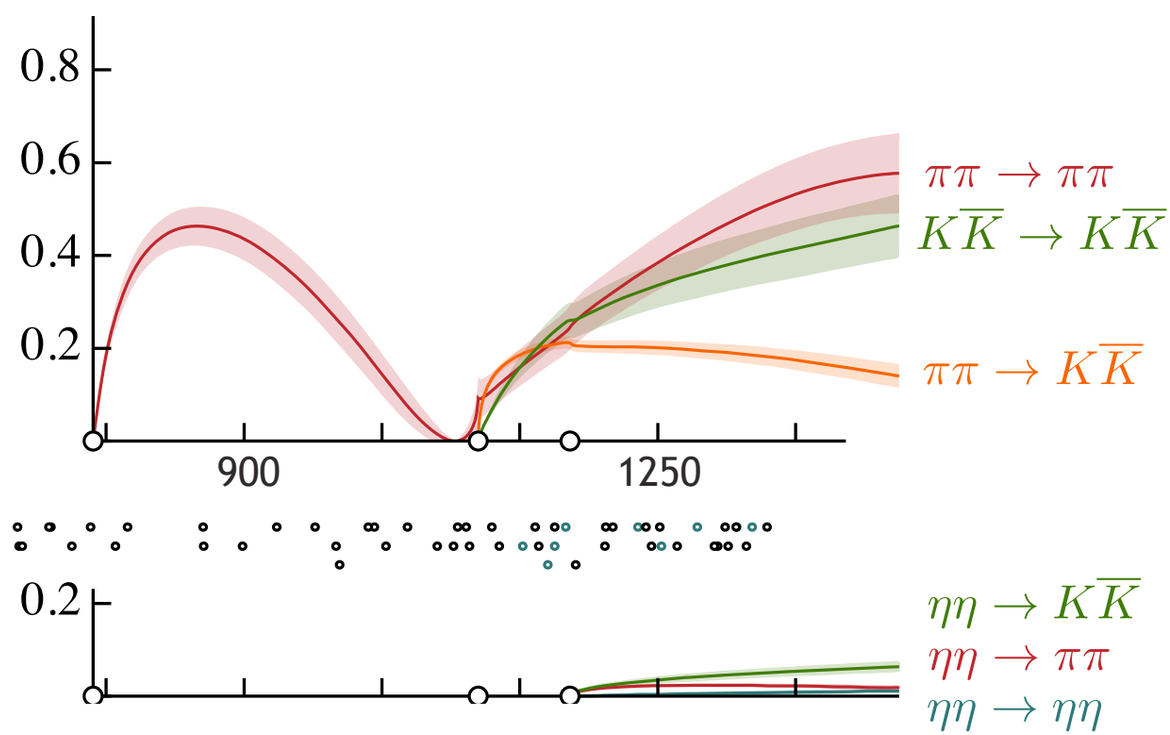
isospin=0 ππ, K$\bar{K}$, ηη



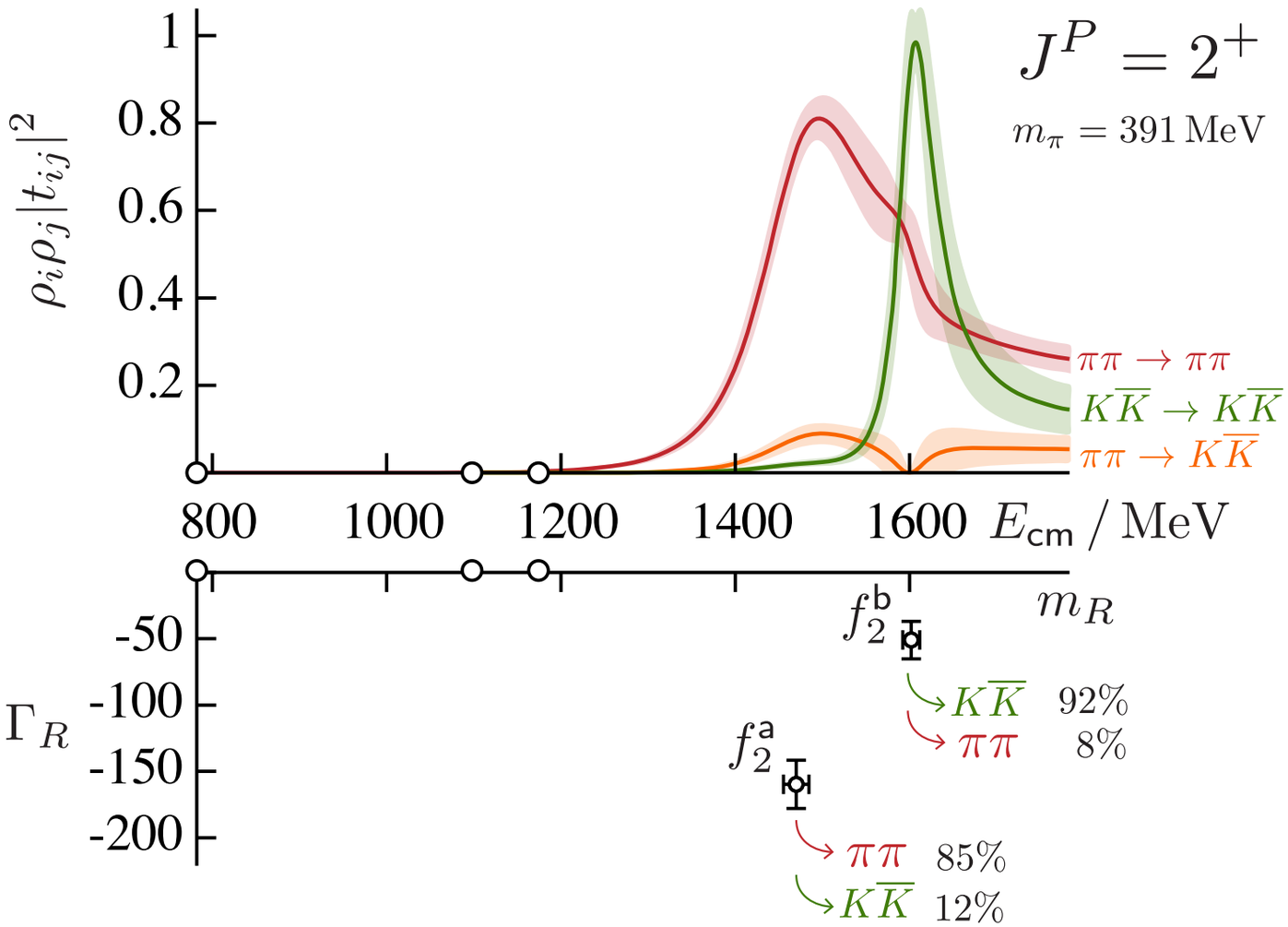


PRD97 054513 (2018)

coupled-channel tensor mesons

D-wave amplitudes & poles

PRD97 054513 (2018)



in this case uncontrolled (but justified?) approximation of neglecting $\pi\pi\eta$, $\pi\pi\pi\pi$

$$f_2^{\mathsf{a}} : \begin{array}{l} m = 1470(15)\,\mathrm{MeV} \\ \Gamma = 160(18)\,\mathrm{MeV} \end{array}$$

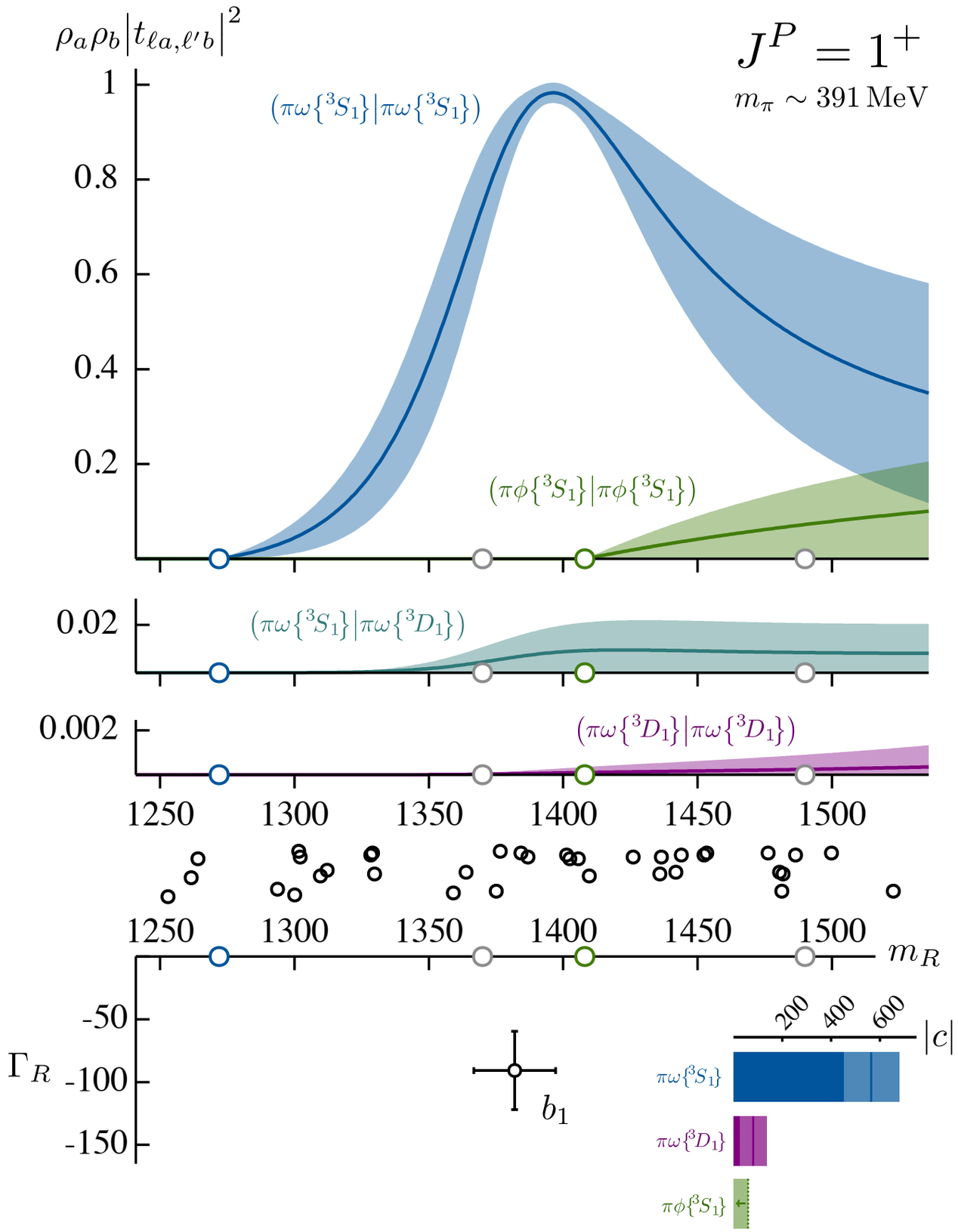
$$f_2^{\mathsf{b}} : \begin{array}{l} m = 1602(10)\,\mathrm{MeV} \\ \Gamma = 54(14)\,\mathrm{MeV} \end{array}$$

scattering with other than pseudoscalars

e.g. ω stable (or very narrow)
down to rather low quark mass

at m_π~391 MeV can study $\pi\omega$ scattering

PRD100 054506 (2019)



$b_1(1235)$ $\quad I^G(J^{PC}) = 1^+(1^{+-})$

Mass $m = 1229.5 \pm 3.2$ MeV (S = 1.6)
Full width $\Gamma = 142 \pm 9$ MeV (S = 1.2)

$b_1(1235)$ DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\omega\pi$	dominant		348
[D/S amplitude ratio $= 0.277 \pm 0.027$]			

so where are the hybrid mesons … ?

e.g. lightest π_1 resonance

why calculate with heavy quark masses ?

remember the f.v. spectrum depends upon all open channels,
and three-meson and higher thresholds will open

e.g. at physical quark mass
if m ~ 1600 MeV > 11 m_π

learning about the unstable π_1 ?

for the first attempt at this, work with u=d=s all at approx the physical strange mass m_π~700 MeV

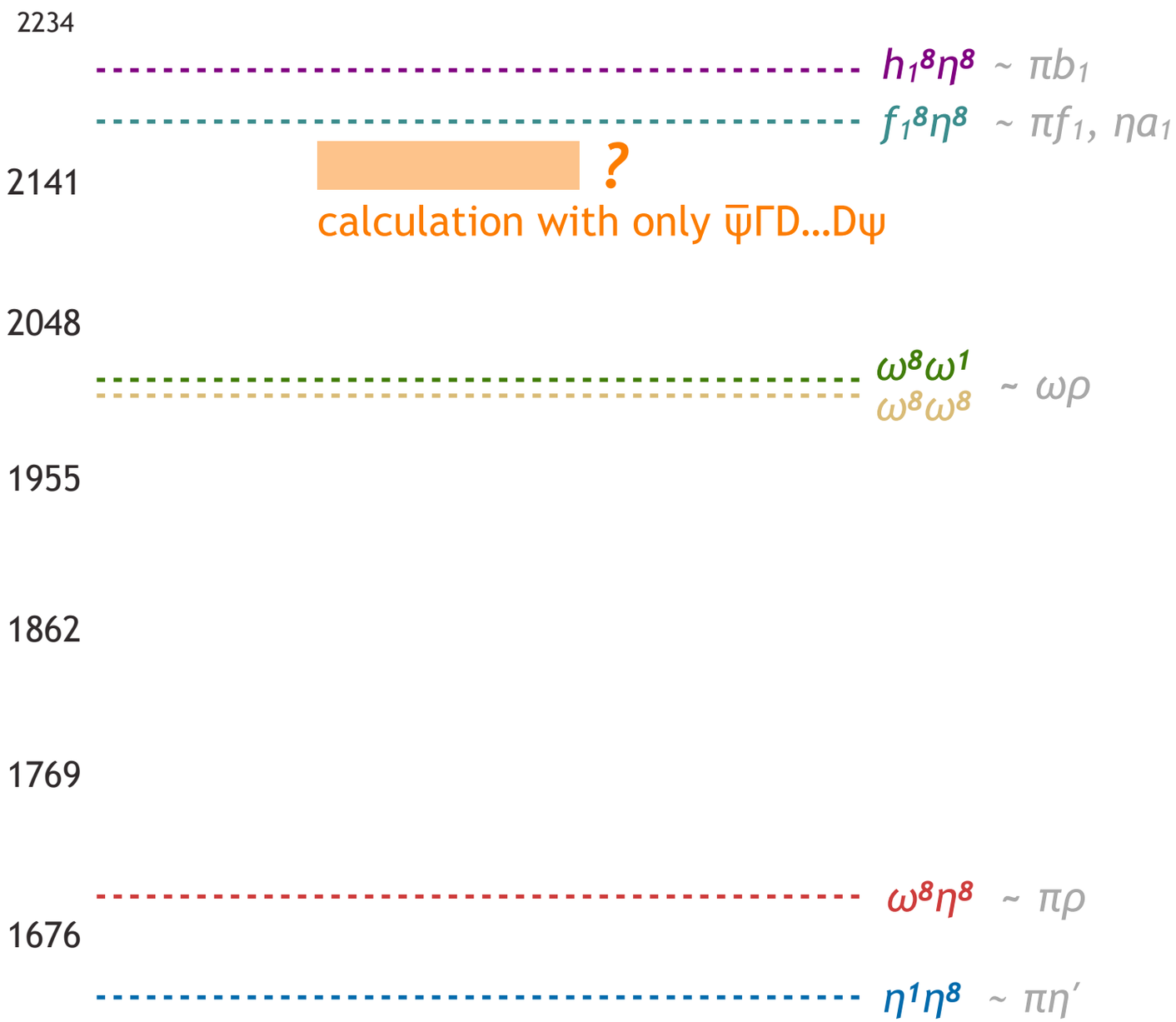
exact SU(3) flavor symmetry — octets & singlets

e.g. $m_\pi = m_K = m_{\eta 8}$

stable hadrons	
η^8 ~ 700 MeV	ω^8 ~ 1020 MeV
η^1 ~ 960 MeV	ω^1 ~ 1030 MeV
f_1^8 ~ 1520 MeV	h_1^8 ~ 1550 MeV

finite-volume spectrum with $J^{PC} = 1^{-+}$

arXiv:2009.10034



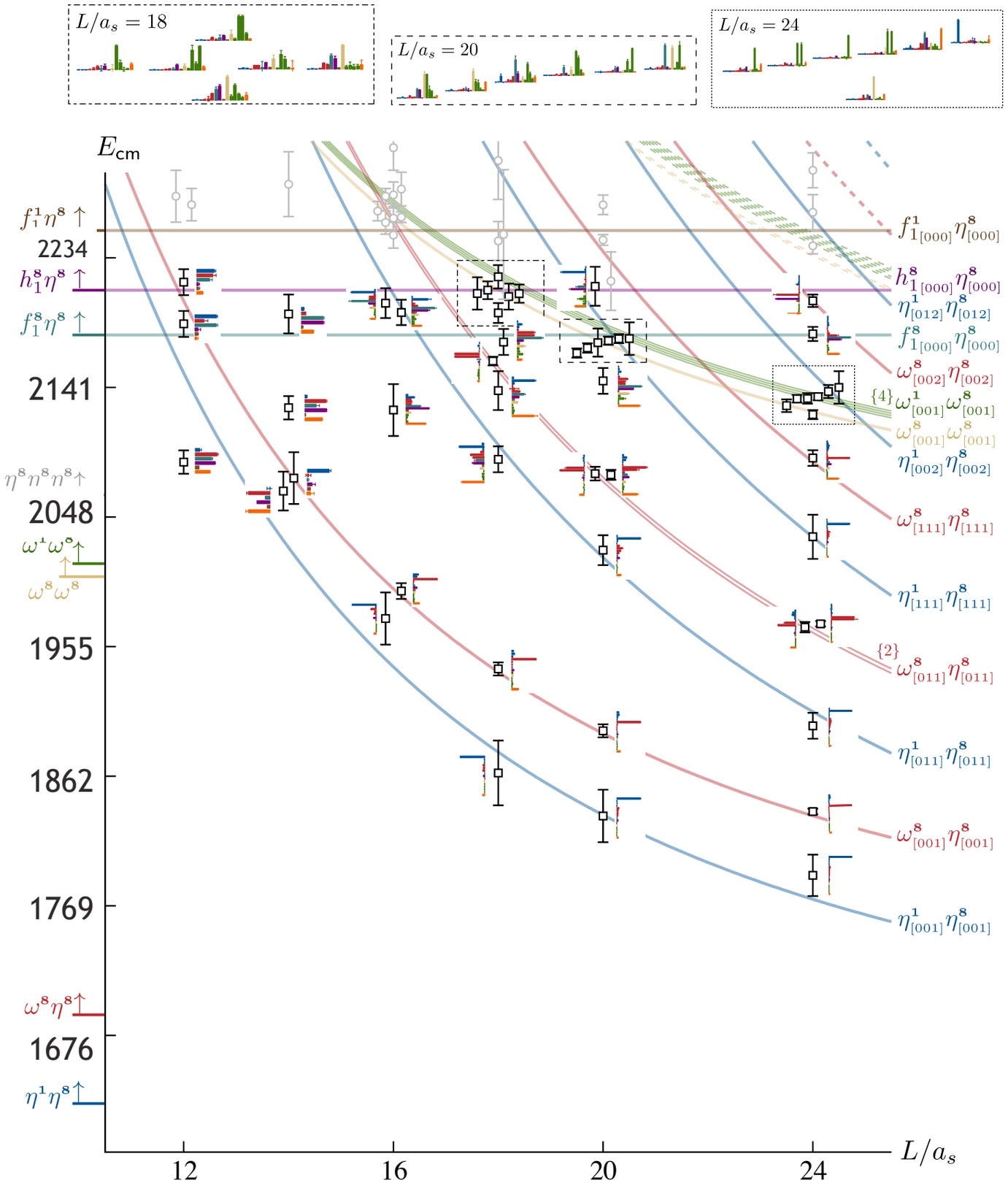
even this simplified case is **very challenging:**

— moving frames impractical (other J^{PC})

— at least 8 coupled channels

finite-volume spectrum with $J^{PC} = 1^{-+}$

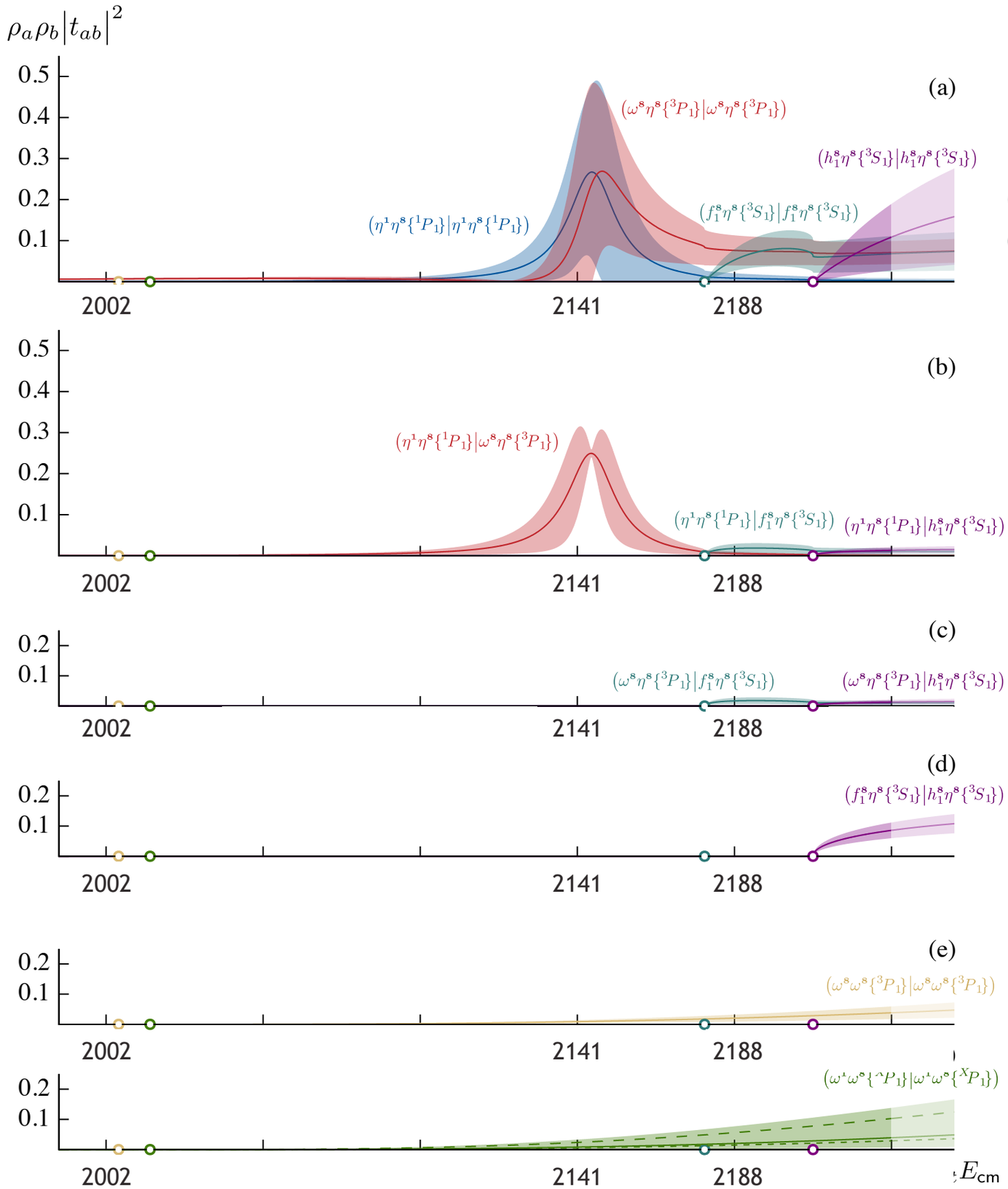
arXiv:2009.10034



try to describe this with an eight channel scattering matrix …

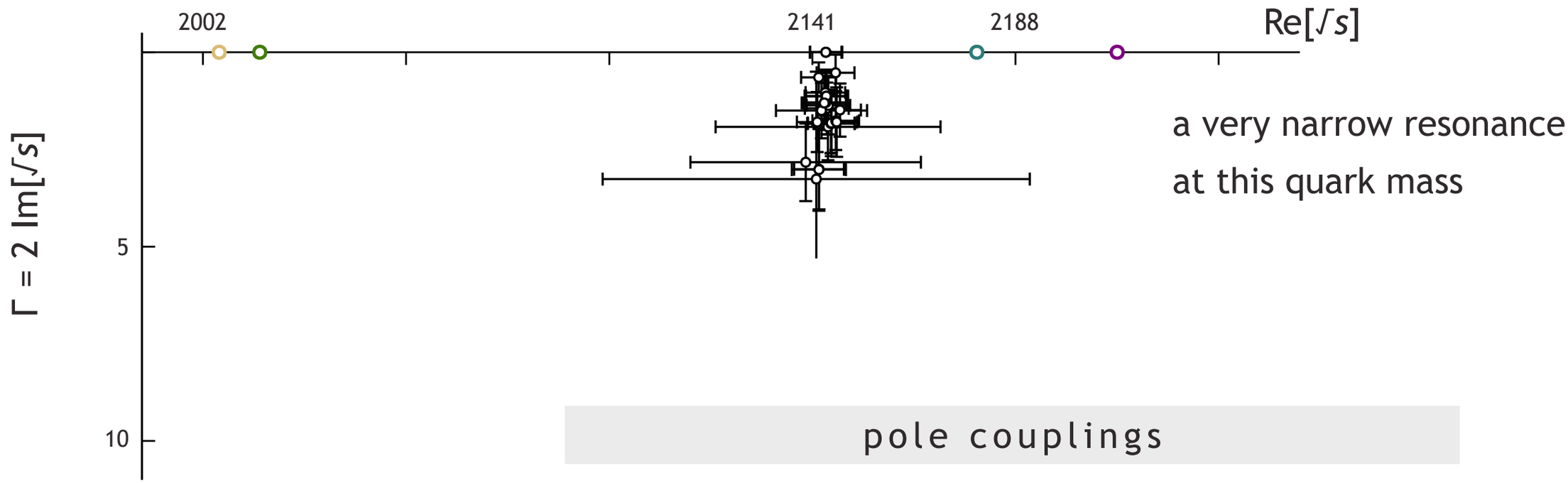
$J^{PC} = 1^{-+}$ scattering amplitudes

arXiv:2009.10034



diagonal elements of t-matrix

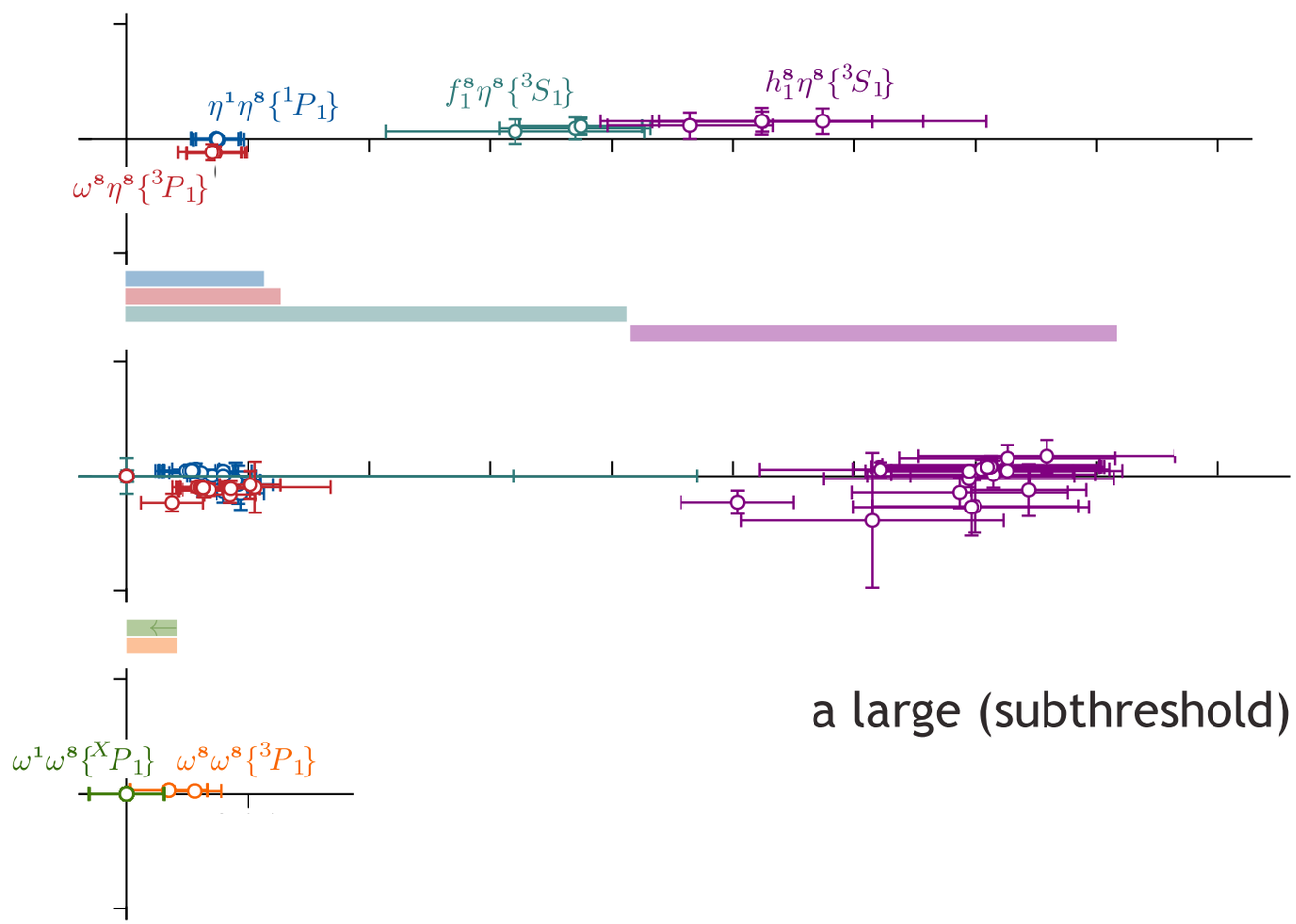
a resonance pole



a very narrow resonance

at this quark mass

pole couplings



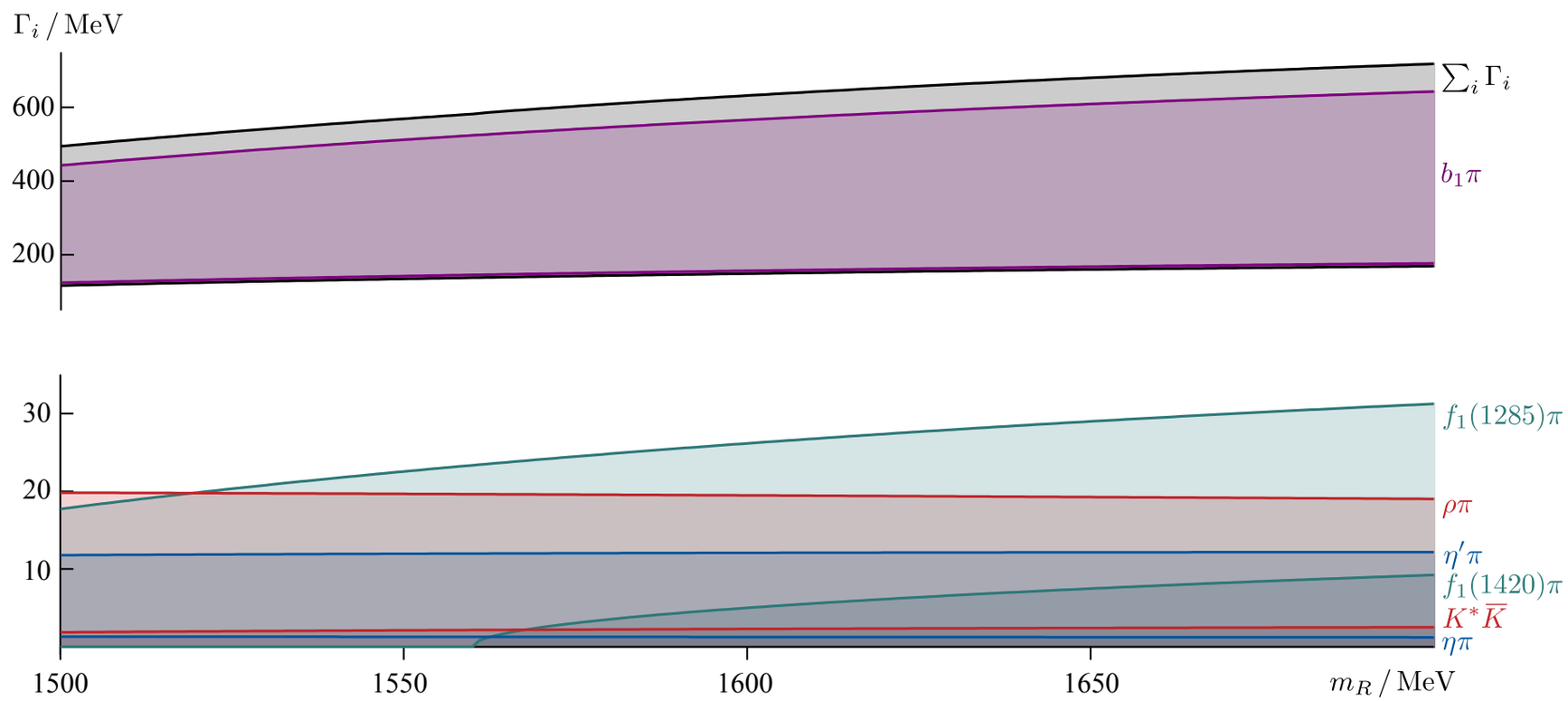
$$t_{ab}(s) \sim \frac{c_a c_b}{s_0 - s}$$

a large (subthreshold) coupling to

$h_1^8\eta^8 \sim \pi b_1$

a crude extrapolation to physical quark mass …

arXiv:2009.10034



Determination of the Pole Position of the Lightest Hybrid Meson Candidate

A. Rodas,[1,*] A. Pilloni,[2,3,†] M. Albaladejo,[2,4] C. Fernández-Ramírez,[5] A. Jackura,[6,7] V. Mathieu,[2] M. Mikhasenko,[8] J. Nys,[9] V. Pauk,[10] B. Ketzer,[8] and A. P. Szczepaniak[2,6,7]

(Joint Physics Analysis Center)

Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
π_1	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$

Investigation of the Lightest Hybrid Meson Candidate with a Coupled-Channel Analysis of $\bar{p}p$-, π^-p- and $\pi\pi$-Data

B. Kopf, M. Albrecht, H. Koch, J. Pychy, X. Qin,* and U. Wiedner
Ruhr-Universität Bochum, 44801 Bochum, Germany

name	pole mass [MeV/c^2]	pole width [MeV]
$a_2(1320)$	$1308.7 \pm 0.4\,^{+2.0}_{-4.2}$	$108.6 \pm 0.4\,^{+1.8}_{-12.9}$
$a_2(1700)$	$1669.2 \pm 1.0\,^{+20.2}_{-4.6}$	$429.0 \pm 1.7\,^{+44.4}_{-9.7}$
π_1	$1561.6 \pm 3.0\,^{+6.6}_{-2.6}$	$388.1 \pm 5.4\,^{+0.2}_{-14.1}$

the state of the art

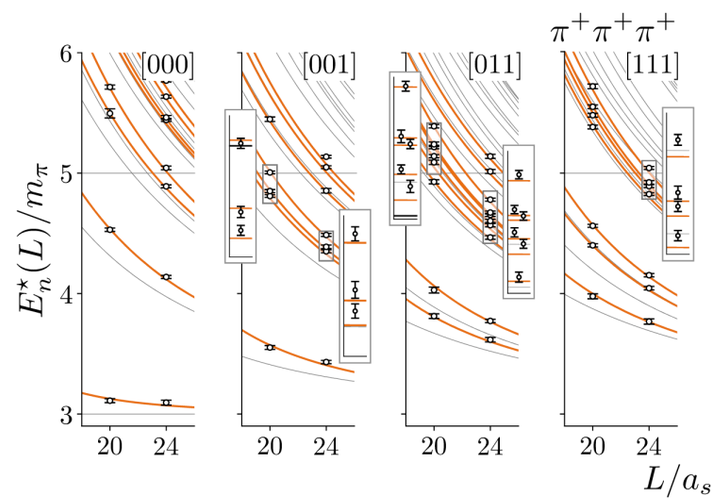
attempting a rigorous approach to study QCD
… but currently often at the wrong light quark mass

emphasis so far on
meson resonances in **coupled-channel meson-meson scattering**

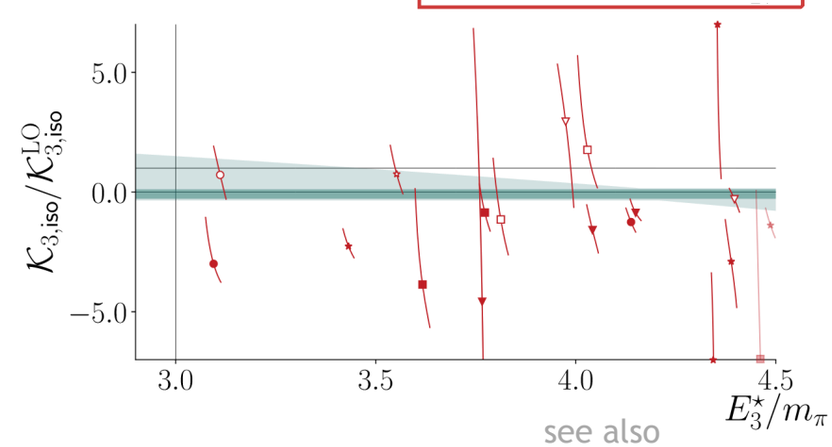
non-obvious scalar mesons
conventional vectors & tensors
axial b_1
1^{-+} hybrid

to study more resonances, and to go to lighter pions, need **three-body decays**

more complicated formalism, currently being tested



arXiv:2009.04931

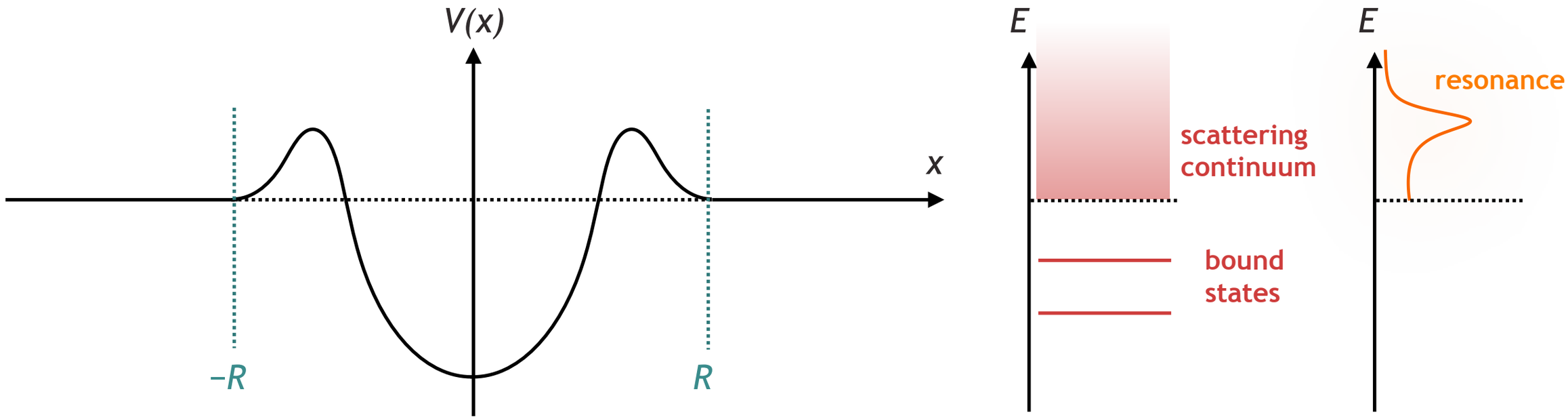


see also
PRL122 062503 (2019)
PRL123 142002 (2019)

learning about internal structure of resonances — **couple to external currents**

maybe distinguish ‘molecules’ by large spatial size ?
maybe even compute parton distributions for resonances ?

resonances illustrated in quantum mechanics

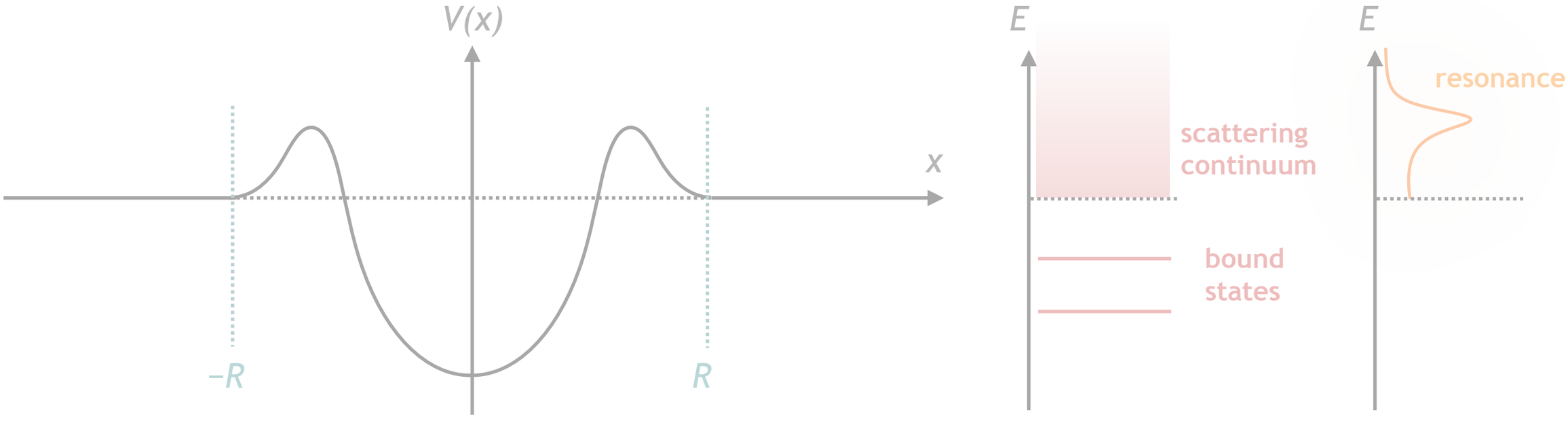


$$\psi(|x| > R) \sim \cos\left(p|x| + \delta(p)\right)$$

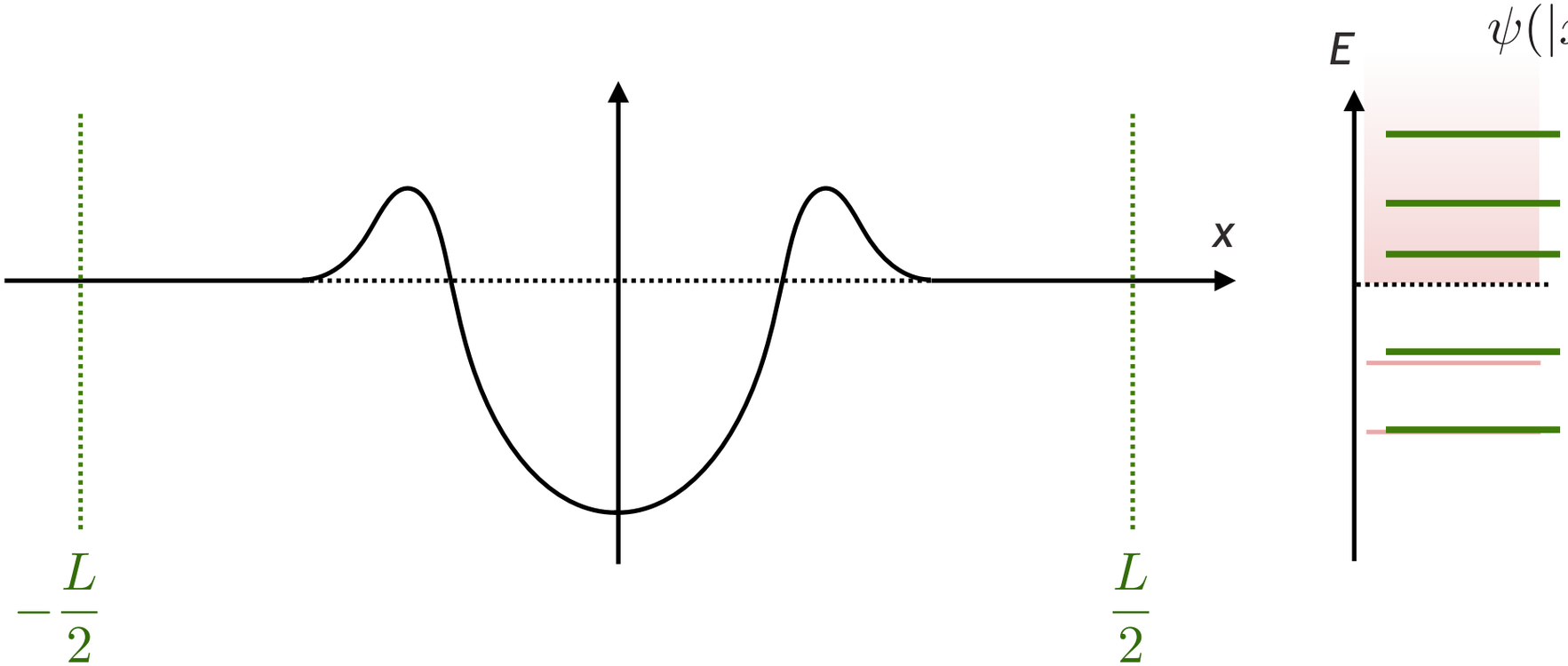
phase shift

resonances lie in the **continuous spectrum of scattering states**

resonances in a finite volume ?



but in a periodic volume …



$$\psi(|x| > R) \sim \cos\left(p|x| + \delta(p)\right)$$

applying the boundary conditions

$$p = \frac{2\pi}{L} n - \frac{2}{L}\delta(p)$$

solved by **discrete** $p_n(L)$

finite-volume spectrum → infinite volume scattering amplitudes

conceptually the same idea applies in 3+1 dim quantum field theory

“the Lüscher method”

some technicalities associated
with mismatch between **rotational symmetry**
and a **cubic spatial boundary**

recent pedagogic review

REVIEWS OF MODERN PHYSICS, VOLUME 90, APRIL–JUNE 2018

Scattering processes and resonances from lattice QCD

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(published 18 April 2018)

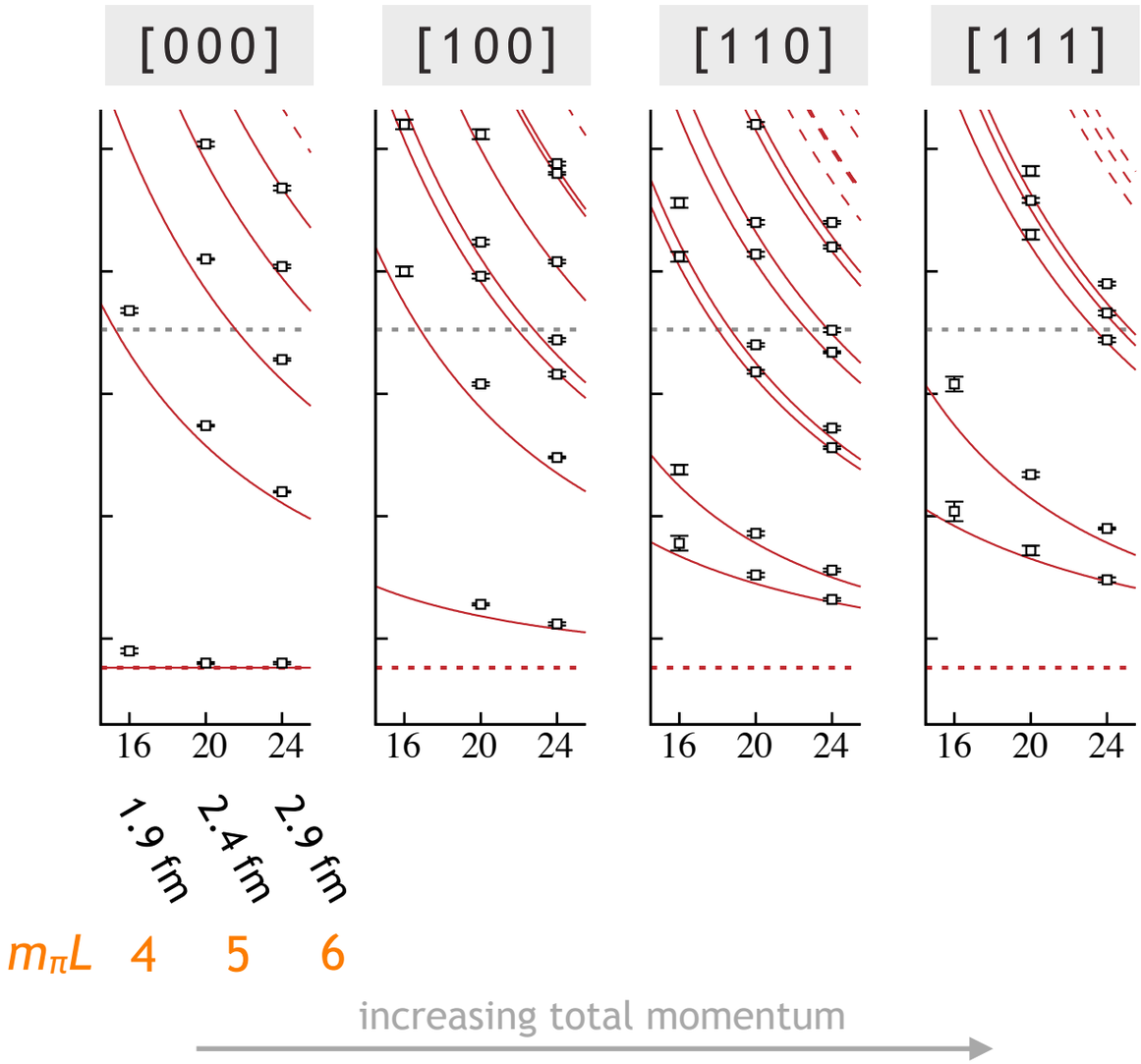
The vast majority of hadrons observed in nature are not stable under the strong interaction; rather they are resonances whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances offers a window into the workings of quantum chromodynamics (QCD) in the low-energy nonperturbative region, and in addition many probes of the limits of the electroweak sector of the standard model consider processes which feature hadron resonances. From a theoretical standpoint, this is a challenging field: the same dynamics that binds quarks and gluons into hadron resonances also controls their decay into lighter hadrons, so a complete approach to QCD is required. Presently, lattice QCD is the only available tool that provides the required nonperturbative evaluation of hadron observables. This article reviews progress in the study of few-hadron reactions in which resonances and bound states appear using lattice QCD techniques. The leading approach is described that takes advantage of the periodic finite spatial volume used in lattice QCD calculations to extract scattering amplitudes from the discrete spectrum of QCD eigenstates in a box. An explanation is given of how from explicit lattice QCD calculations one can rigorously garner information about a variety of resonance properties, including their masses, widths, decay couplings, and form factors. The challenges which currently limit the field are discussed along with the steps being taken to resolve them.

DOI: 10.1103/RevModPhys.90.025001

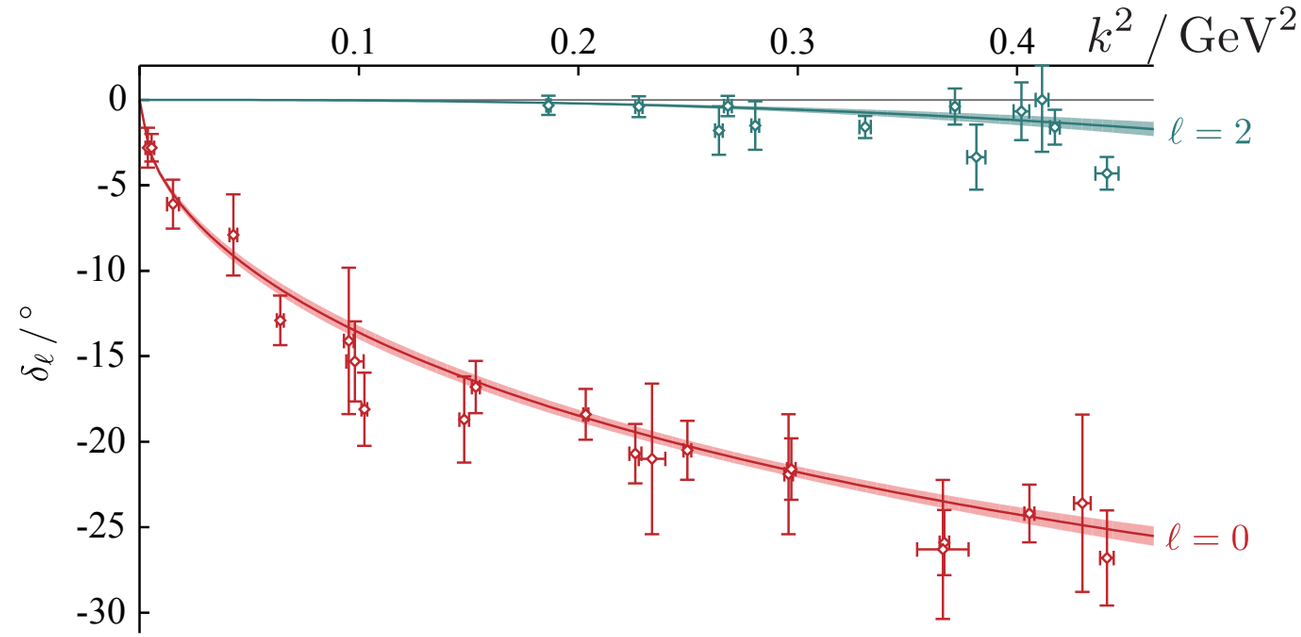
non-resonant elastic scattering — $\pi^+\pi^+$ (isospin=2)

PRD86 034031 (2012)

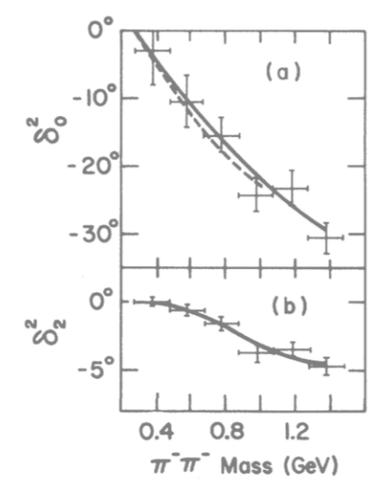
m_π ~ 391 MeV



scattering phase-shifts



repulsive scattering, no resonances

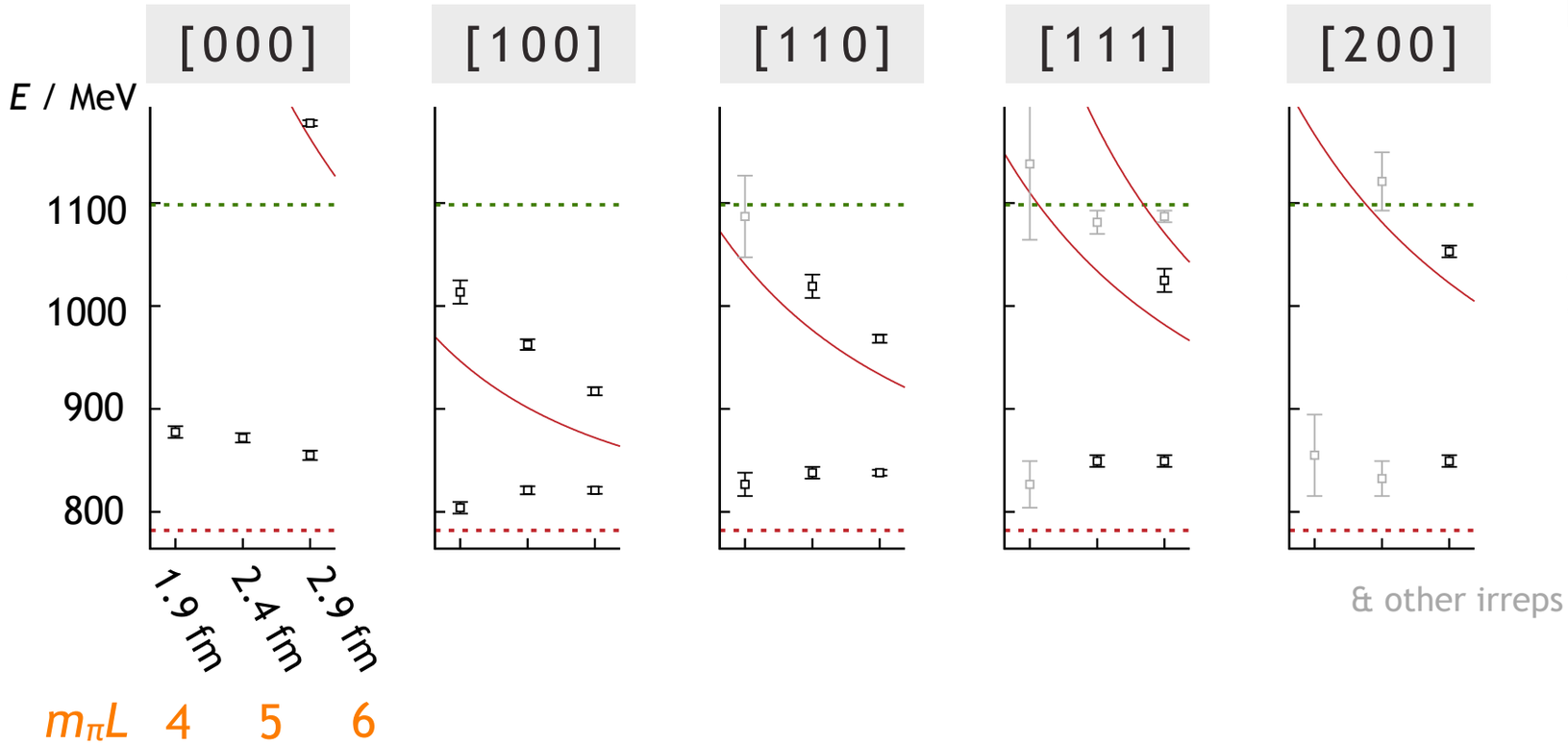


Cohen 1972

an elastic resonance — the ρ in $\pi\pi$ (isospin=1)

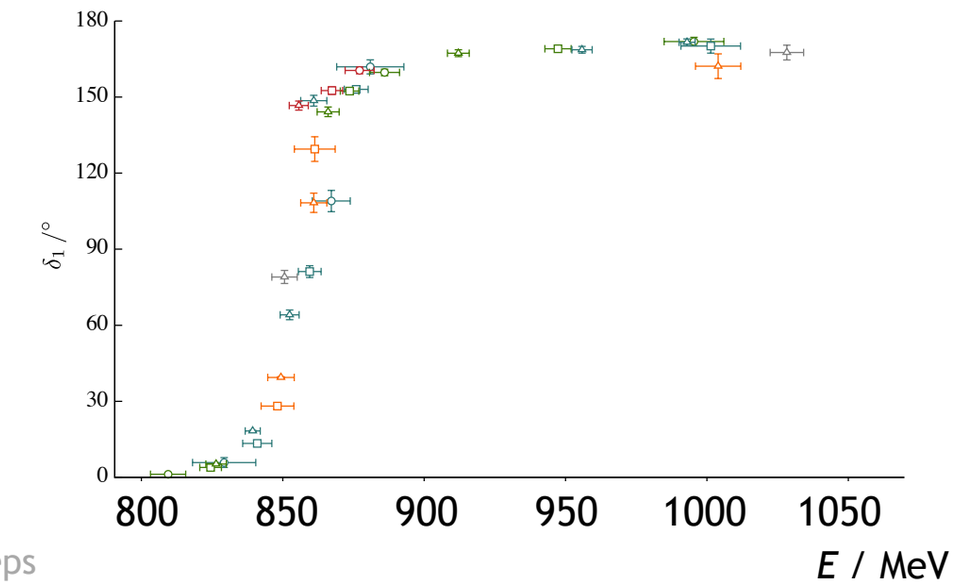
PRD87 034505 (2013)

m_π ~ 391 MeV



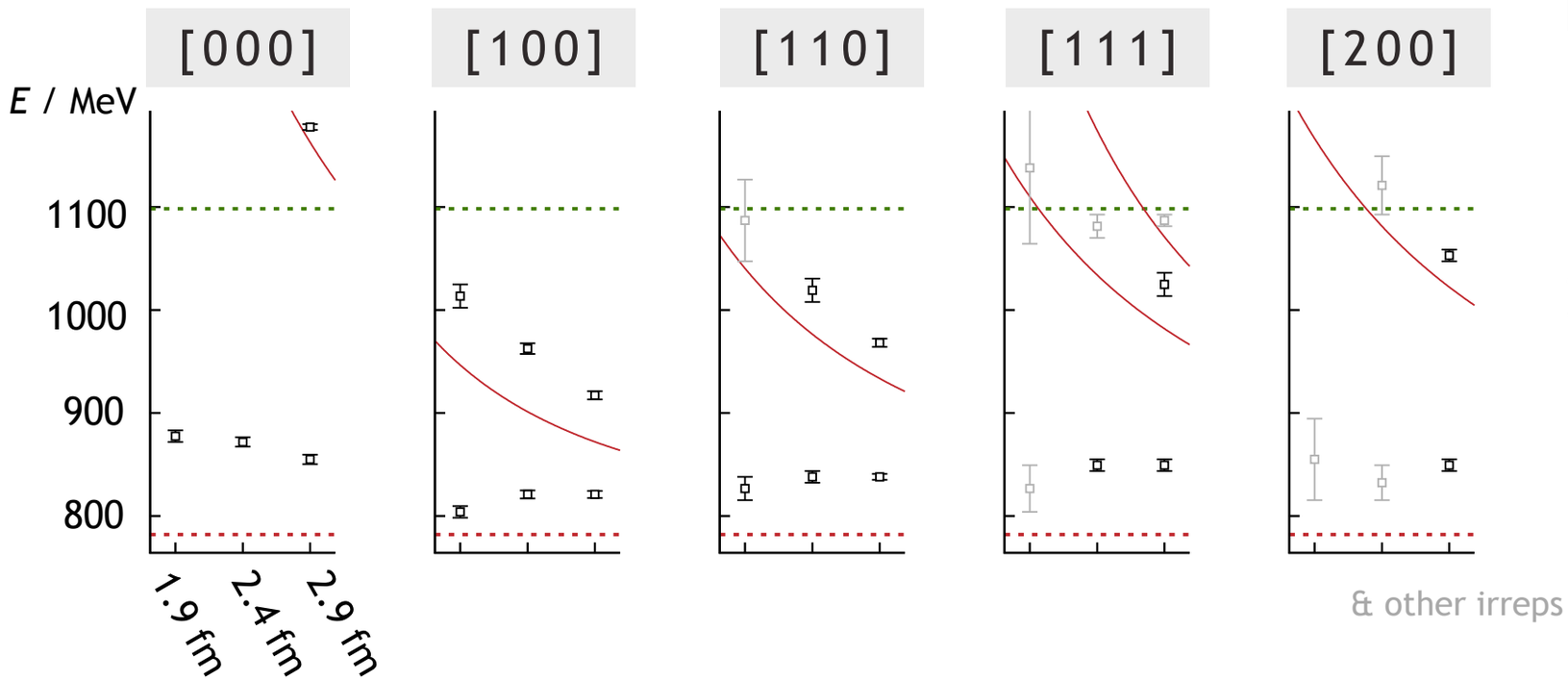
& other irreps

scattering phase-shift

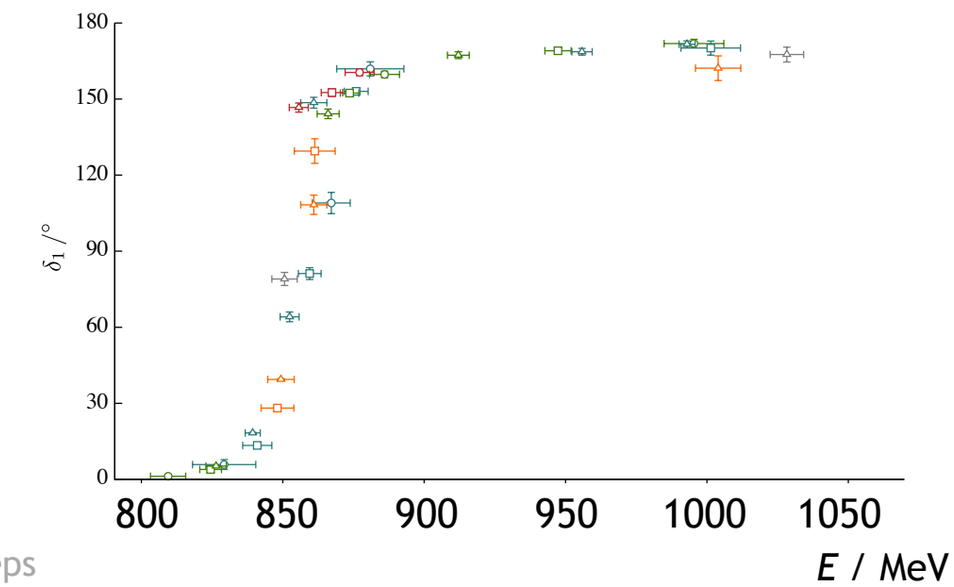


an elastic resonance — the ρ in $\pi\pi$ (isospin=1)

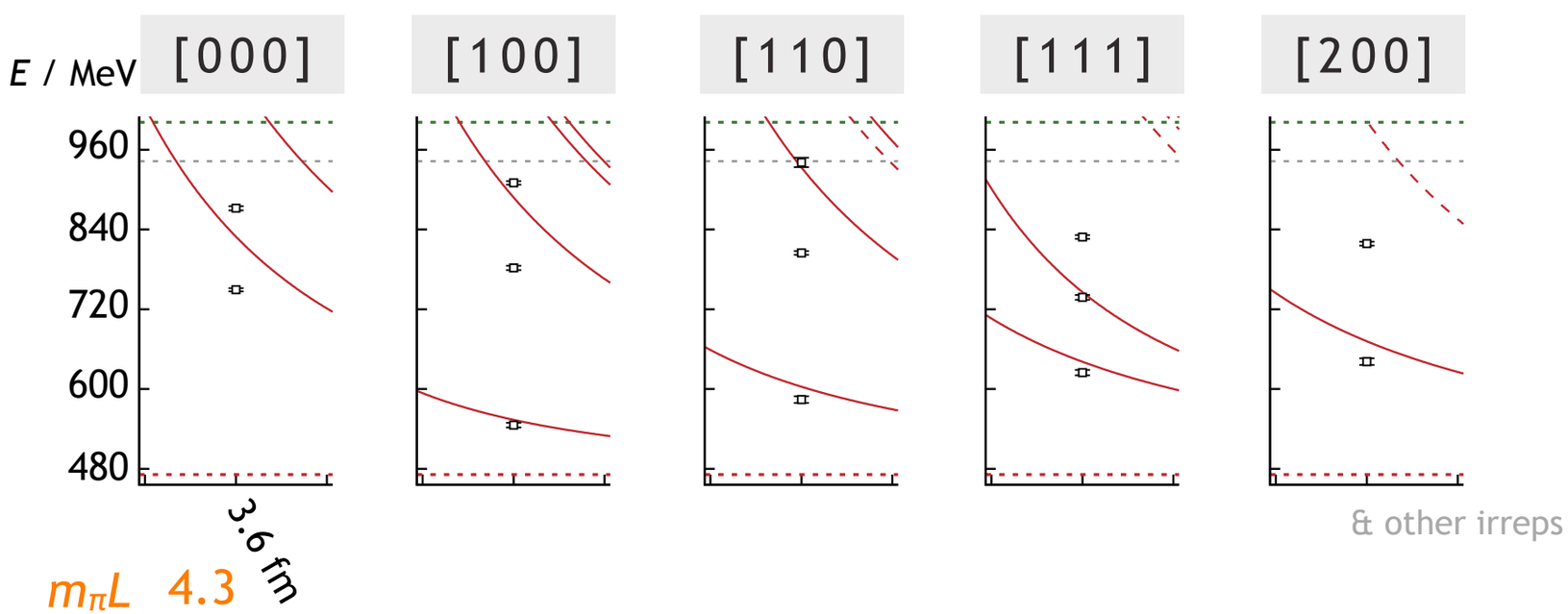
PRD87 034505 (2013) m_π ~ 391 MeV



scattering phase-shift

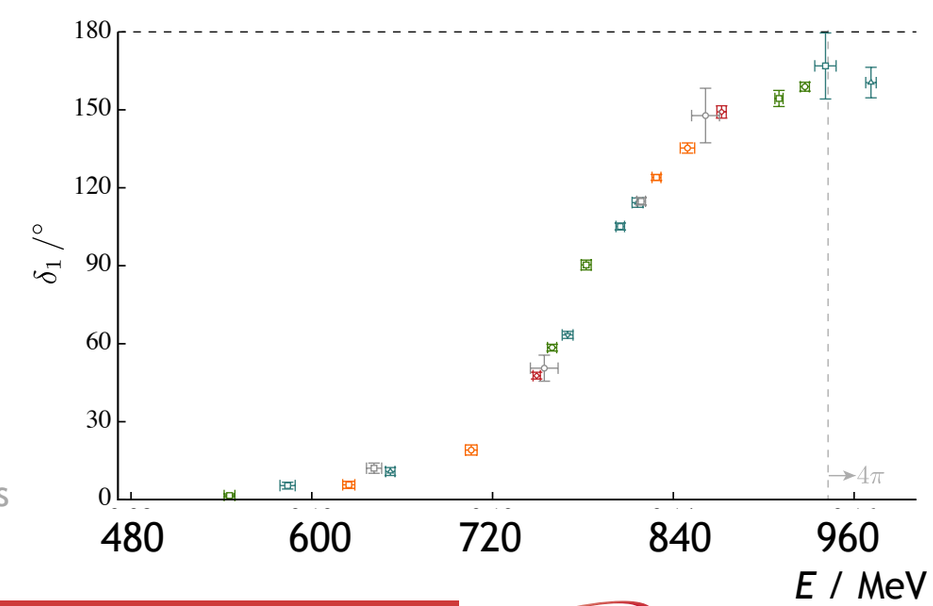


PRD92 094502 (2015) m_π ~ 236 MeV



$m_\pi L$ 4.3

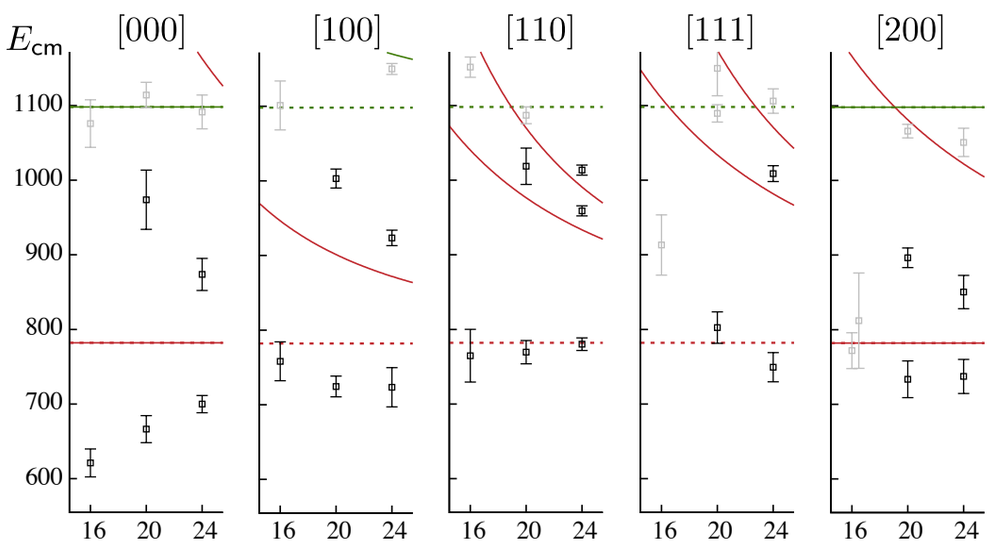
scattering phase-shift



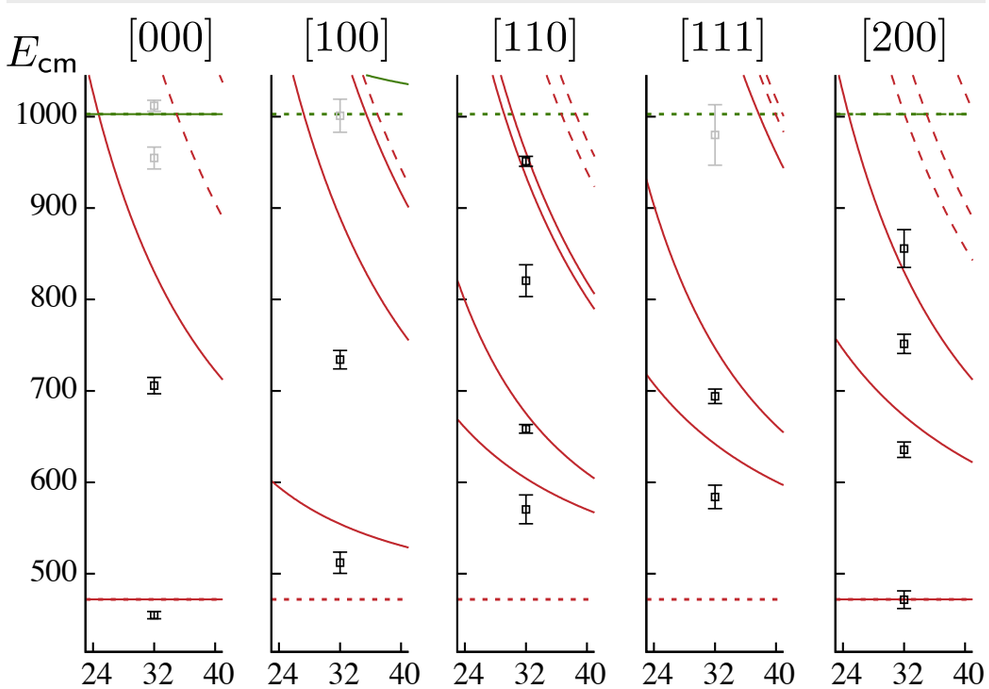
$\pi\pi$ isospin=0

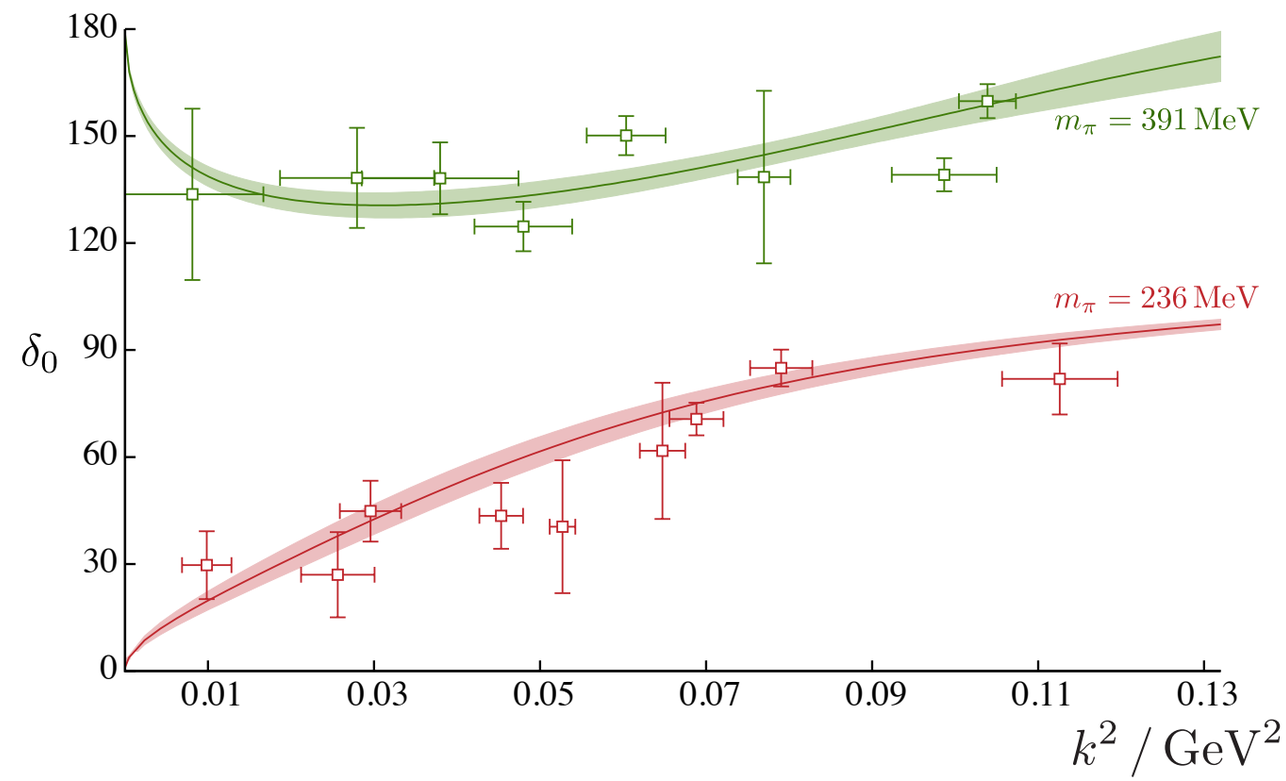
PRL118 022002 (2017)

$m_\pi \sim 391$ MeV



$m_\pi \sim 236$ MeV





heavier quark mass — **a bound-state**

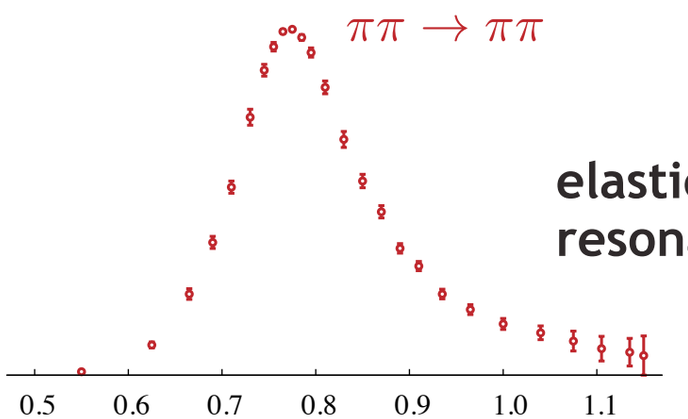
lighter quark mass — attraction, maybe a **broad resonance** ?

c.f. the experimental σ resonance …

coupled-channel resonances

most resonances can decay into more than one final state

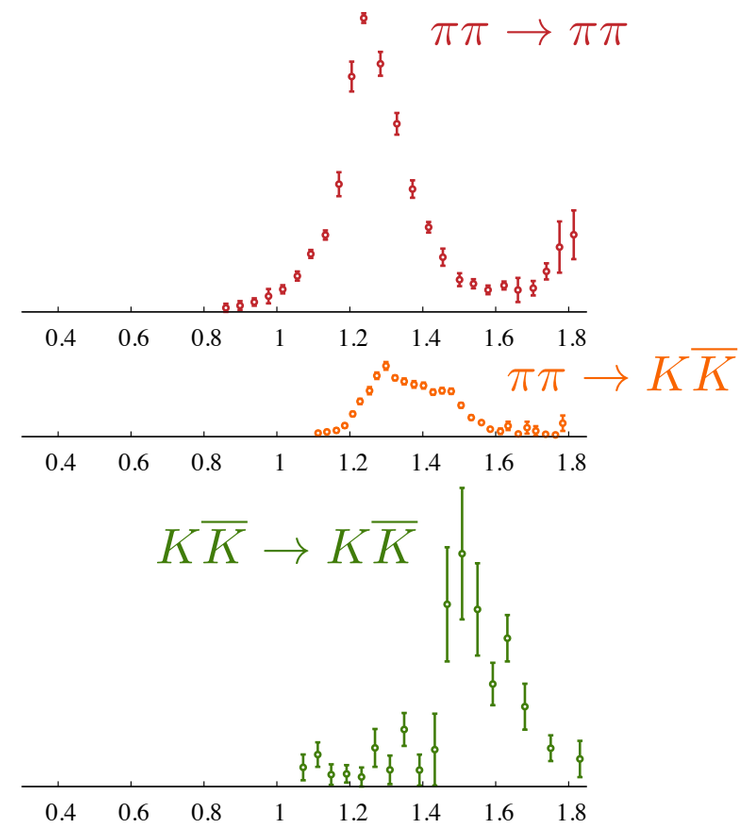
e.g. $\rho(770)$ $\pi\pi$



elastic resonance

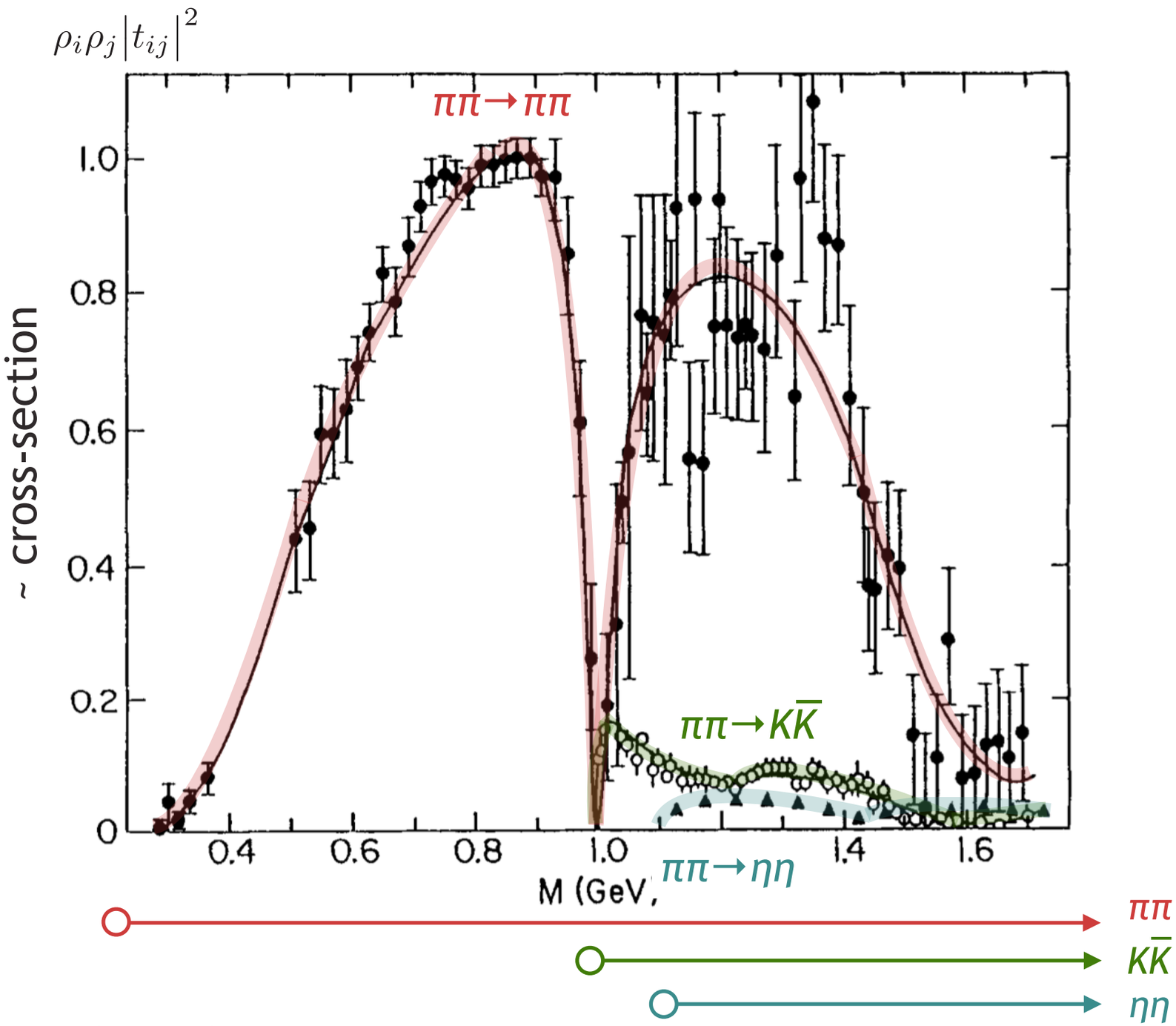
$f_2(1270)$ $\pi\pi, \pi\pi\pi\pi, K\overline{K}, \eta\eta$

$f_2(1525)$ $\pi\pi, K\overline{K}, \eta\eta$



coupled-channel resonances

resonances aren't always bumps



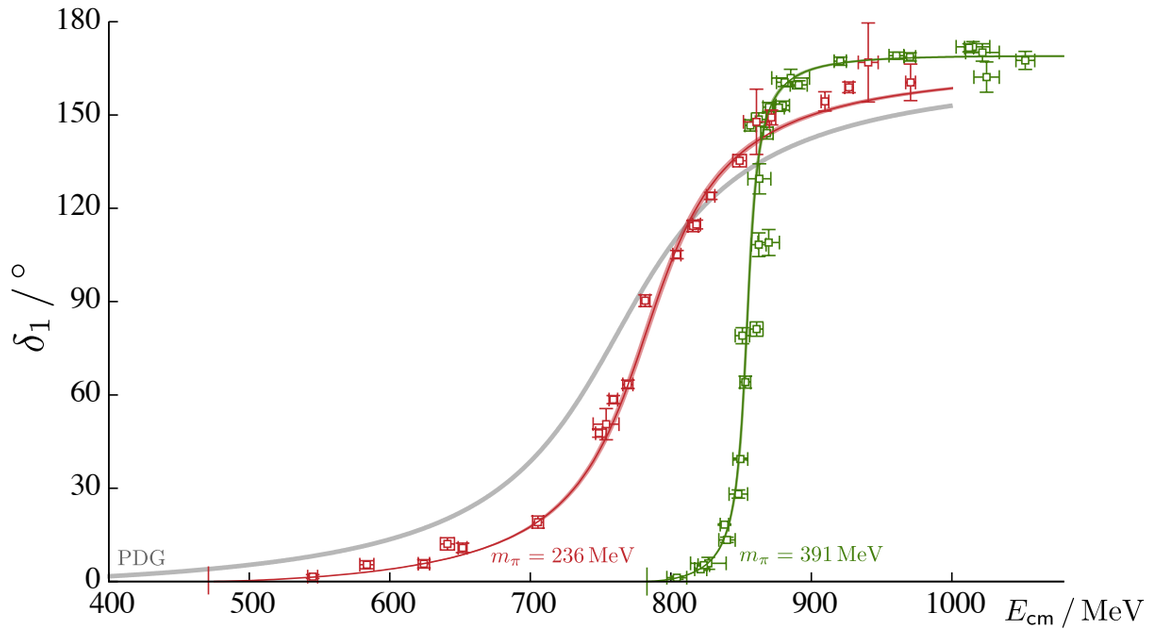
normalization of $\pi\pi \to KK$ slightly uncertain …

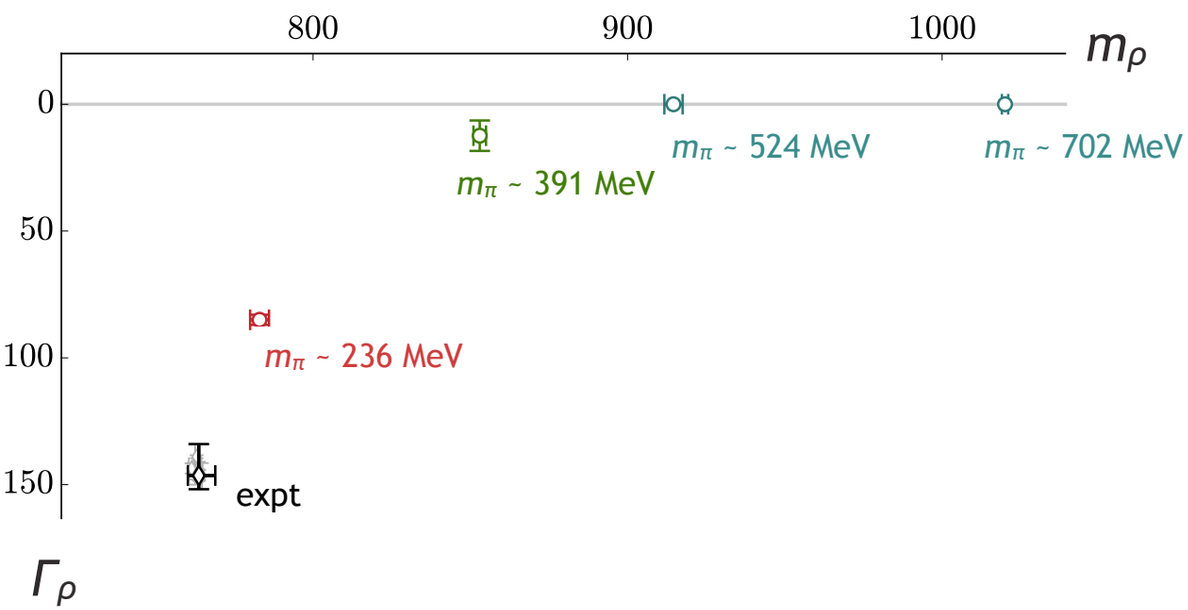
experimentally quite difficult to fill out the whole t-matrix

$$\mathbf{t} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \pi\pi & K\bar{K} & \eta\eta \end{array} \\ \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ & \square & \square \\ & & \square \end{pmatrix} \end{array}\begin{array}{c} \\ \pi\pi \\ K\bar{K} \\ \eta\eta \end{array}$$

isolating kaon exchange hard & η beams don't exist

ρ pole with changing quark mass from lattice QCD

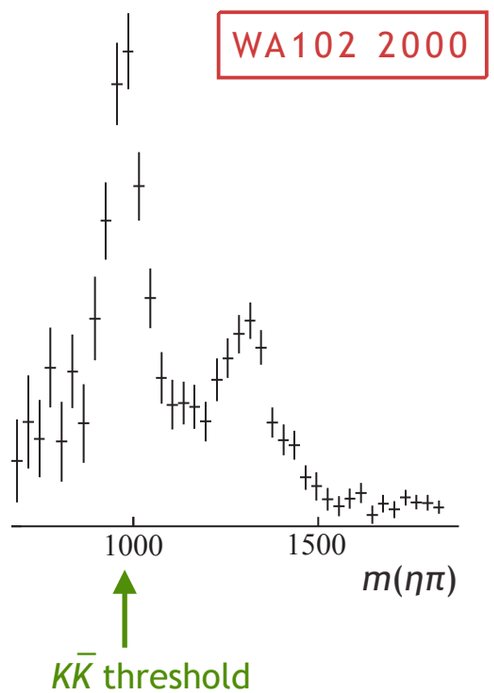




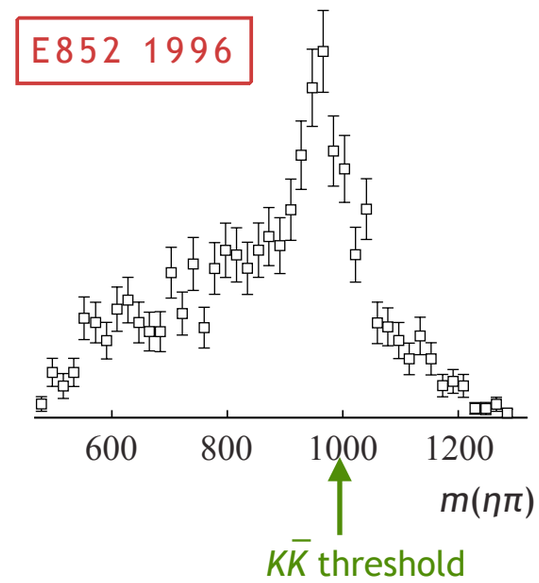
experimental $a_0(980)$ in $\pi\eta$

e.g.

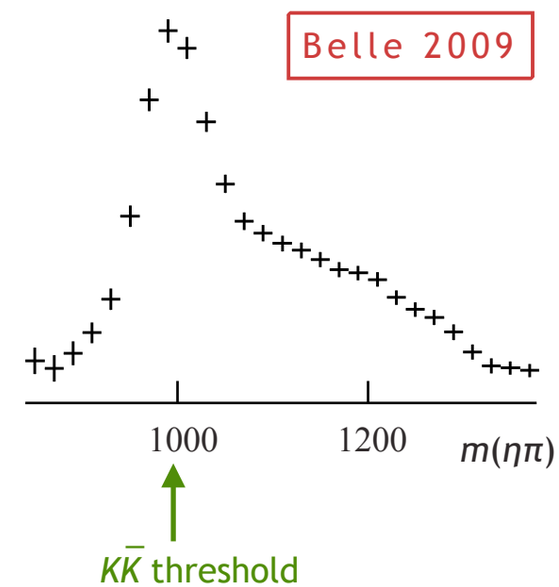
$pp \to p\,\eta\pi^0\,p$



$\pi^- p \to \eta\pi^+\pi^- n$



$\gamma\gamma \to \eta\pi^0$

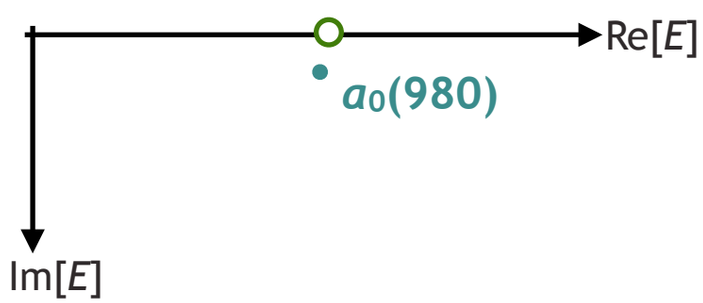


right at the $K\bar{K}$ threshold

$a_0(980)$ [j]

$I^G(J^{PC}) = 1^-(0^{++})$

Mass $m = 980 \pm 20$ MeV
Full width $\Gamma = 50$ to 100 MeV



coupled-channel scattering from lattice QCD ?

one possible approach — mimic 'ideal' experiment: determine $t_{ab}^{(\ell)}(E)$ for real energies

maybe can then be analytically continued to complex energies to find poles ?

scattering matrix determines the finite-volume spectrum:

$$0 = \det\Big[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\big)\Big]$$

a generalized Lüscher equation

for a given scattering matrix, has a discrete set of solutions $E_n(L)$

the finite-volume spectrum

one approach:

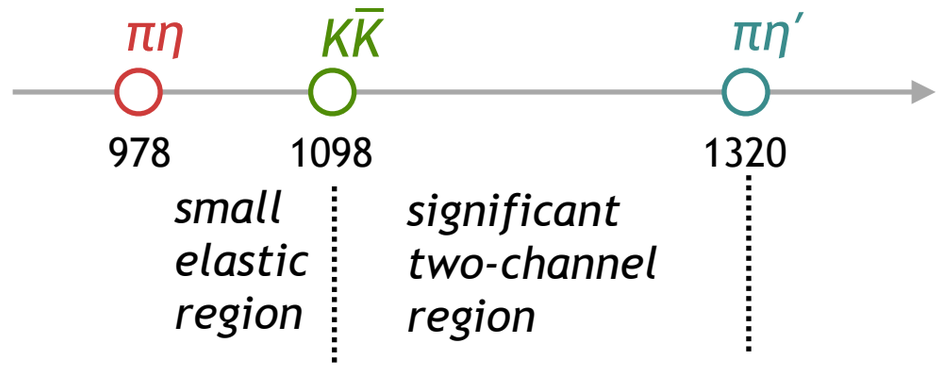
- parameterize the energy dependence of $\boldsymbol{t}(E)$
- solve $0 = \det\Big[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\big)\Big]$
- compare 'model' spectrum to lattice spectrum …

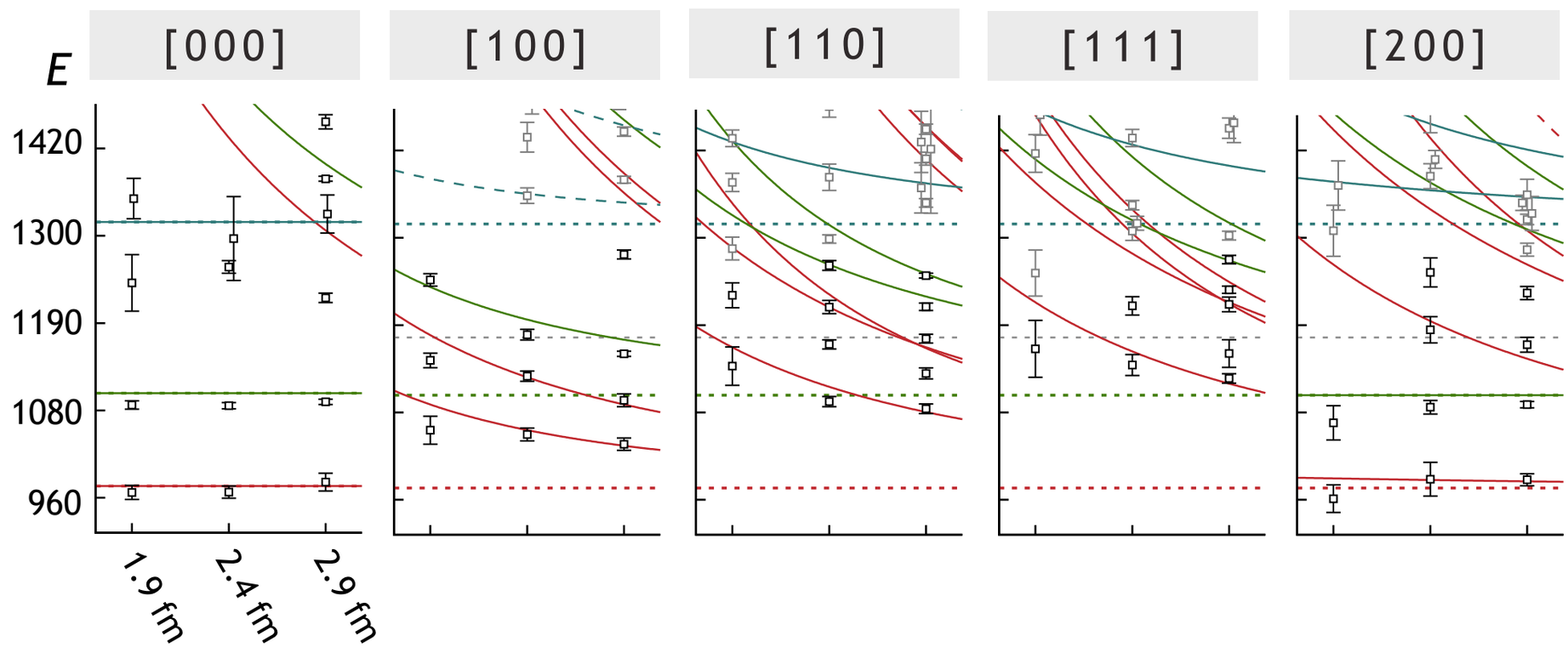
ensure important features are independent of parameterization details by varying parameterization …

$\pi\eta/K\bar{K}$ scattering in lattice QCD

PRD93 094506 (2016)

m_π ~ 391 MeV
m_K ~ 549 MeV
m_η ~ 587 MeV



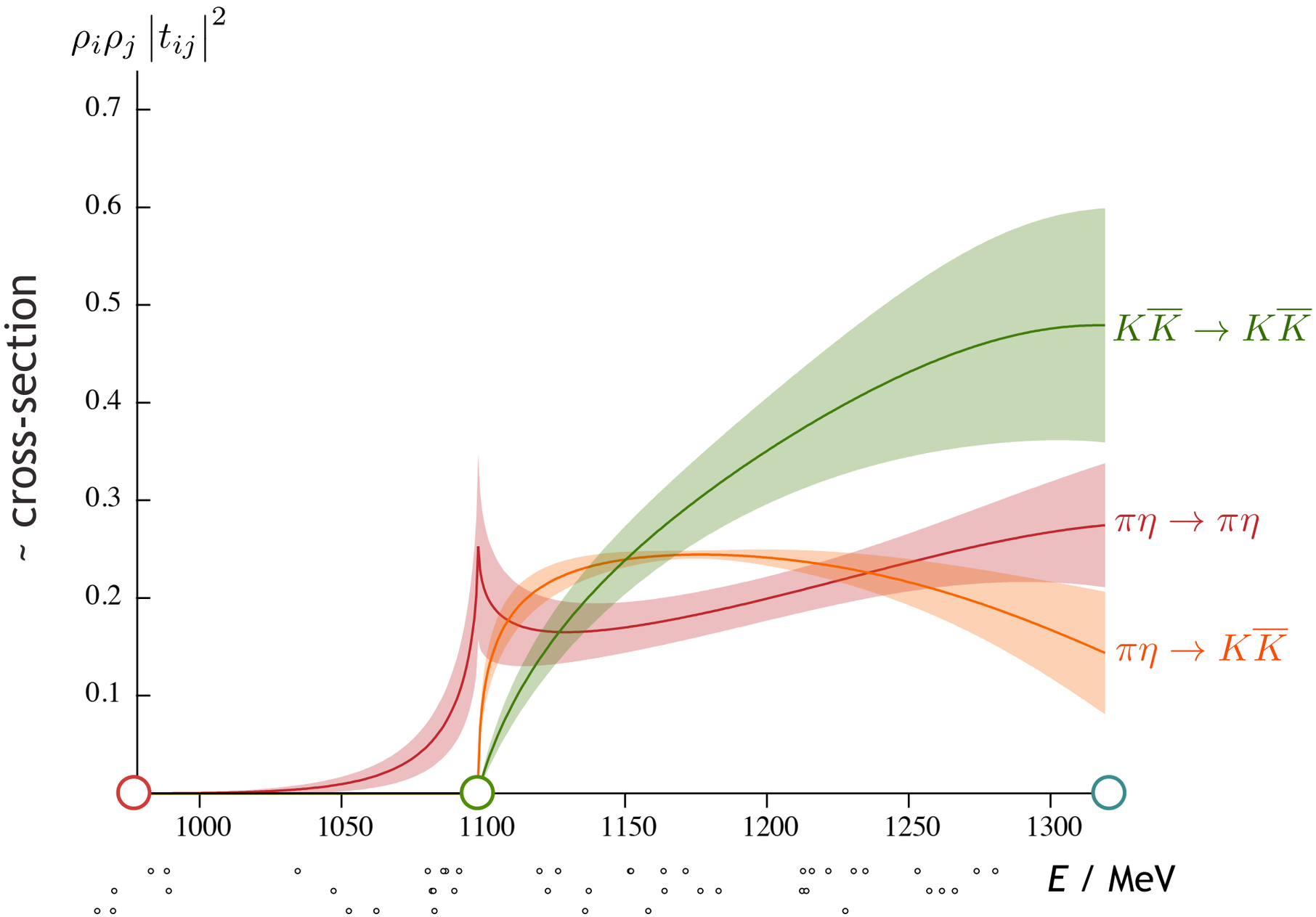


$\pi\eta/K\bar{K}$ scattering in lattice QCD

m_π ~ 391 MeV

PRD93 094506 (2016)

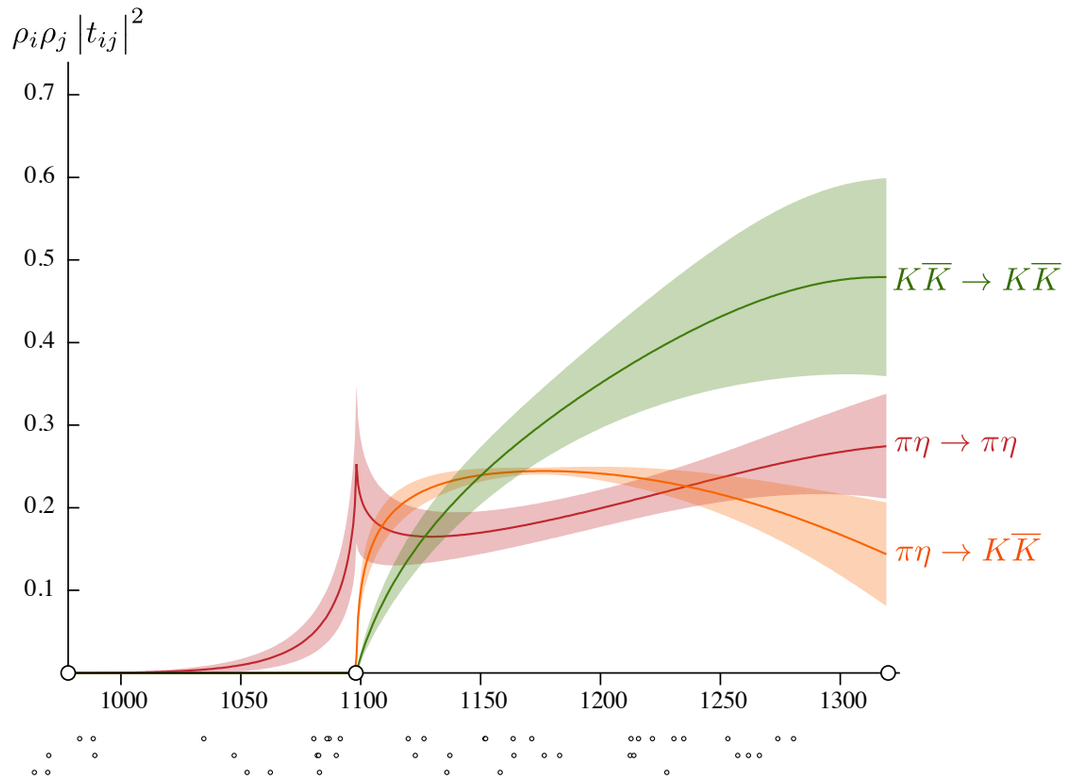
πη/KK̄ coupled-channel scattering amplitudes



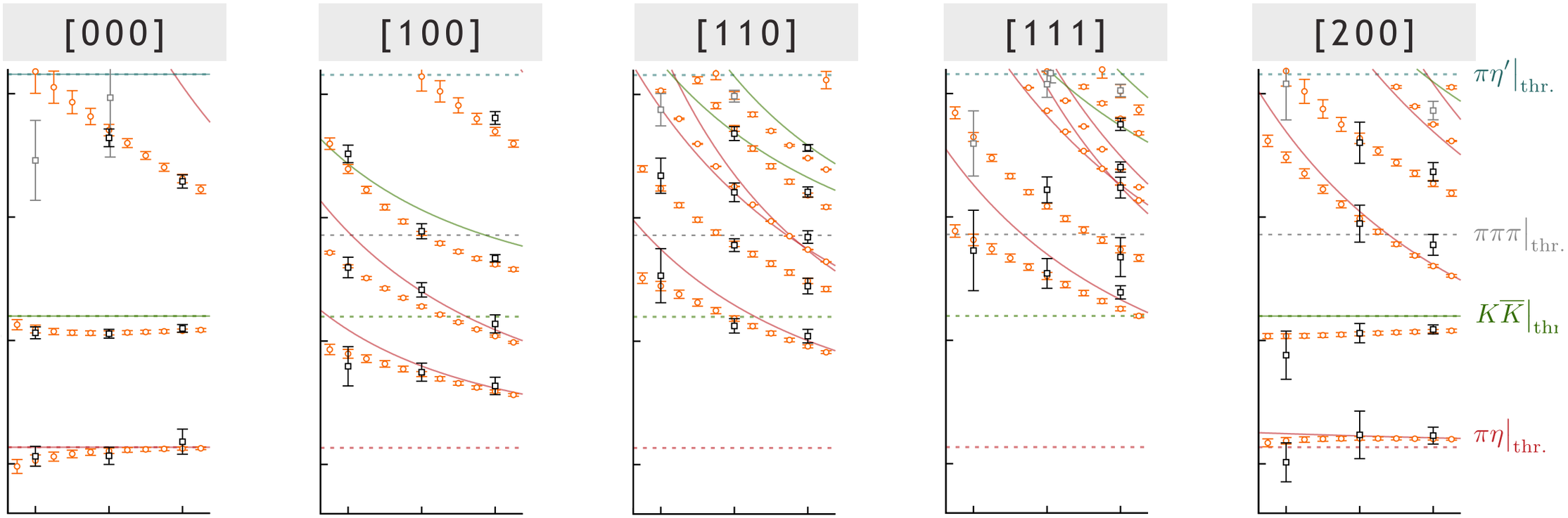
$$\mathbf{K} = \frac{1}{m^2 - s}\begin{bmatrix} g^2_{\pi\eta} & g_{\pi\eta}g_{K\bar{K}} \\ g_{\pi\eta}g_{K\bar{K}} & g^2_{K\bar{K}} \end{bmatrix} + \begin{bmatrix} \gamma_{\pi\eta,\pi\eta} & \gamma_{\pi\eta,K\bar{K}} \\ \gamma_{\pi\eta,K\bar{K}} & \gamma_{K\bar{K},K\bar{K}} \end{bmatrix}$$

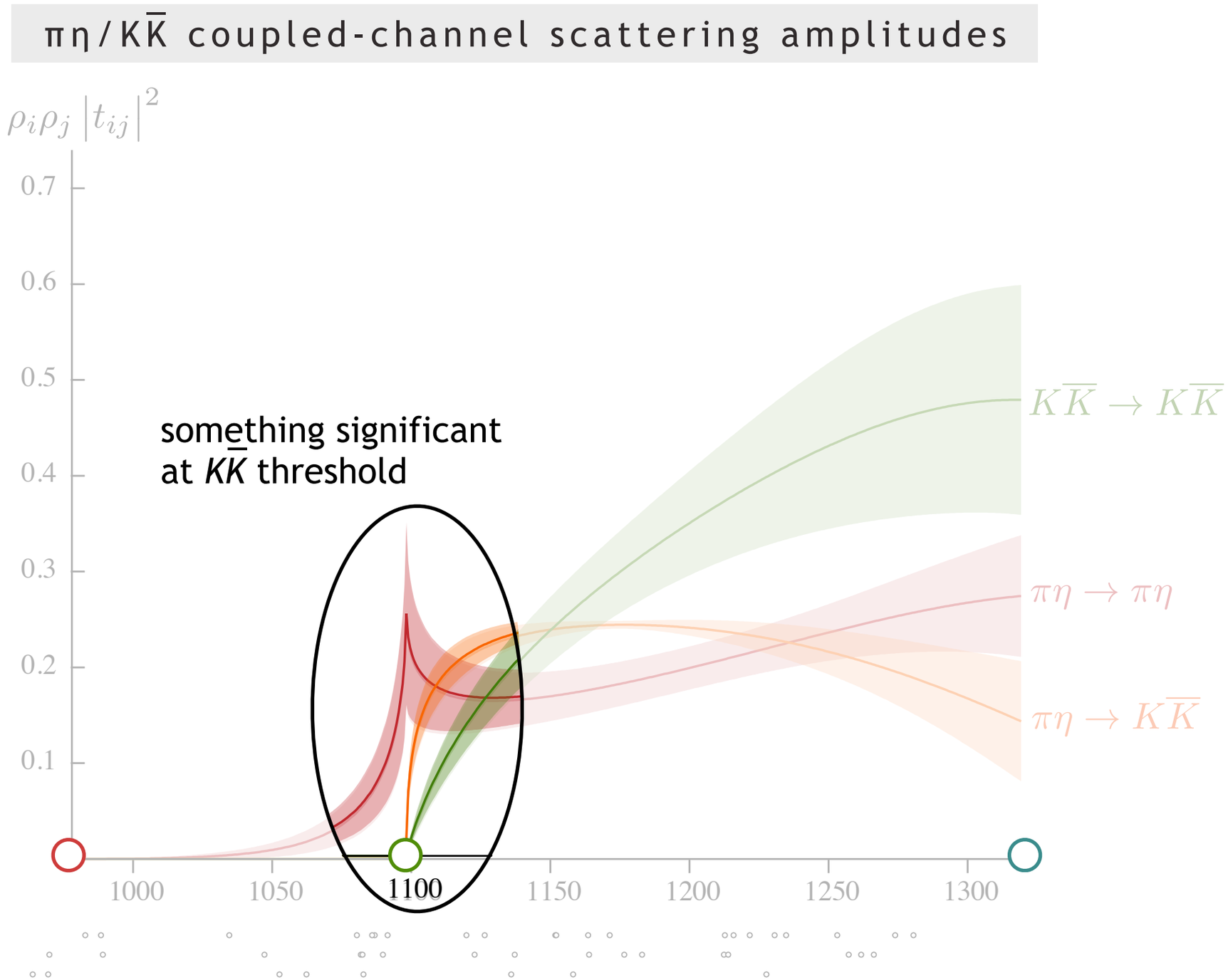
& Chew-Mandelstam phase-space

a good description of the finite-volume spectrum



$$\chi^2/N_{\rm dof} = \frac{58.0}{47-6} = 1.41$$





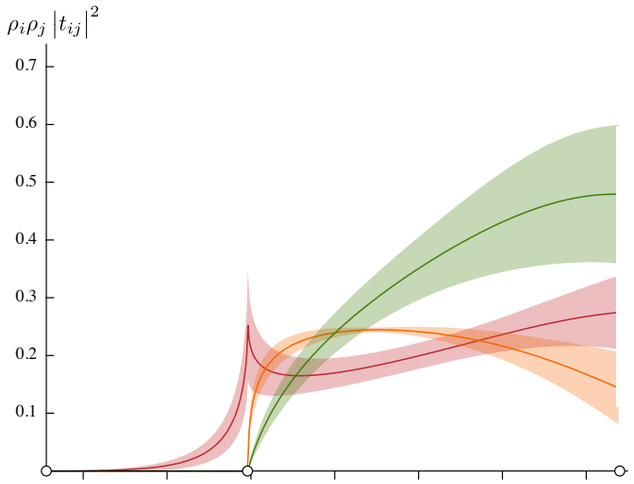
is there a resonance causing this ?

examine the amplitude for nearby poles …

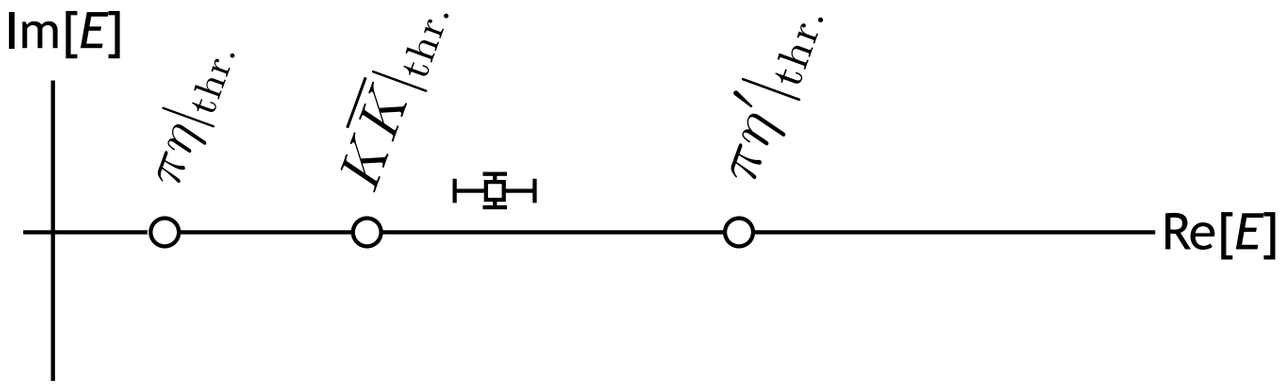
continue the amplitude into the complex plane

PRD93 094506 (2016)

m_π ~ 391 MeV



pole position

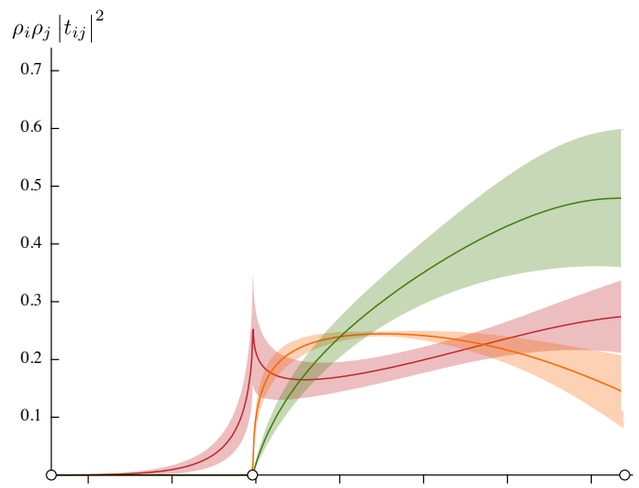


but is this sensitive to the parameterization choice ?

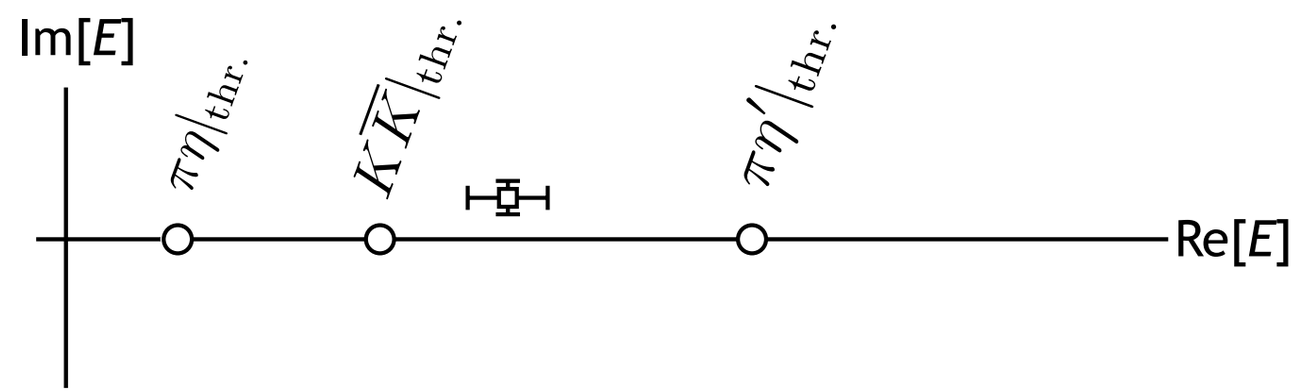
continue the amplitude into the complex plane

PRD93 094506 (2016)

m_π ~ 391 MeV

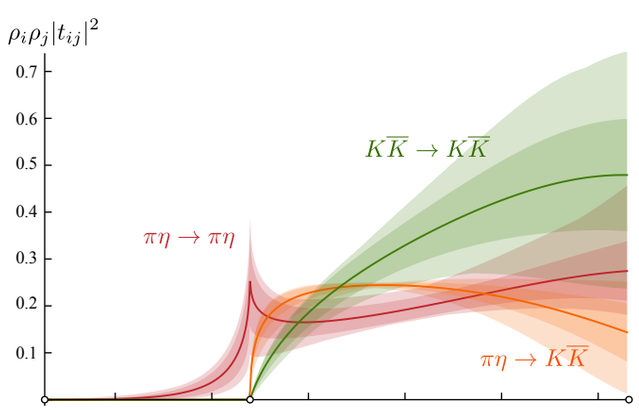


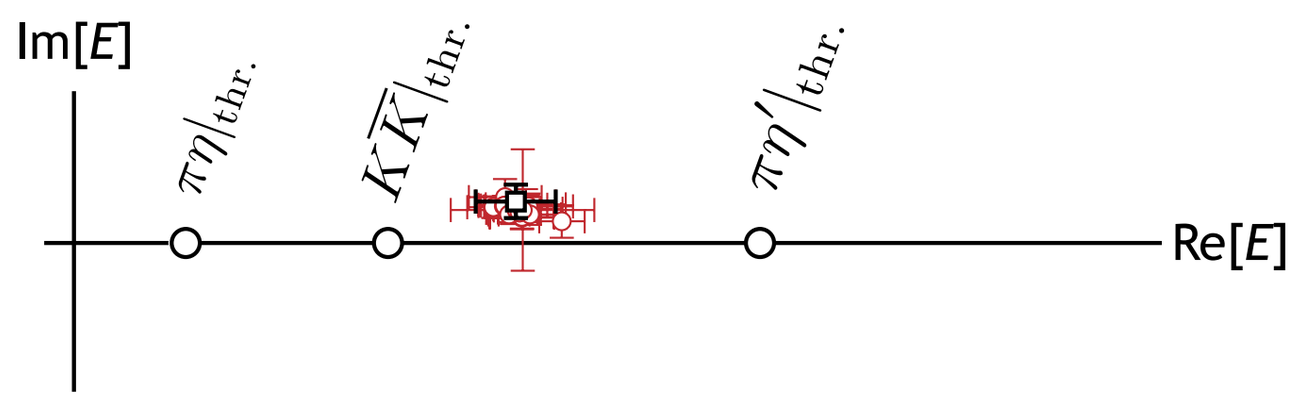
pole position



but is this sensitive to the parameterization choice ?

no



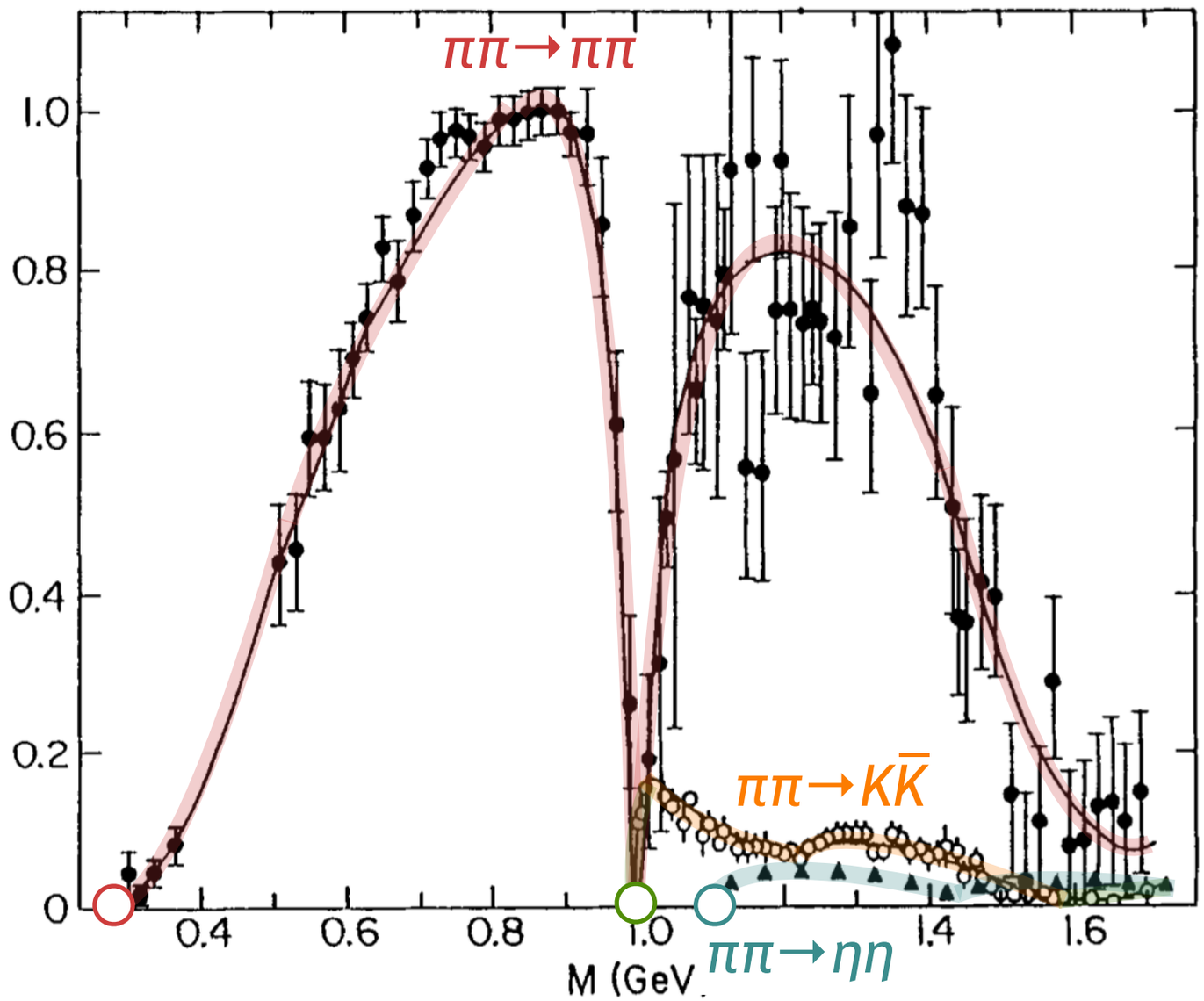


$$m_{a_0} = 1177(27)\,\text{MeV} \qquad \left|\frac{c_{K\bar{K}}}{c_{\pi\eta}}\right|^2 = 1.7(6)$$

$$\Gamma_{a_0} = 49(33)\,\text{MeV}$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering

experimental amplitudes



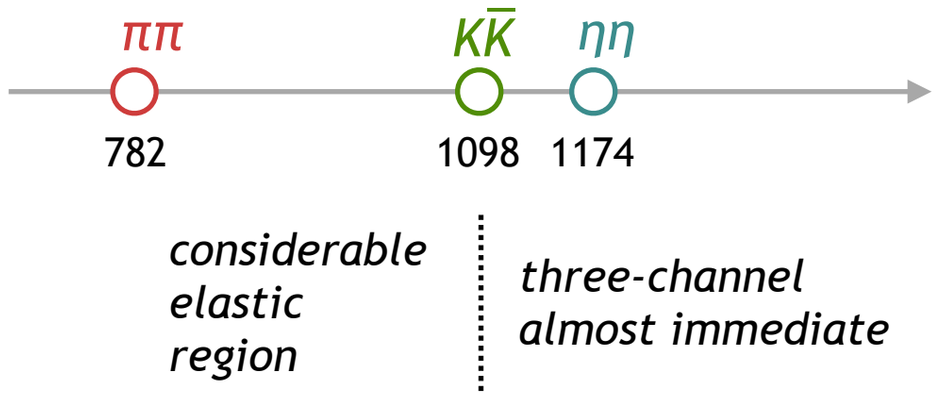
combination of broad σ resonance and narrow $f_0(980)$ at $K\bar{K}$ threshold

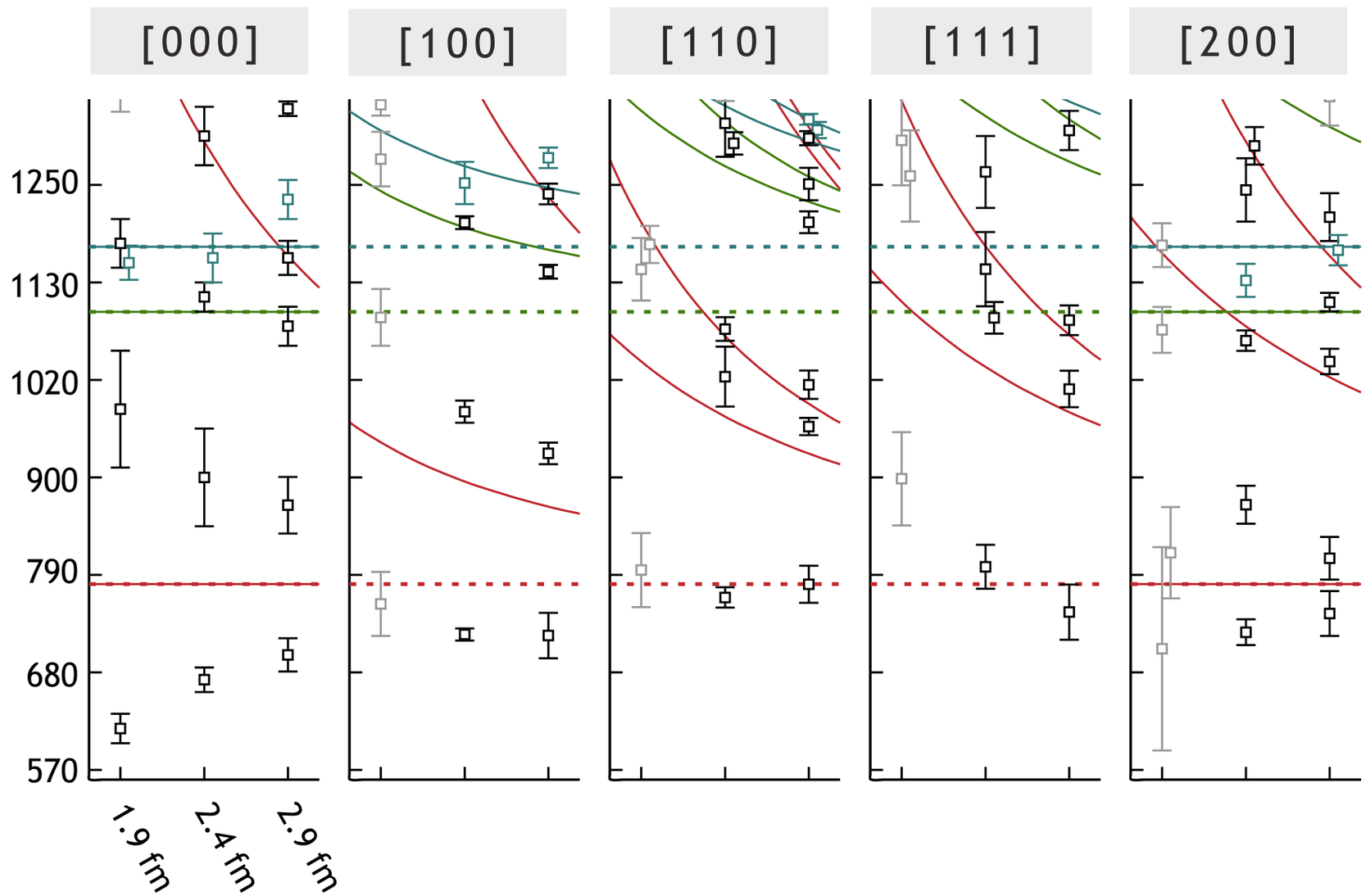
two low lying resonances …

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering in lattice QCD

PRD97 054513 (2018)

m_π ~ 391 MeV
m_K ~ 549 MeV
m_η ~ 587 MeV



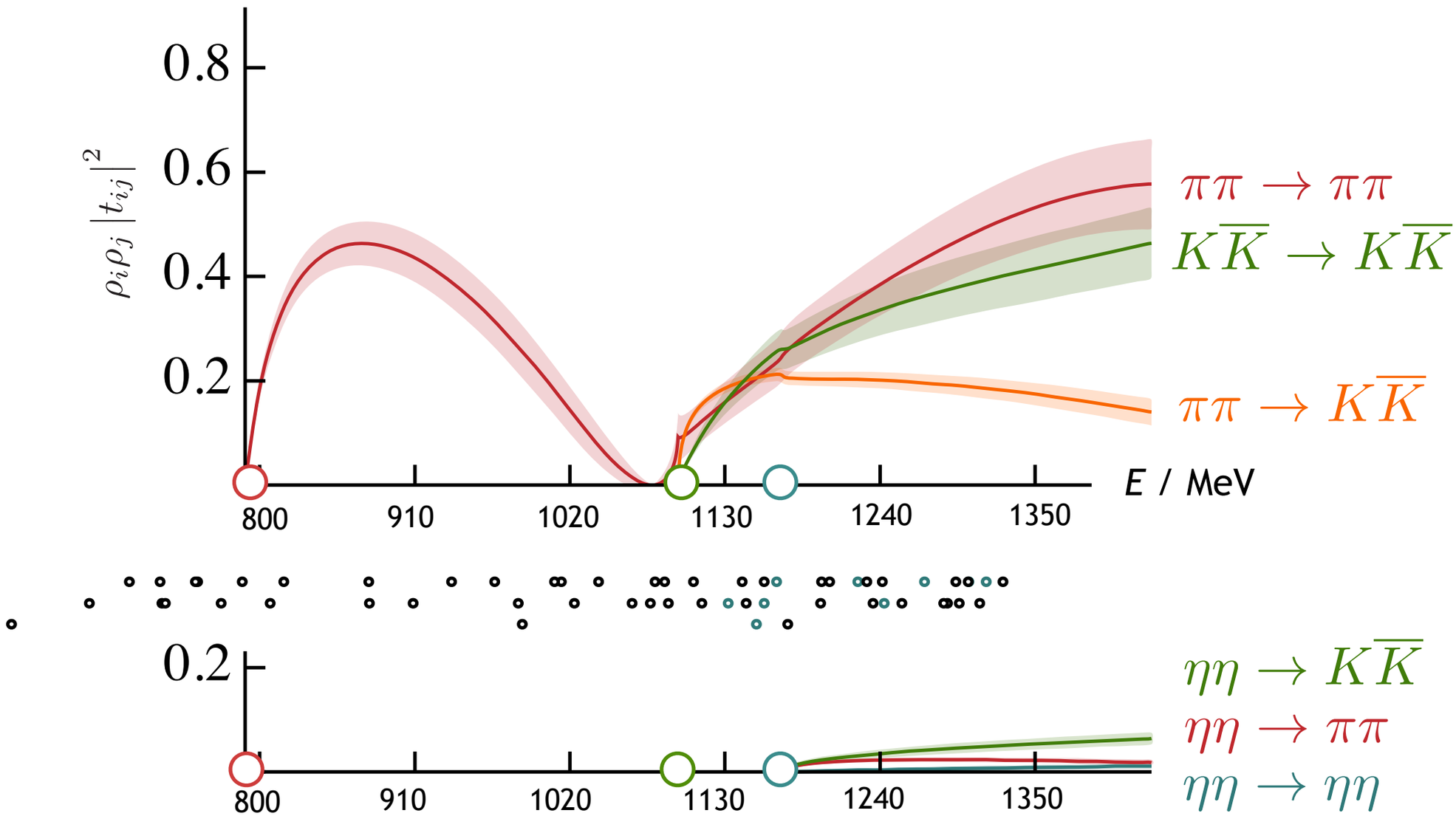


$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering in lattice QCD

PRD97 054513 (2018)

m_π ~ 391 MeV
m_K ~ 549 MeV
m_η ~ 587 MeV

ππ/KK̄/ηη coupled-channel scattering amplitudes

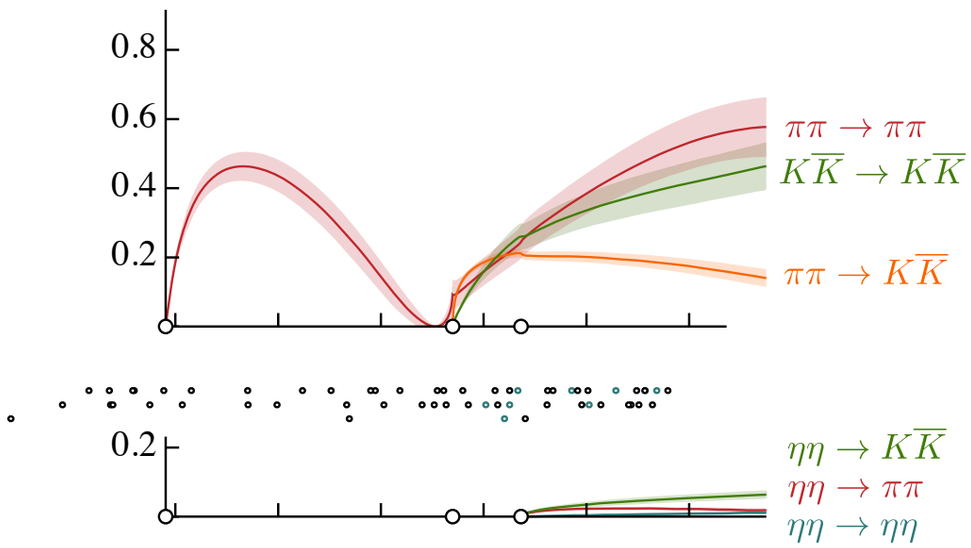


nothing much happening in $\eta\eta$

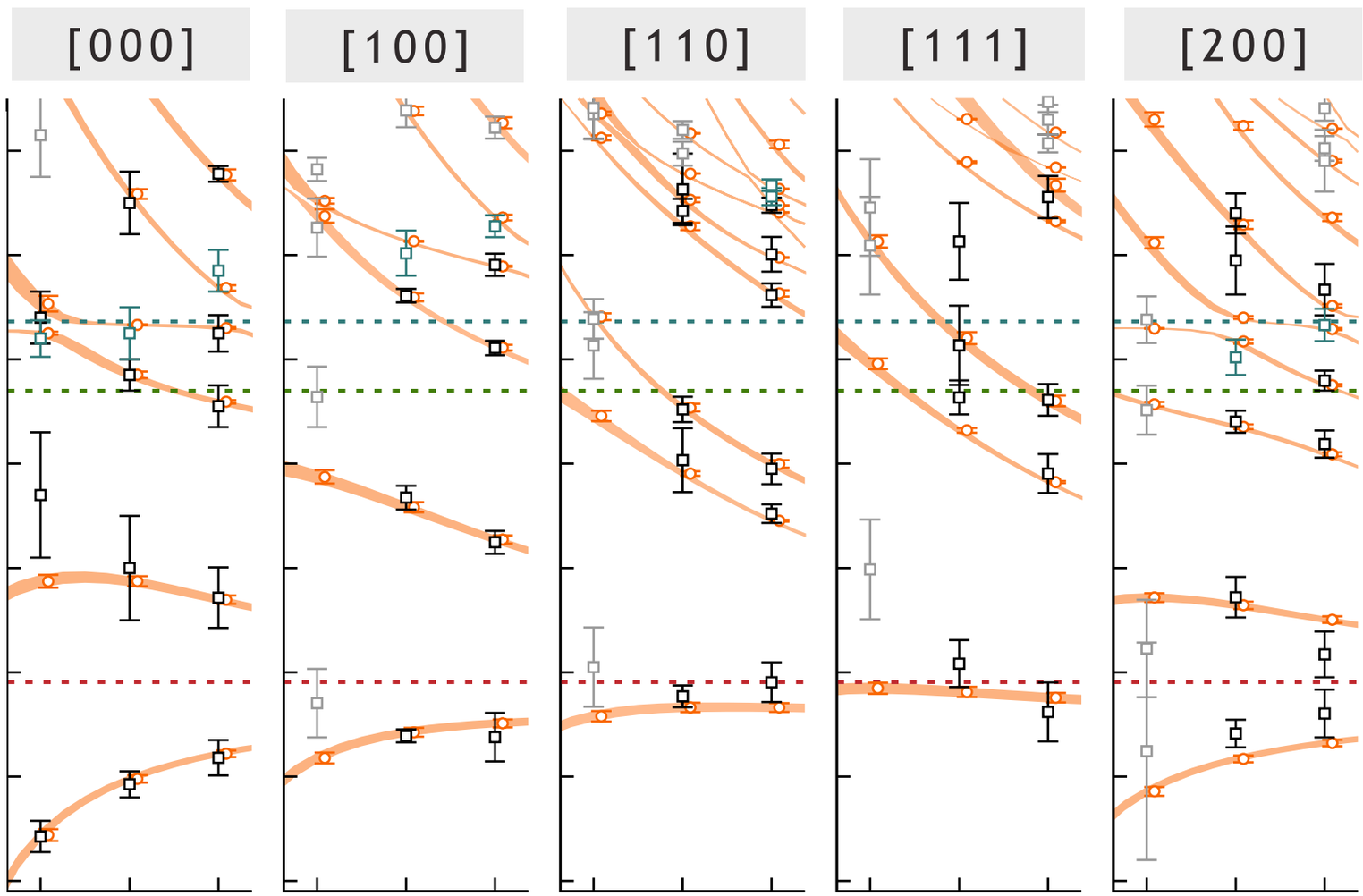
$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

a good description of the finite-volume spectrum

PRD97 054513 (2018)



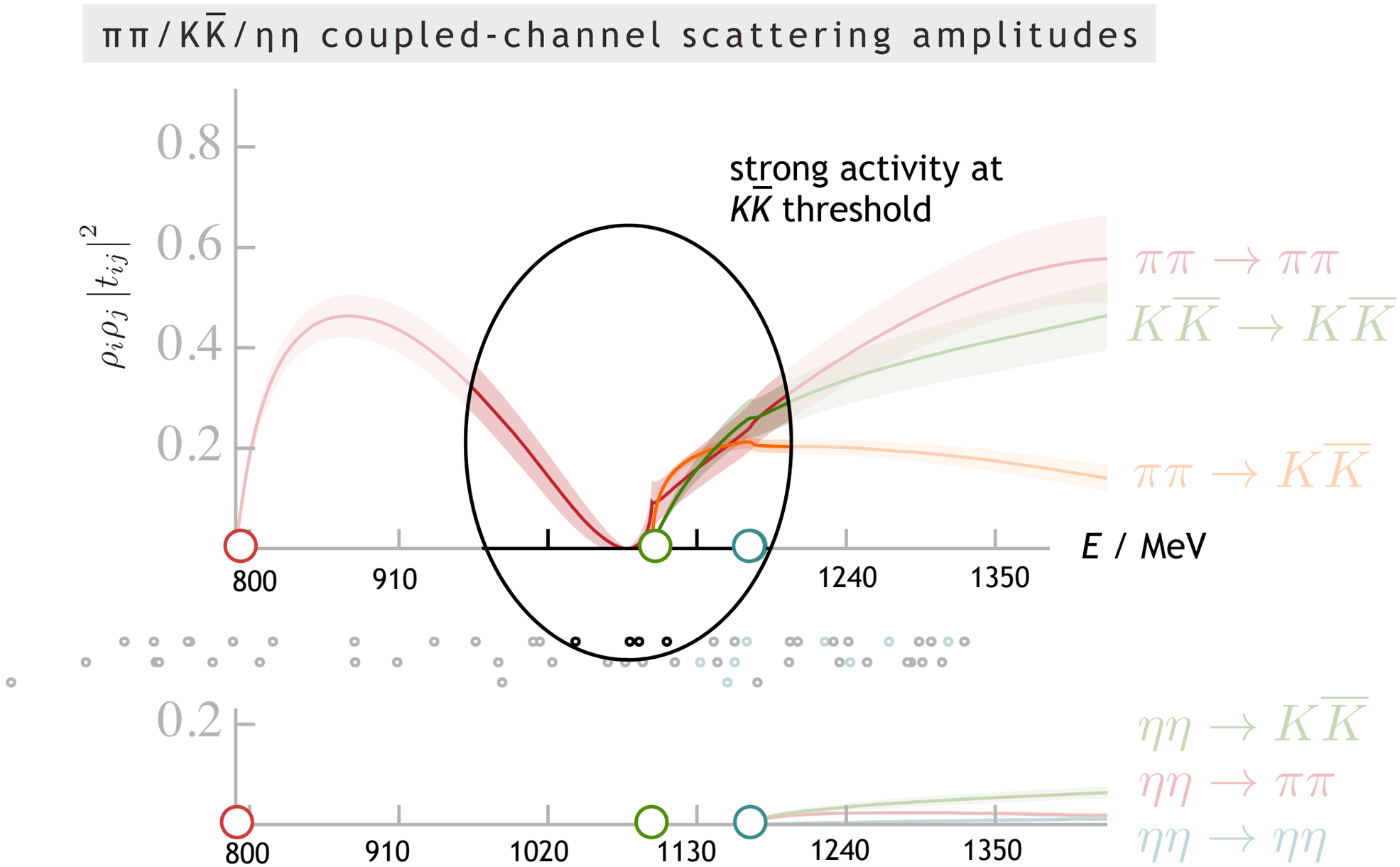
$$\chi^2/N_{\rm dof} = \frac{44.0}{57-8} = 0.90$$



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering in lattice QCD

PRD97 054513 (2018)

m_π ~ 391 MeV
m_K ~ 549 MeV
m_η ~ 587 MeV



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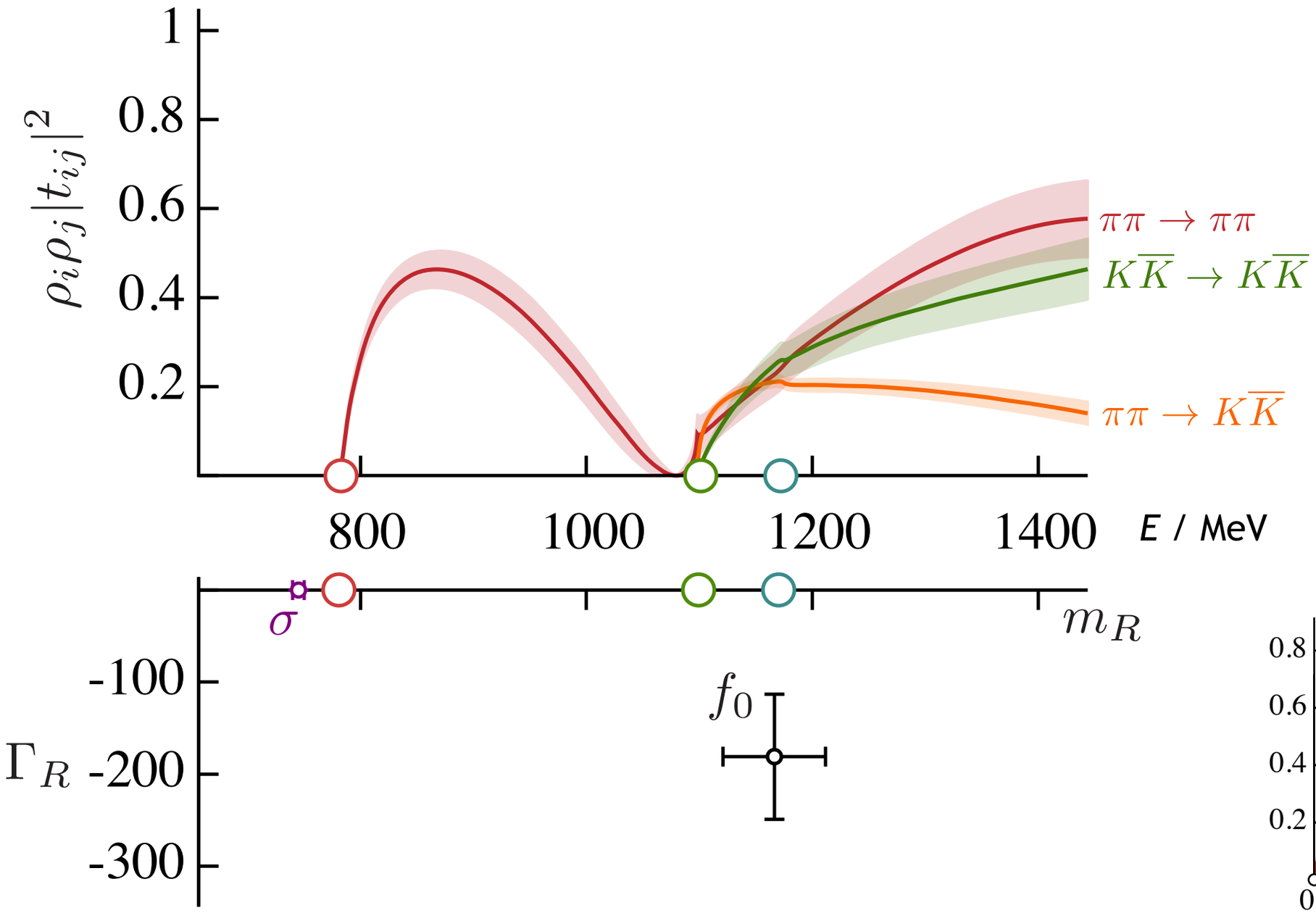

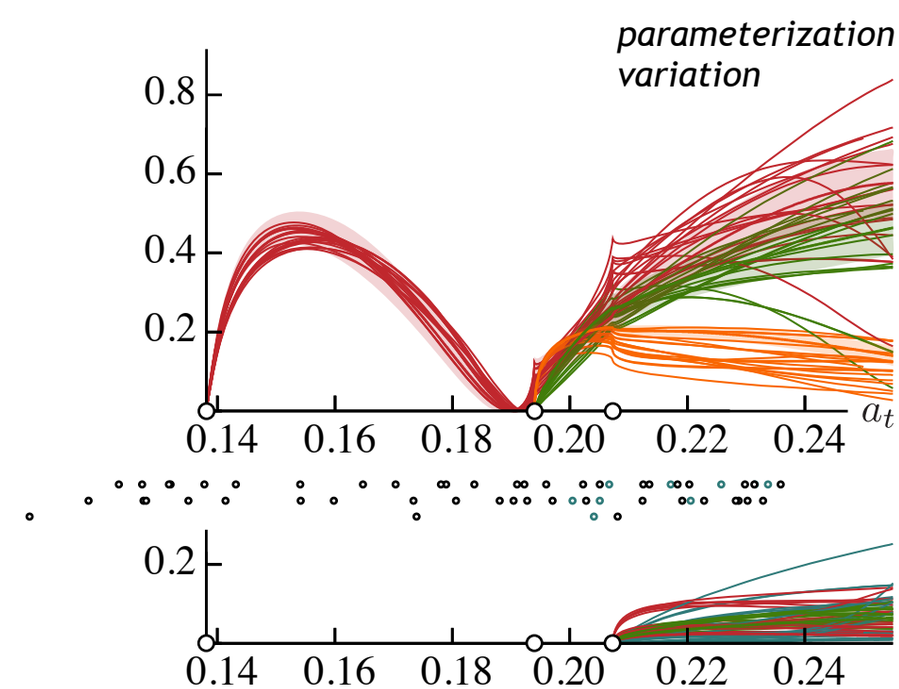

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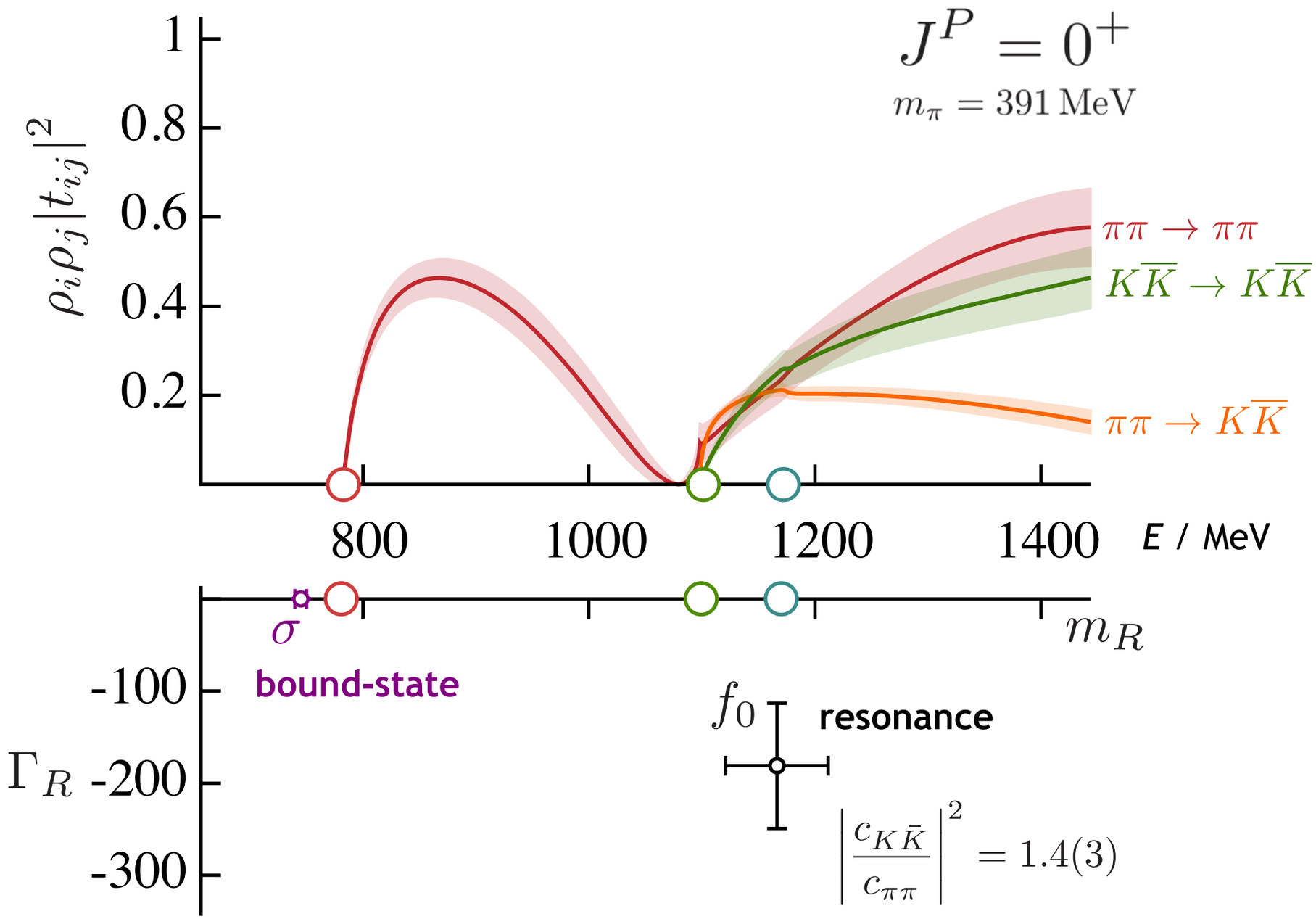


$\pi\omega$ scattering

except at very low quark masses, the ω is a stable state

the non-zero spin of the ω introduces new features, e.g. $J^P = 1^+$ in **two** partial-waves $\begin{pmatrix} {}^3S_1 \\ {}^3D_1 \end{pmatrix}$

expect a b_1 **resonance**

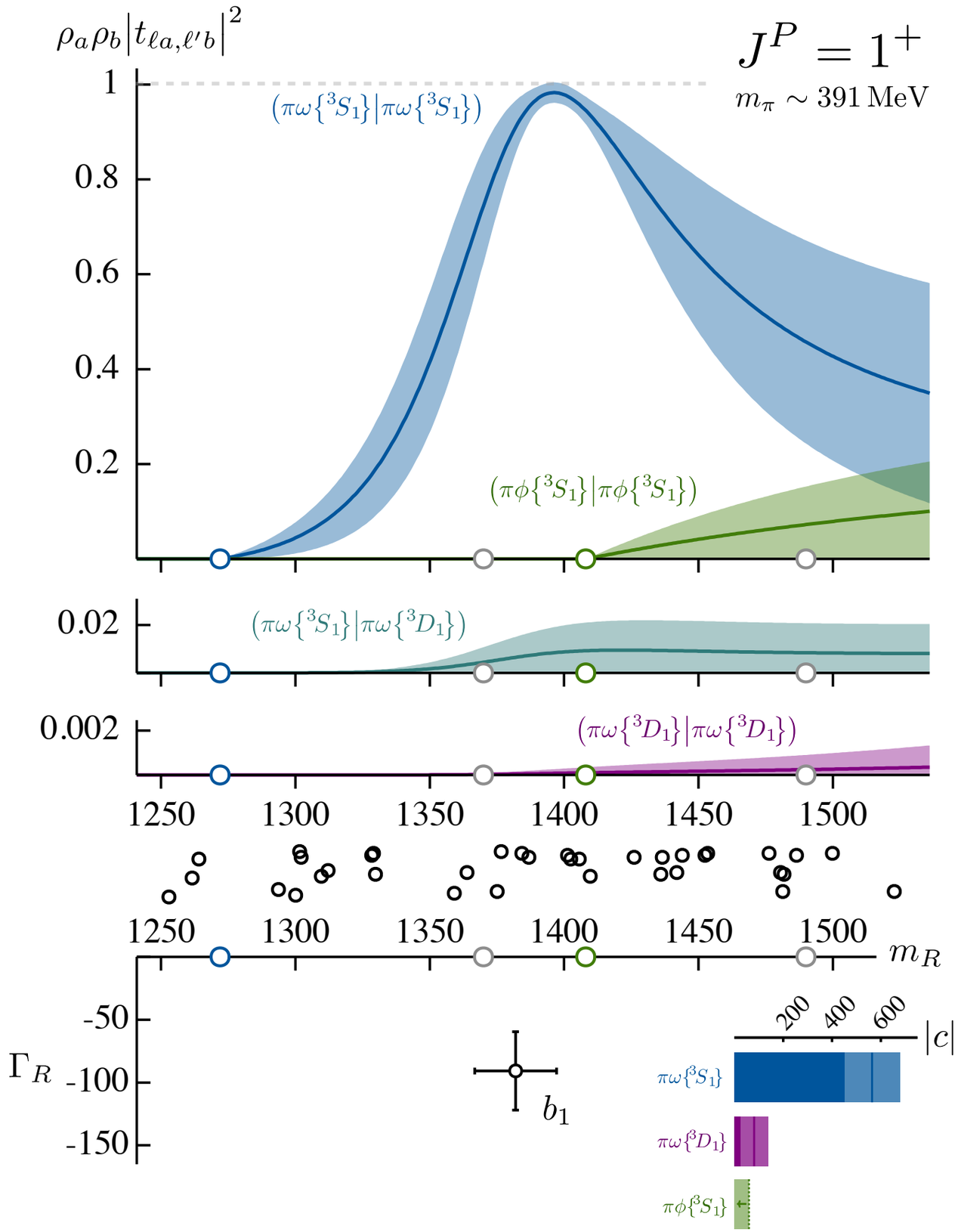
$b_1(1235)$ $\quad I^G(J^{PC}) = 1^+(1^{+-})$

Mass $m = 1229.5 \pm 3.2$ MeV (S = 1.6)
Full width $\Gamma = 142 \pm 9$ MeV (S = 1.2)

$b_1(1235)$ DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\omega\pi$	dominant		348
[D/S amplitude ratio $= 0.277 \pm 0.027$]			

in the quark model it's the $u\bar{d}(^1P_1)$ state

$\pi\omega/\pi\varphi$ scattering

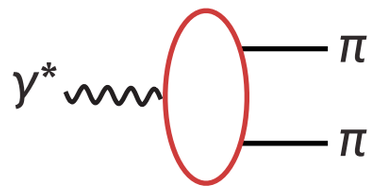


clear b_1 resonance

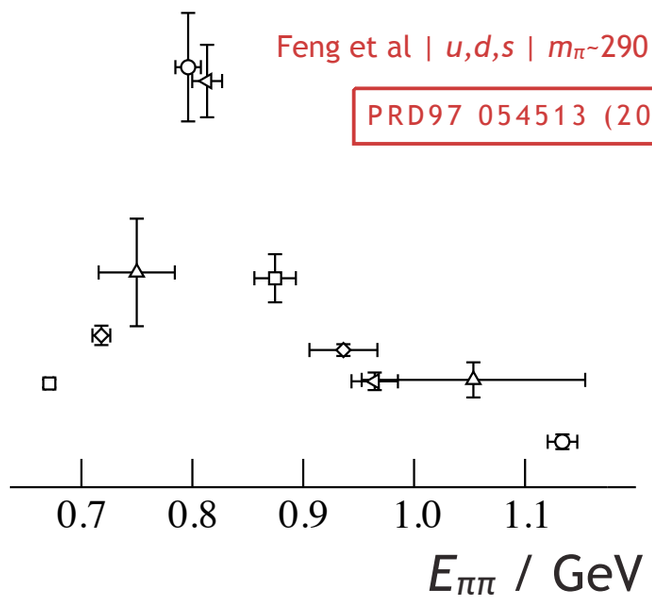
- strong $\pi\omega$ S-wave
- weak $\pi\omega$ D-wave
- negligible $\pi\varphi$

coupling resonances to currents

$\gamma^* \to \pi\pi$

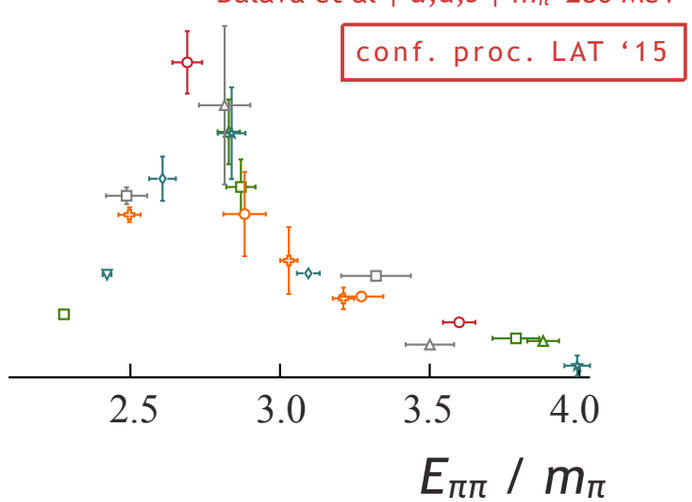


$F_\pi(E_{\pi\pi})$



Feng et al | *u,d,s* | m_π~290 MeV

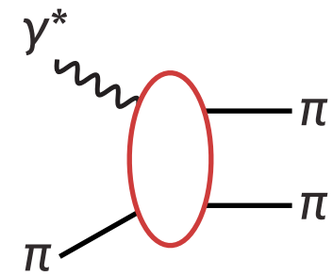
PRD97 054513 (2018)



Bulava et al | *u,d,s* | m_π~280 MeV

conf. proc. LAT '15

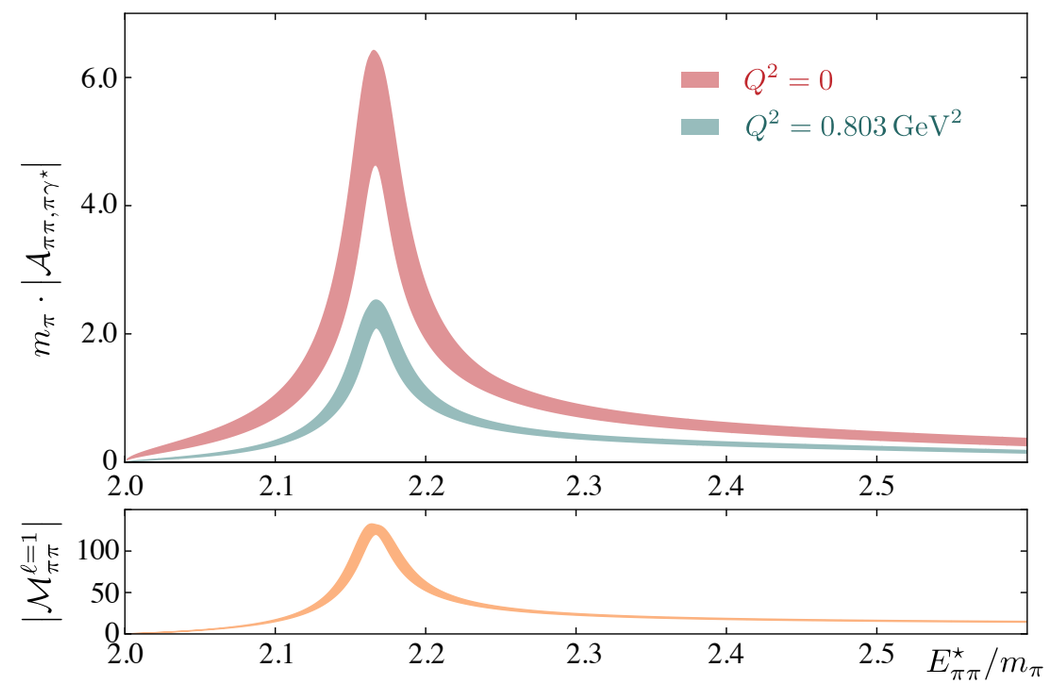
$\gamma^*\pi \to \pi\pi$

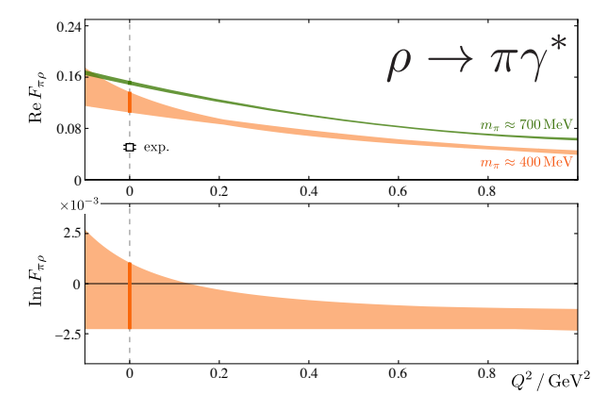


$\mathcal{A}(E_{\pi\pi}, Q^2)$

Briceno et al | *u,d,s* | m_π~391 MeV

PRL115 242001 (2015)





what do you calculate

calculate correlation functions

e.g. $\langle 0|\mathcal{O}_i(t)\mathcal{O}_j(0)|0\rangle$ where the operators are constructed from quark and gluon fields and have the quantum numbers of the hadronic system you want to study

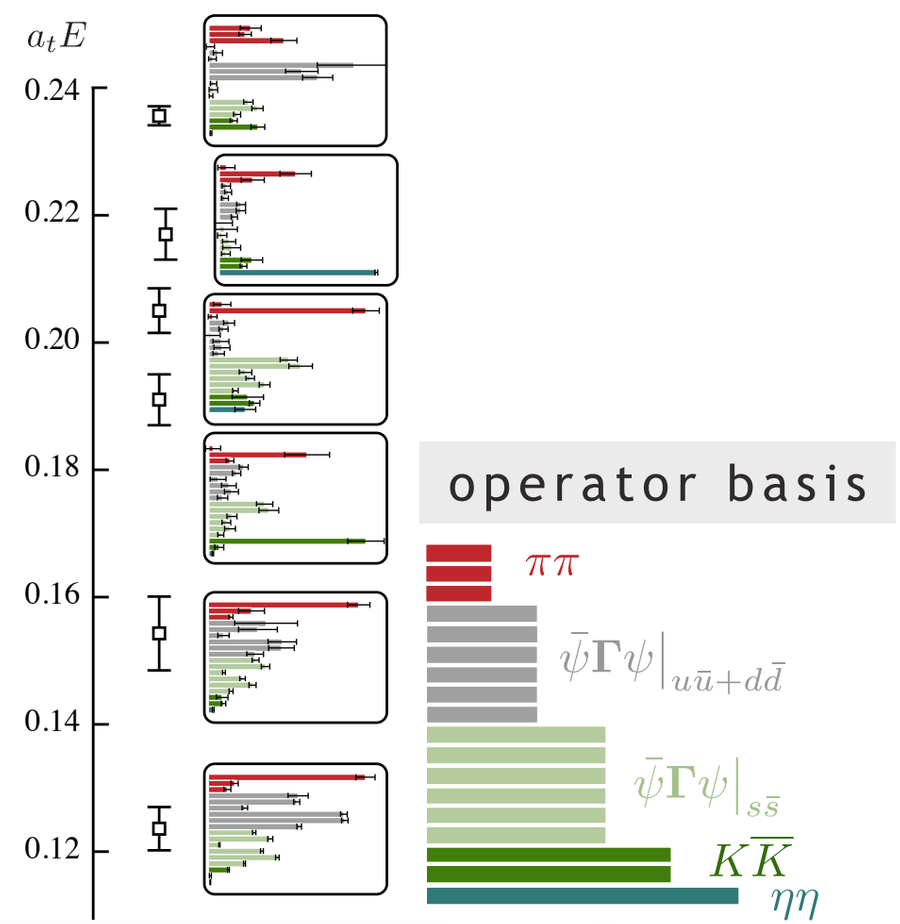
$$\langle 0|\mathcal{O}_i(t)\mathcal{O}_j(0)|0\rangle = \sum_n \langle 0|\mathcal{O}_i|n\rangle \langle n|\mathcal{O}_j|0\rangle \, e^{-E_n t}$$

a superposition of the (finite-volume) eigenstates of QCD

powerful approach:

- use a large basis of operators*
- form a matrix of correlation functions
- diagonalize this matrix

e.g. [000] A_1^+ 24^3



* could give a whole interesting talk on the construction of these operators

operator basis — I=0 $\pi\pi$, $K\bar{K}$, $\eta\eta$

operator basis: ‘single-meson’ + ‘meson-meson’

$$\overline{\psi}\mathbf{\Gamma}\psi$$

(& if you like,
tetraquark & …)

$$\sum_{\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2} C(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p})\, M_1(\mathbf{p}_1)\, M_2(\mathbf{p}_2)$$

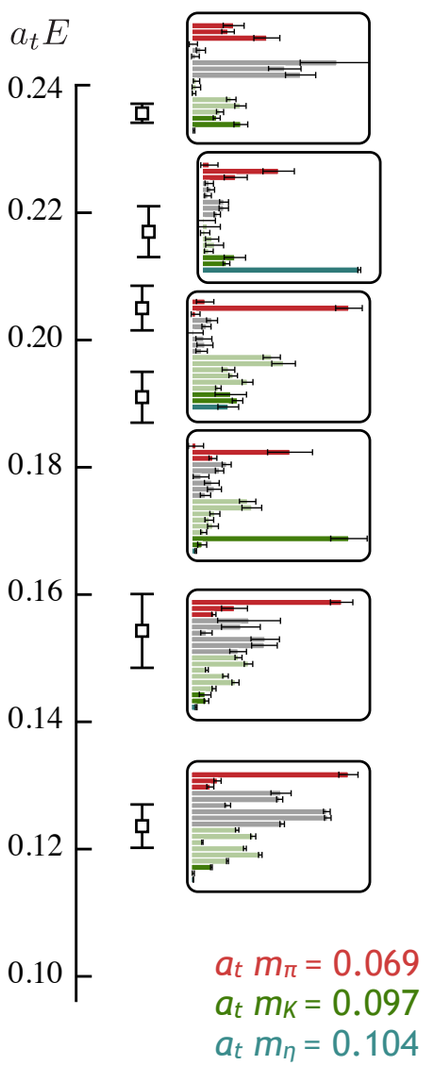
$$\mathbf{p} = \frac{2\pi}{L}[n_x, n_y, n_z]$$

maximum momentum
guided by non-interacting
energies

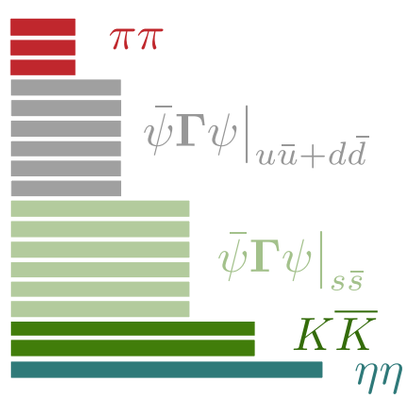
$$\sqrt{m_1^2+\mathbf{p}_1^2}+\sqrt{m_2^2+\mathbf{p}_2^2}$$

solutions of the det equation
when **t** = **0**

[000] A_1^+ 24^3



operator basis



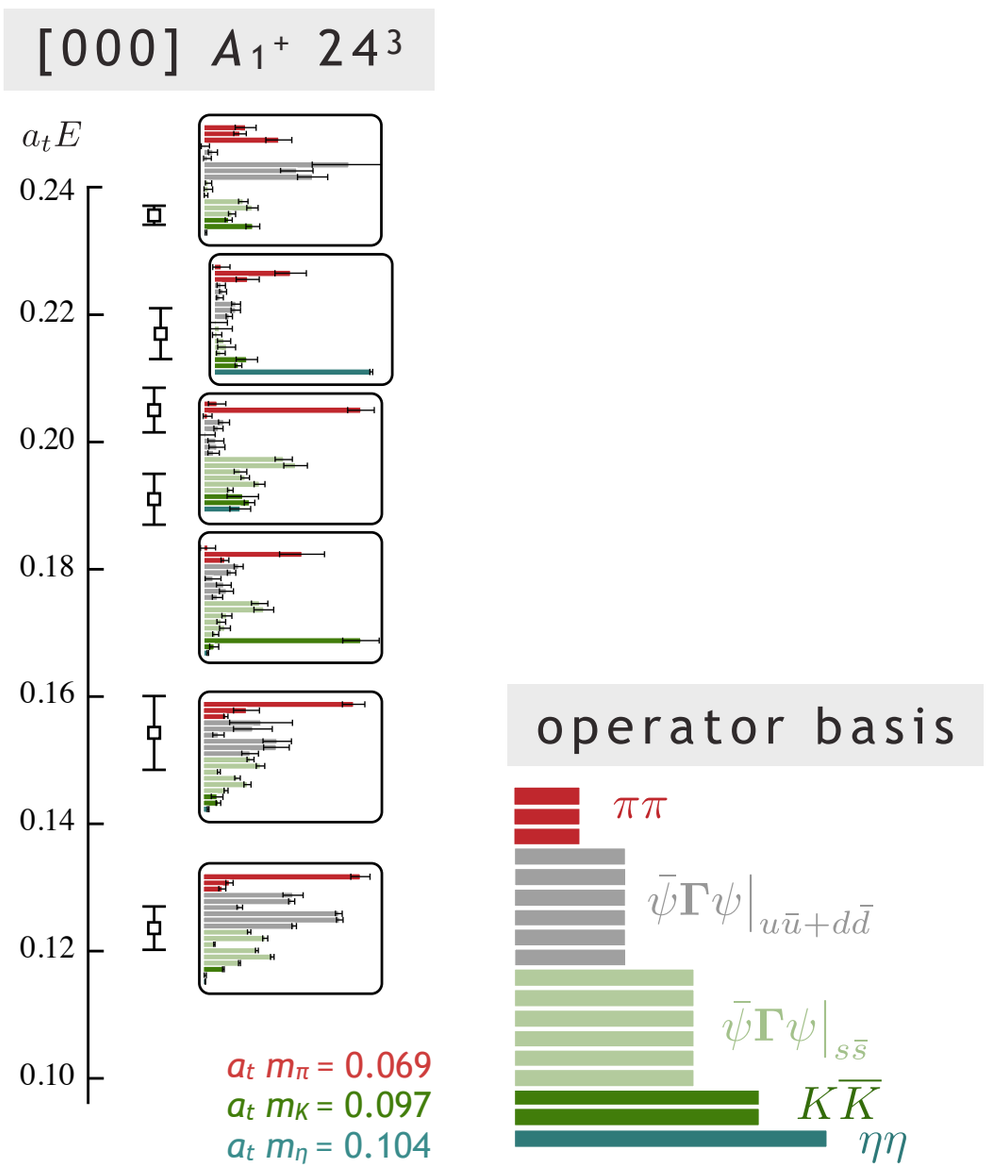
operator basis

operator basis: ‘single-meson’ + ‘meson-meson’

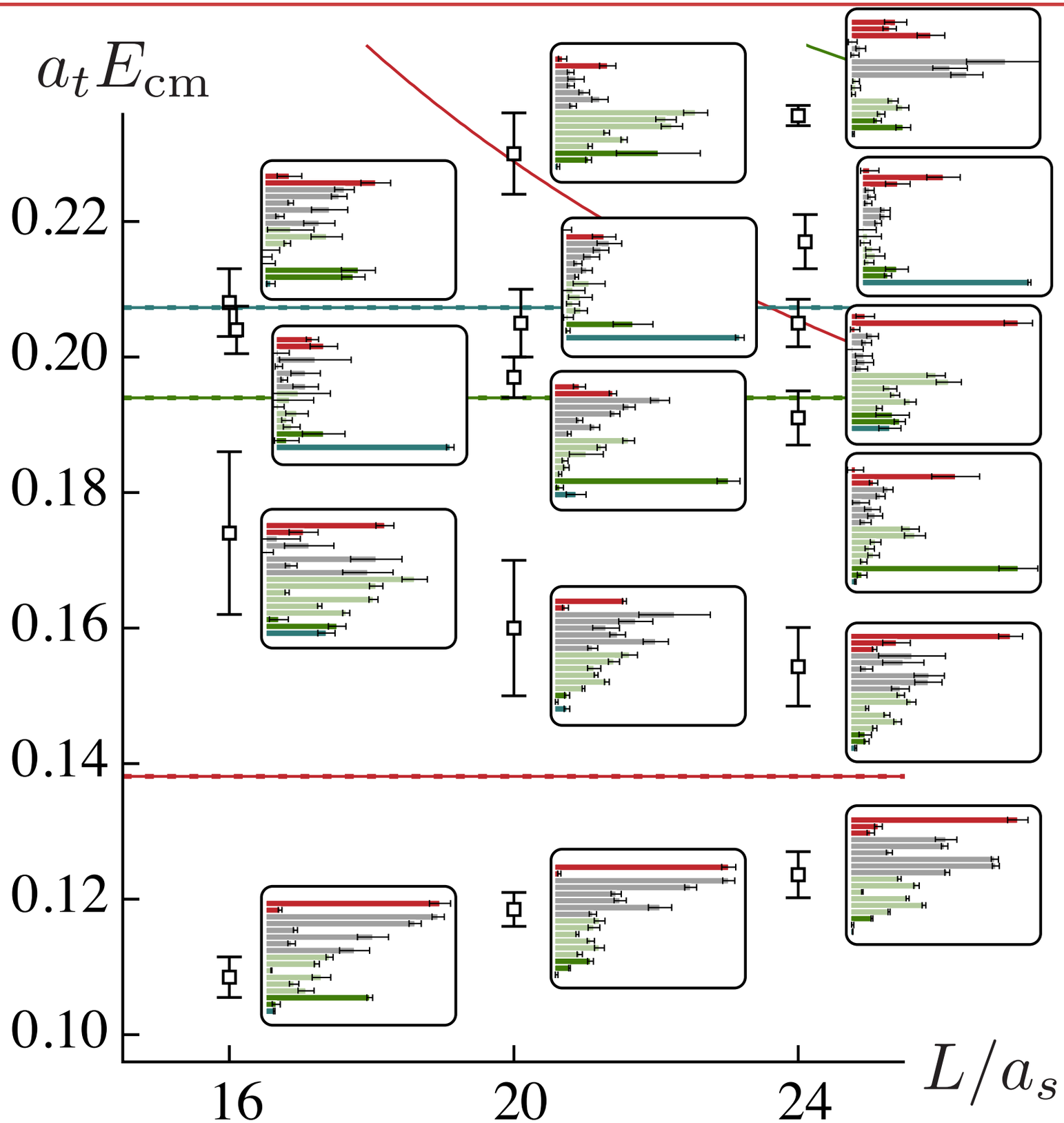
$$\overline{\psi}\mathbf{\Gamma}\psi \qquad \sum_{\hat{\mathbf{p}}_1,\hat{\mathbf{p}}_2} C(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p})\, M_1(\mathbf{p}_1)\, M_2(\mathbf{p}_2)$$

$$\sum_{\mathbf{x}} e^{i\mathbf{p}_1\cdot\mathbf{x}}\,\bar{\psi}_{\mathbf{x}}\mathbf{\Gamma}\psi_{\mathbf{x}}\ \sum_{\mathbf{y}} e^{i\mathbf{p}_2\cdot\mathbf{y}}\,\bar{\psi}_{\mathbf{y}}\mathbf{\Gamma}'\psi_{\mathbf{y}}$$

sampling the whole lattice volume



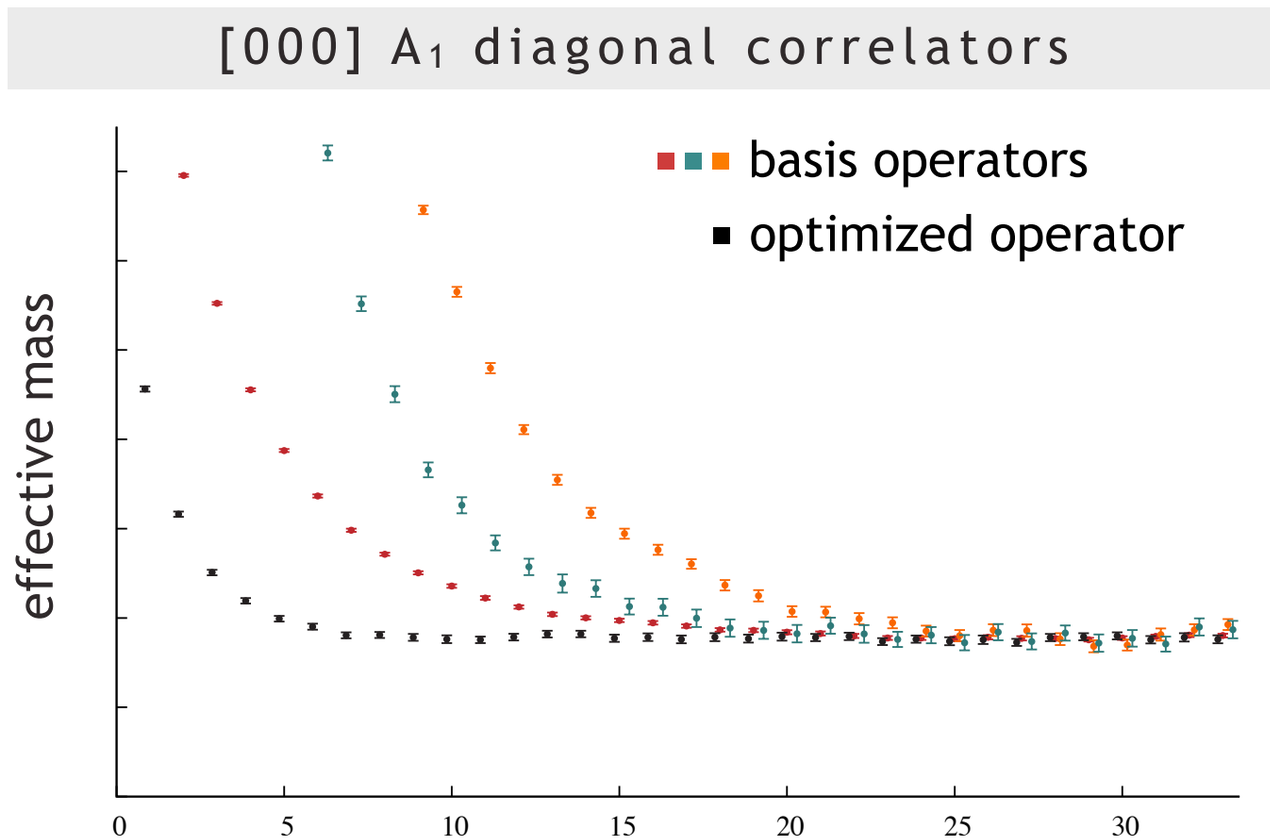
prefer to use
optimized single-meson operators …


$a_t E_{\rm cm}$
0.22
0.20
0.18
0.16
0.14
0.12
0.10
16
20
24
L/a_s

some technical stuff — 'meson-meson'-like operators

then each single-meson operator can be the **variationally optimized** one for that $\boldsymbol{p}$, Λ

three example operators shown, actual basis much larger



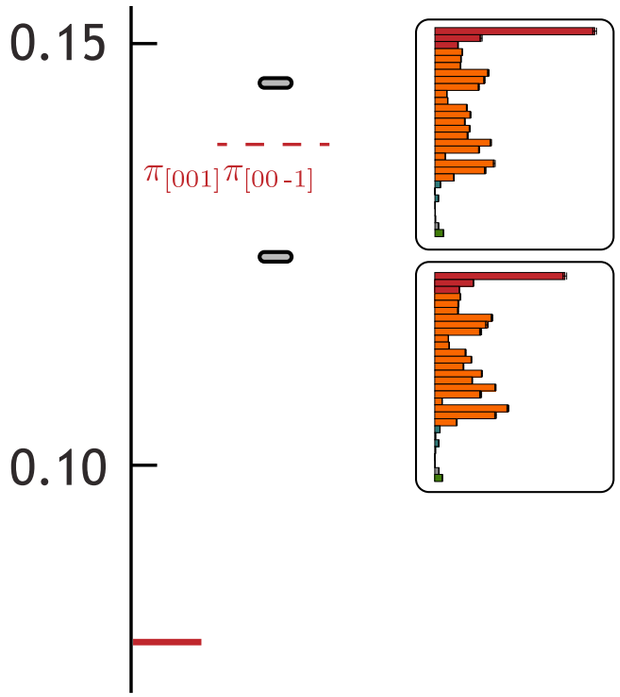
optimized operator saturated by the pion by timeslice 7

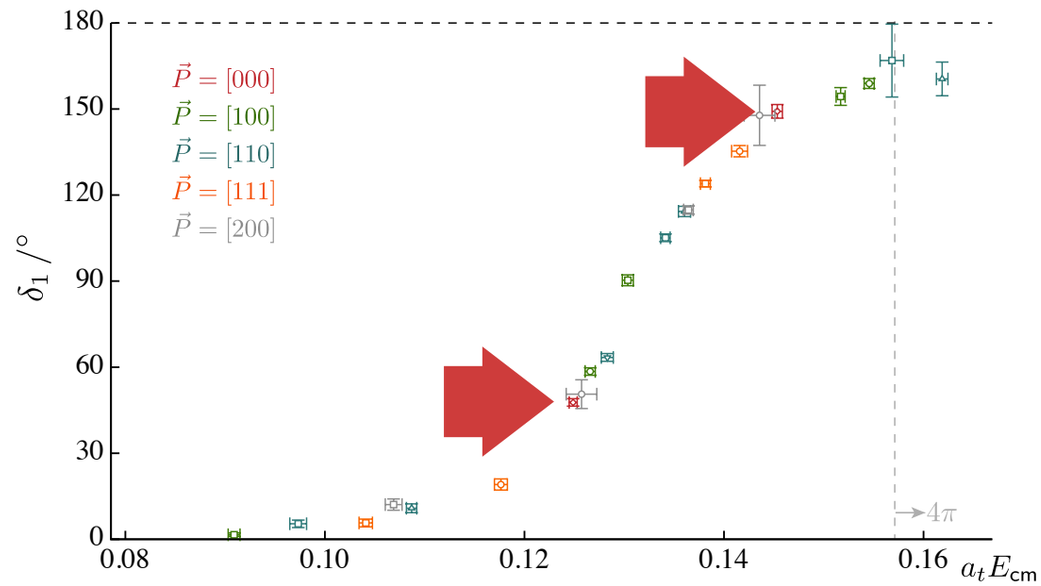
‘dropping’ ops in $\rho \to \pi\pi$

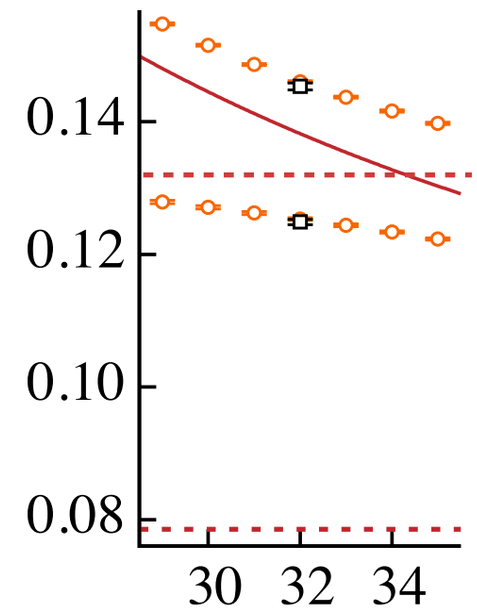
$m_\pi = 0.039$
$m_K = 0.083$
$L \sim 3.8$ fm

what's happening here ?

focus on the lowest two states





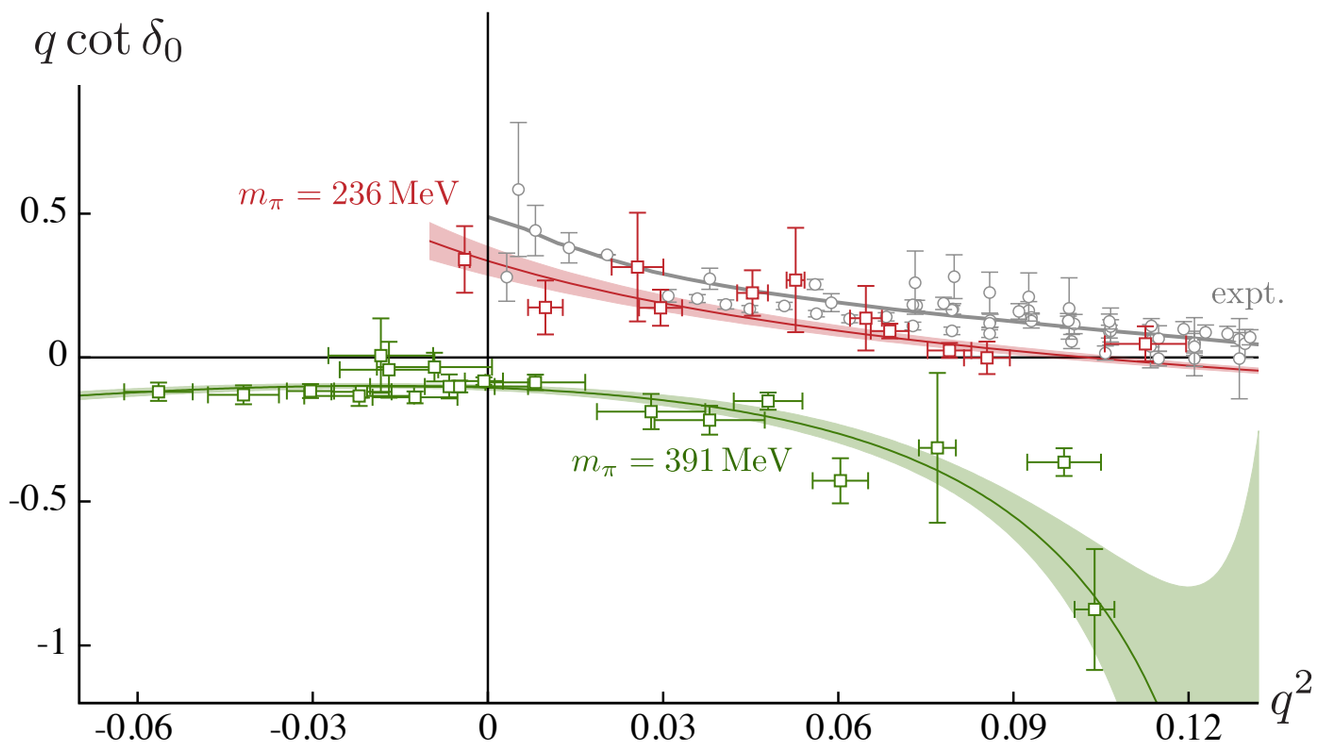


an avoided level crossing

quark mass dependence of σ meson

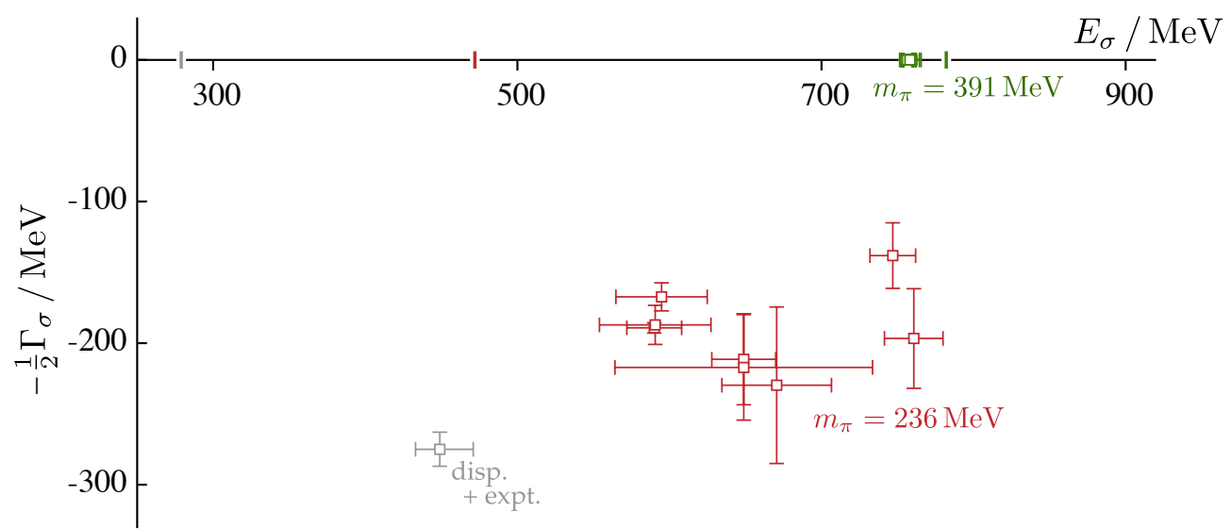
elastic part of the calculation also done on a lighter quark mass

ππ isospin=0 elastic scattering



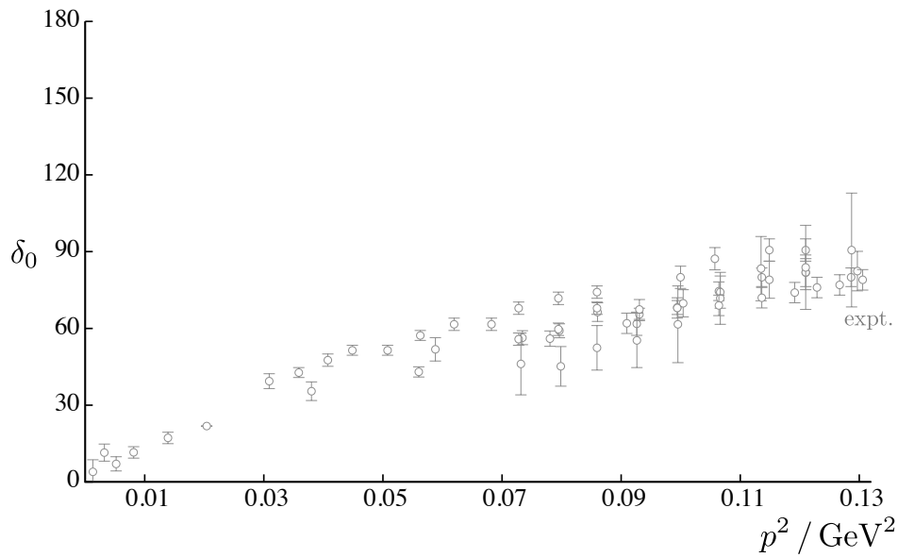
σ pole singularity

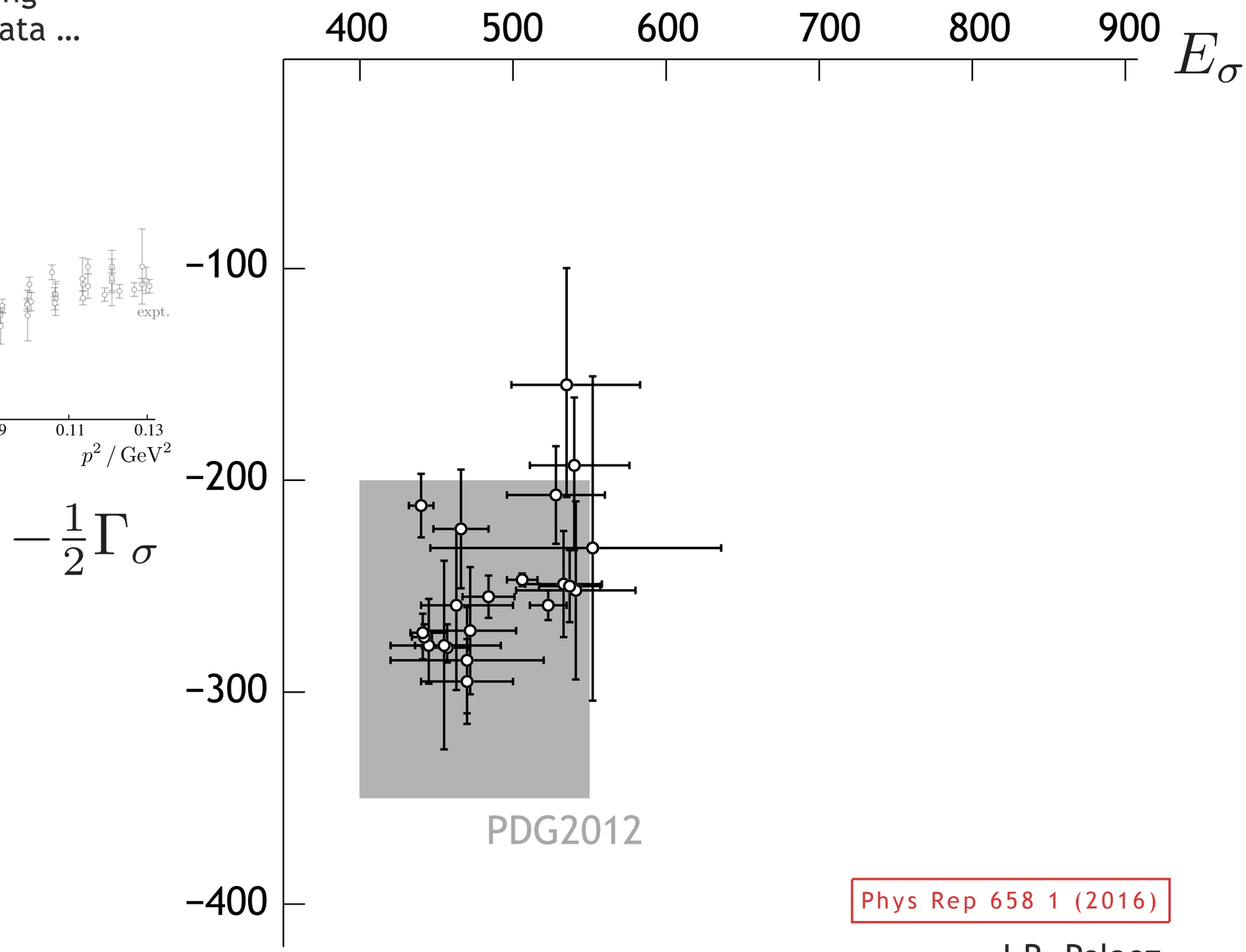
evolves from a bound-state to a broad resonance as the pion mass decreases



σ pole scatter from experimental phase-shift data

actually same problem using experimental scattering data …

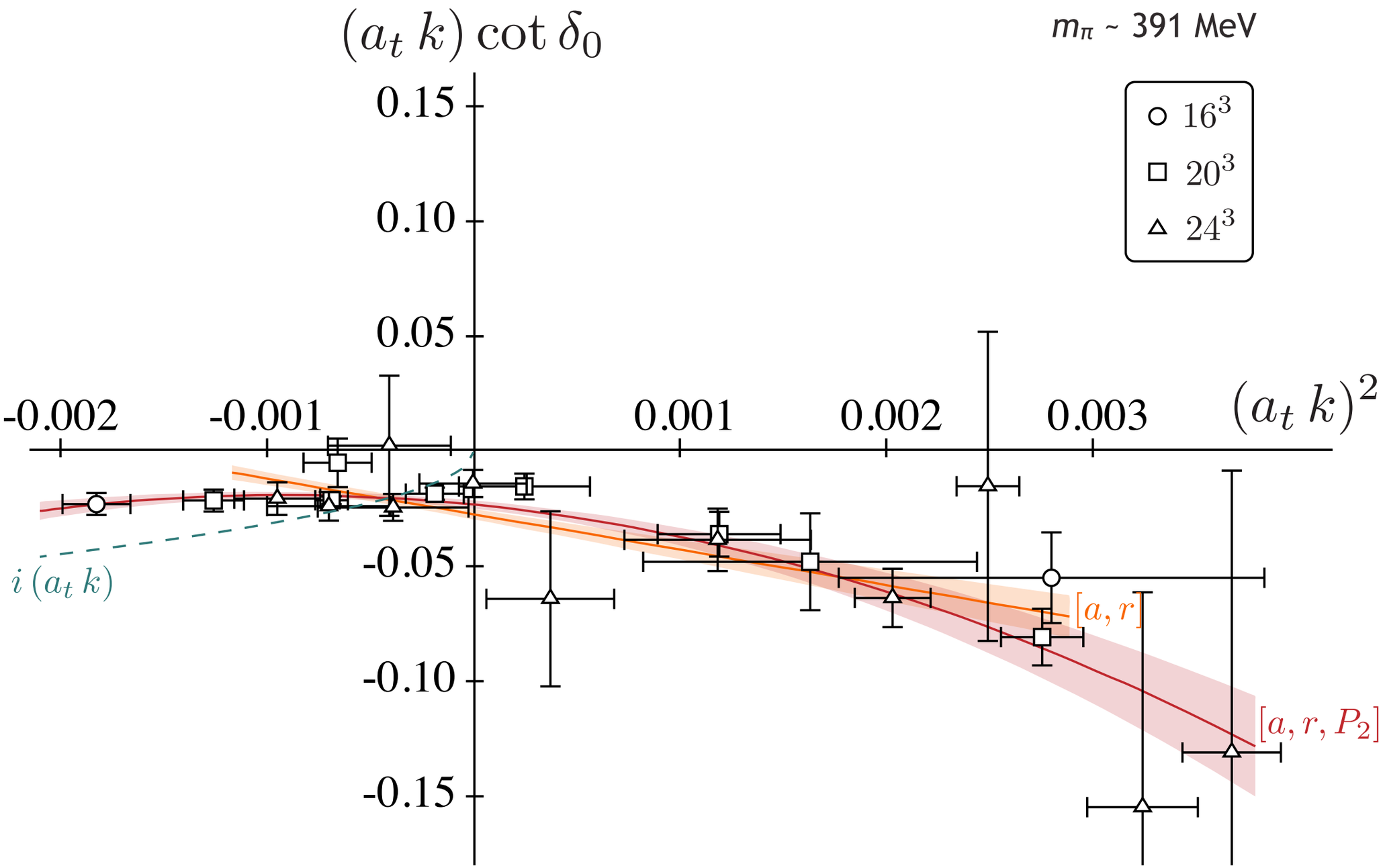




Phys Rep 658 1 (2016)

J.R. Pelaez

σ bound state — elastic and subthreshold region



Weinberg compositeness criterion — σ at m_π~391 MeV

relates scattering length, effective range
to compositeness measure, Z

$$a = -2\frac{1-Z}{2-Z}\frac{1}{\sqrt{m_\pi \epsilon}} + \dots$$
$$r = -2\frac{Z}{1-Z}\frac{1}{\sqrt{m_\pi \epsilon}} + \dots$$

with corrections whose size is set by
the range of the interaction

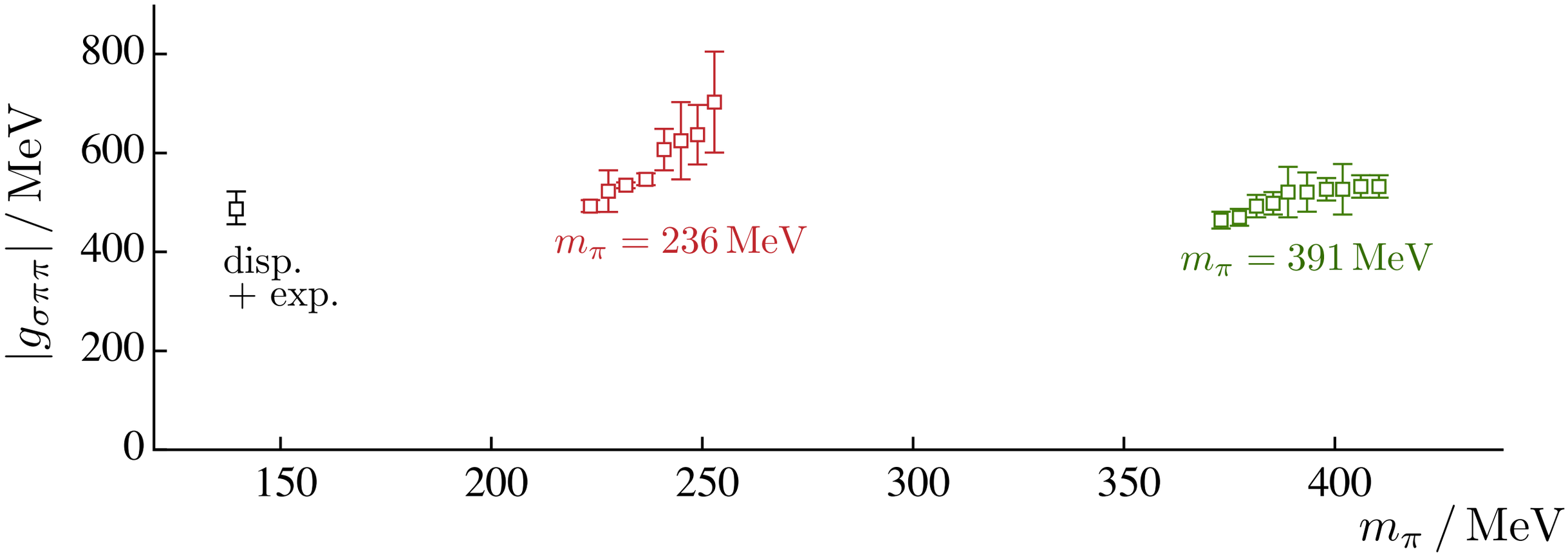
Z=1 compact state
Z=0 a $\pi\pi$ molecule

$$\epsilon = 37(5)\,\mathrm{MeV}$$

$$a = -0.0071(11)\,\mathrm{MeV}^{-1} = -1.4(2)\,\mathrm{fm}$$
$$r = -0.0041(14)\,\mathrm{MeV}^{-1} = -0.8(3)\,\mathrm{fm}$$

$$\boxed{Z \sim 0.3(1)}$$

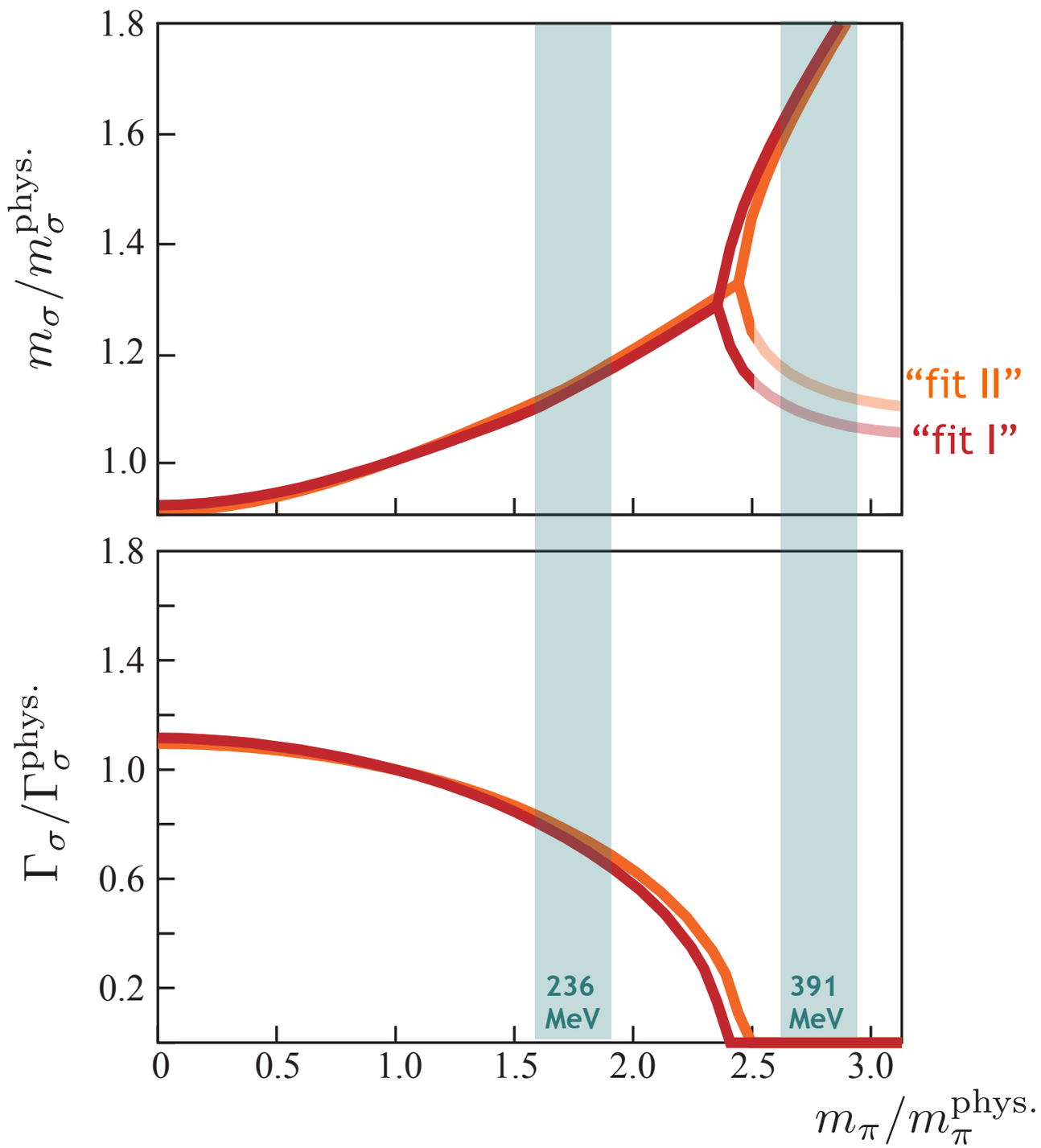
$\sigma \to \pi\pi$ coupling from the pole residue



σ pole in unitarized chiral perturbation theory

J.R. Pelaez ...

PRD81 054035 (2010)



resonance becomes a **virtual bound state** near m_π~350 MeV ...

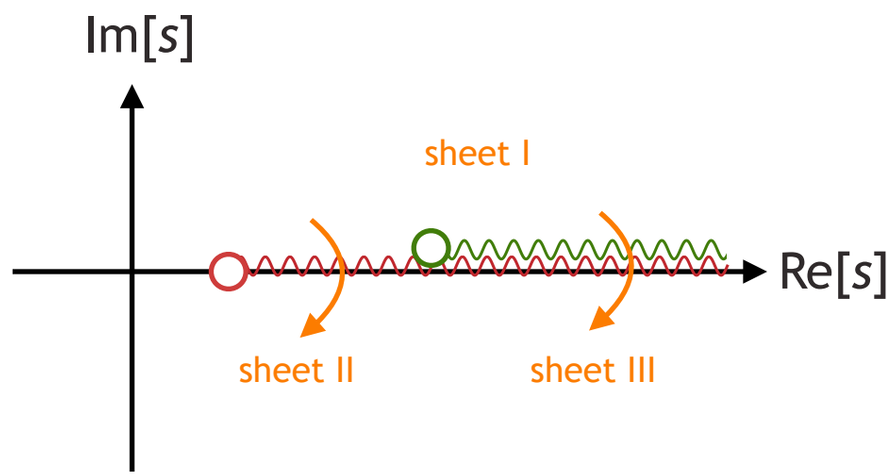
... then a **bound state** near m_π~420 MeV

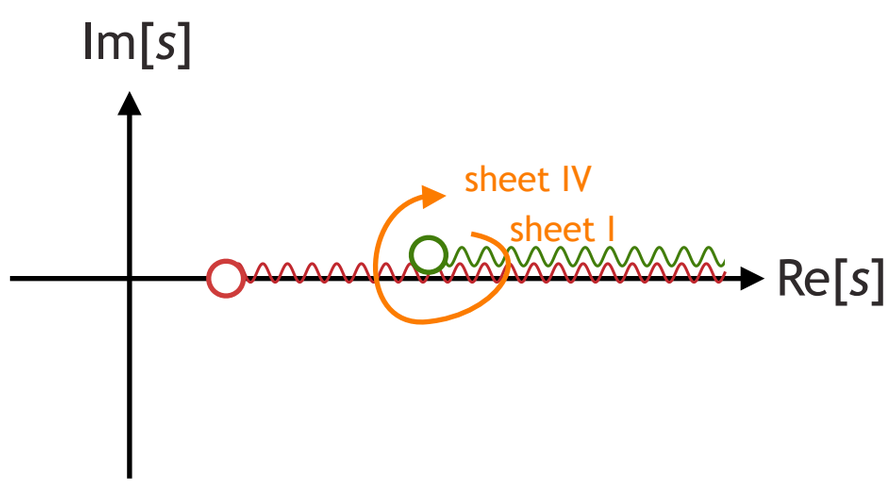
"the exact m_π value when this happens is not very reliable"

coupled-channel Riemann sheet structure

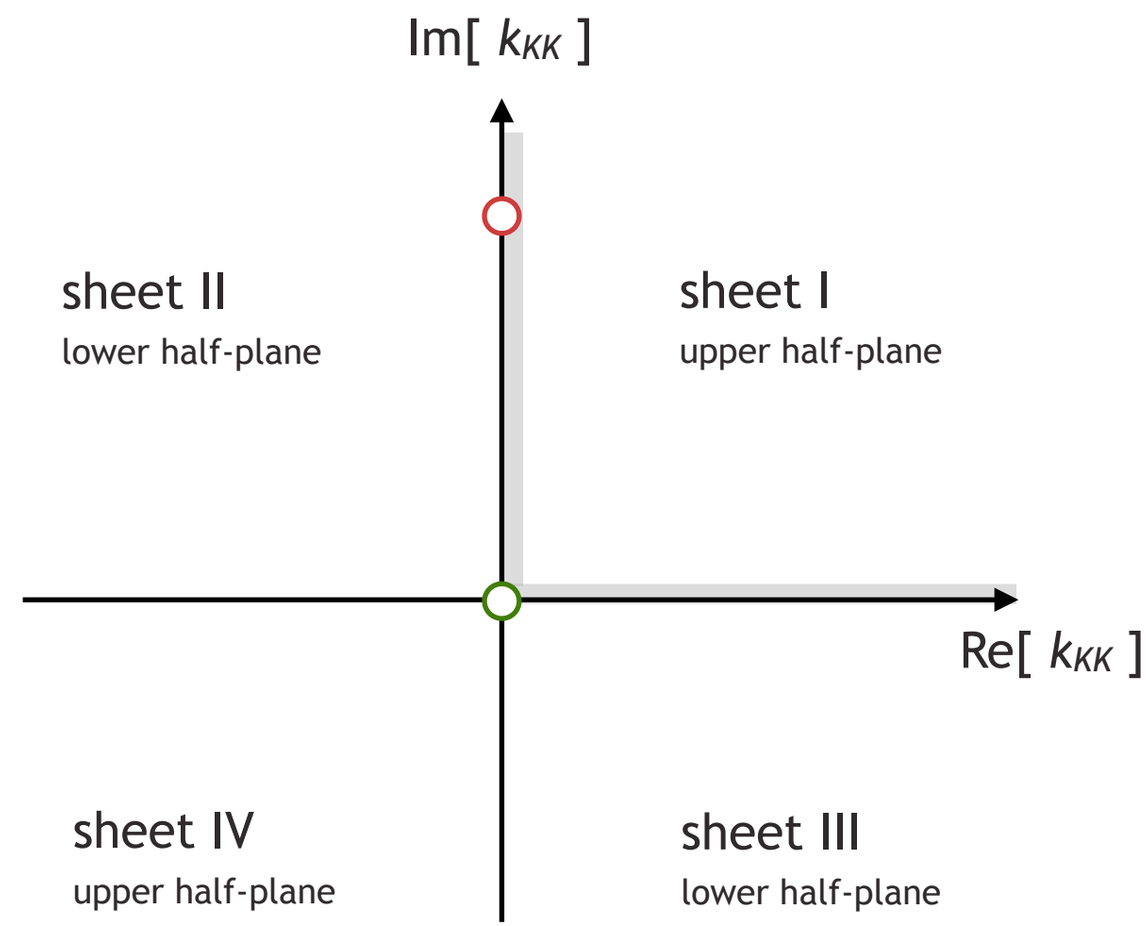
for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

e.g. two channels



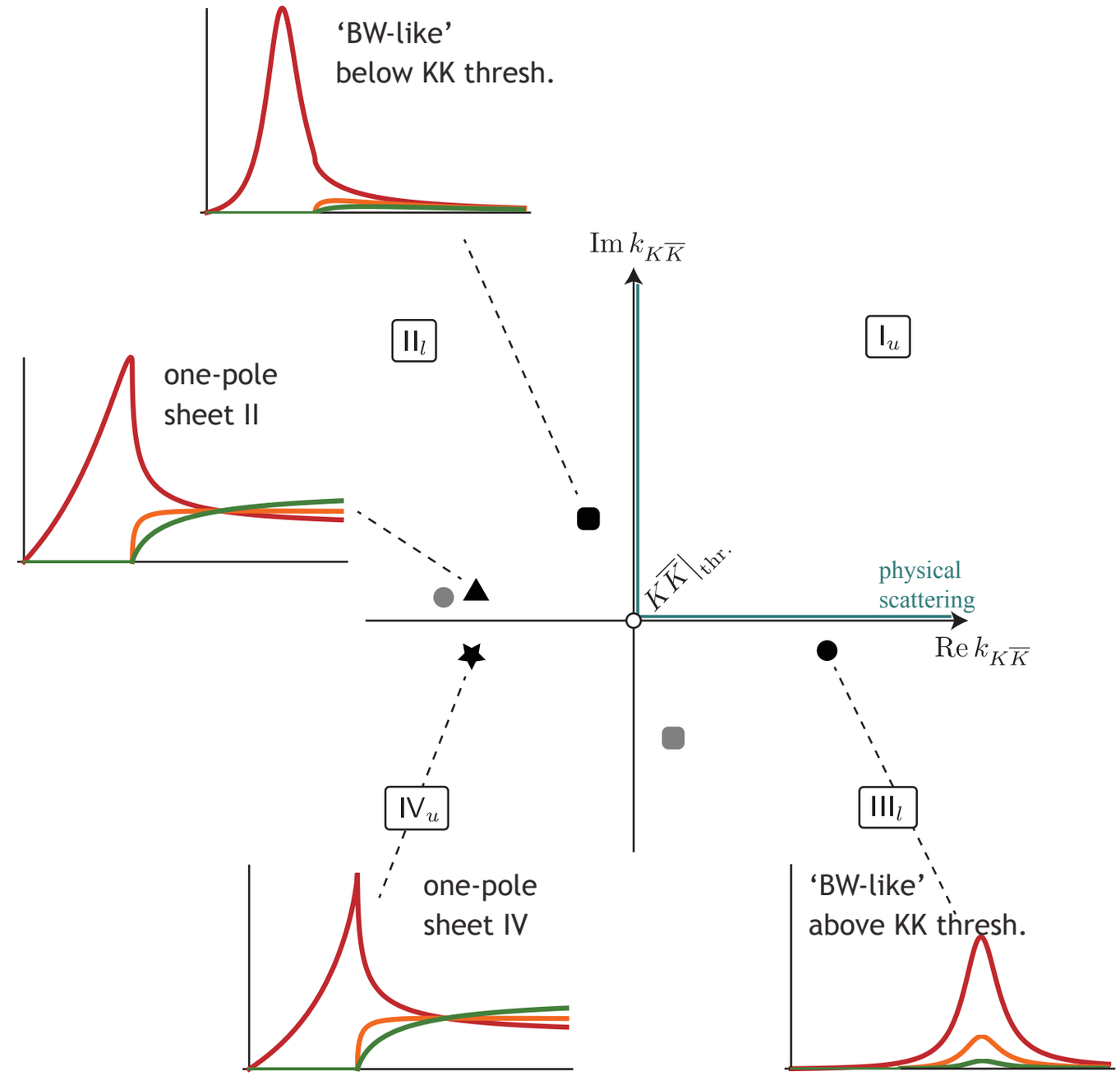


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−



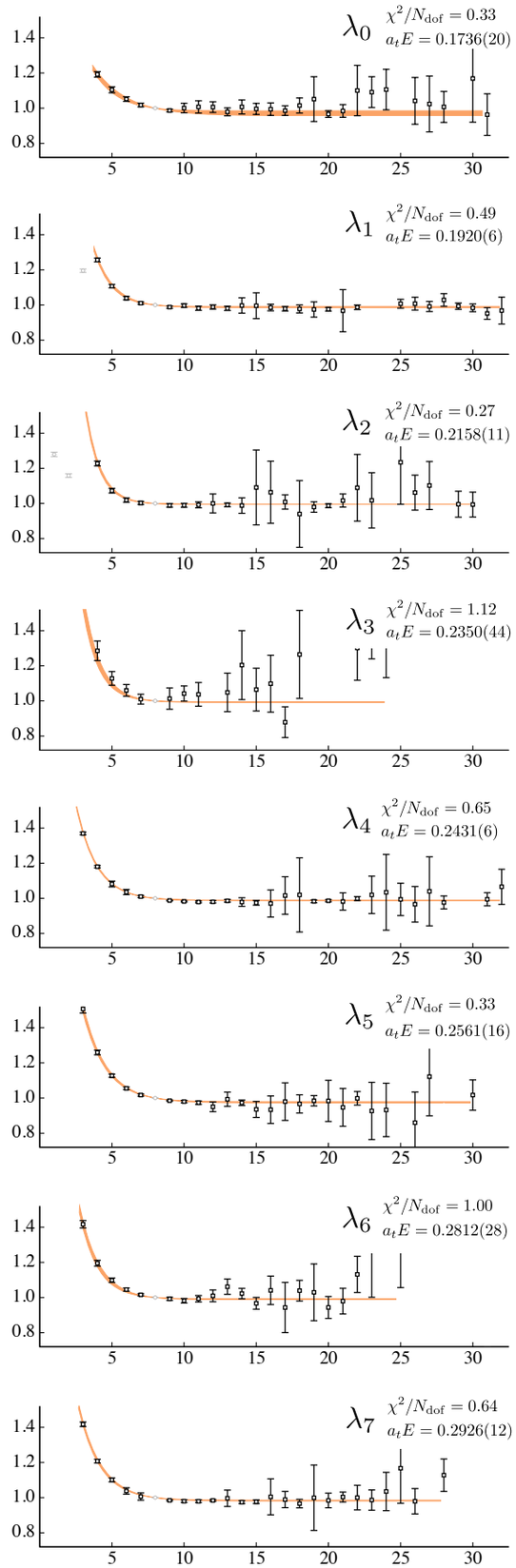
poles on different sheets

the distribution of poles across sheets
does have visible effects

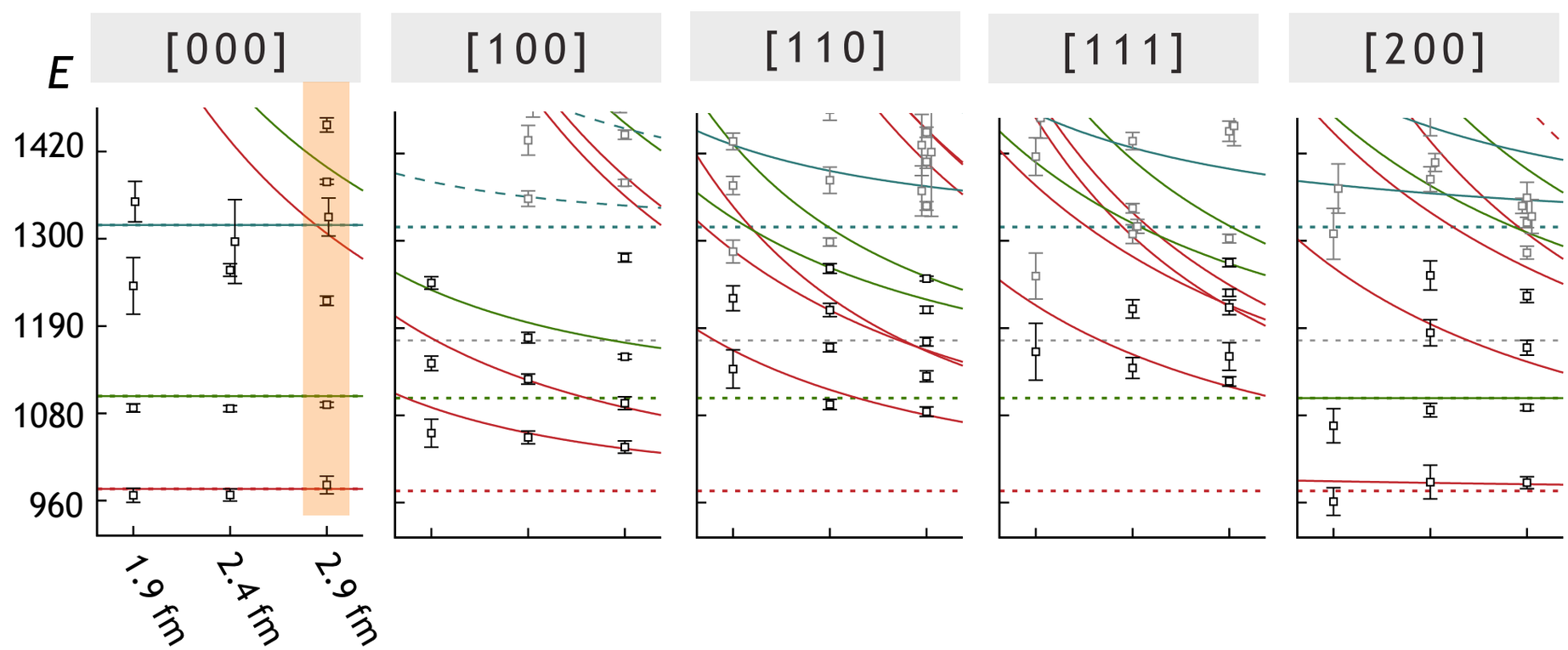


and there are (model-dependent) suggestions that
the distribution is related to the resonance structure

principal correlators — $\pi\eta/K\bar{K}$

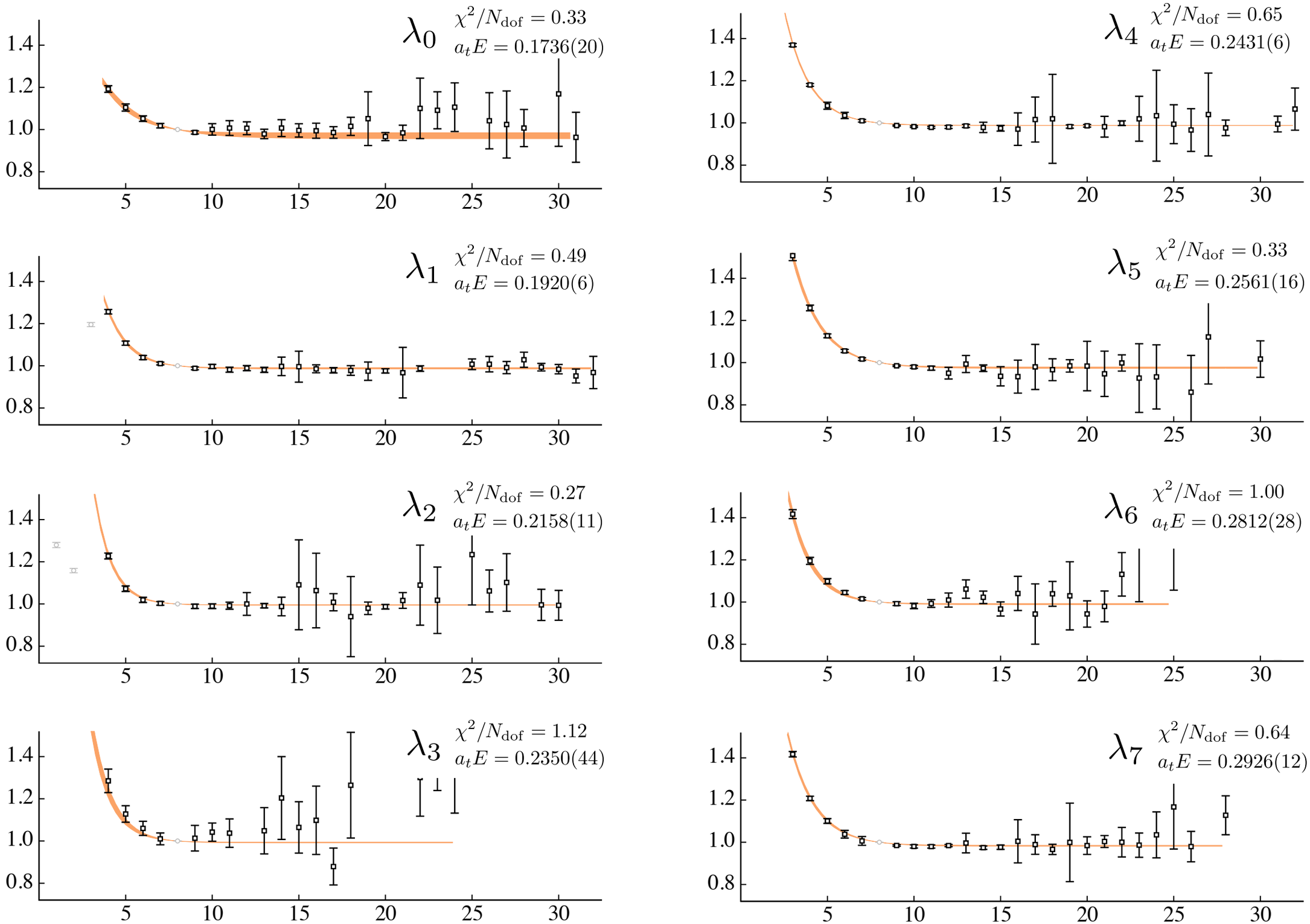


$$\lambda_{\mathfrak{n}}(t) \cdot e^{E_{\mathfrak{n}}(t-t_0)}$$

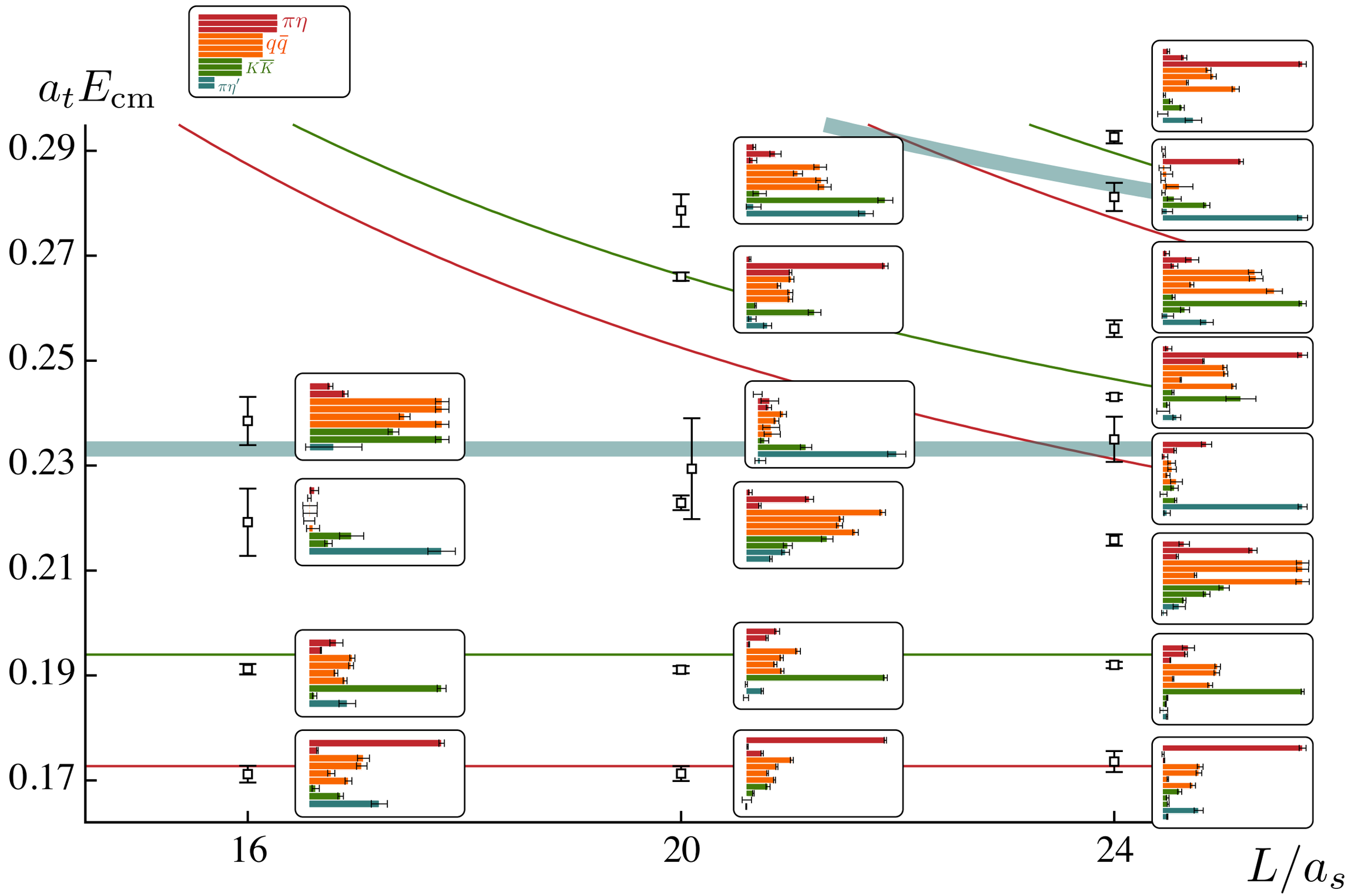


principal correlators — $\pi\eta/K\bar{K}$

$\lambda_n(t) \cdot e^{E_n(t-t_0)}$

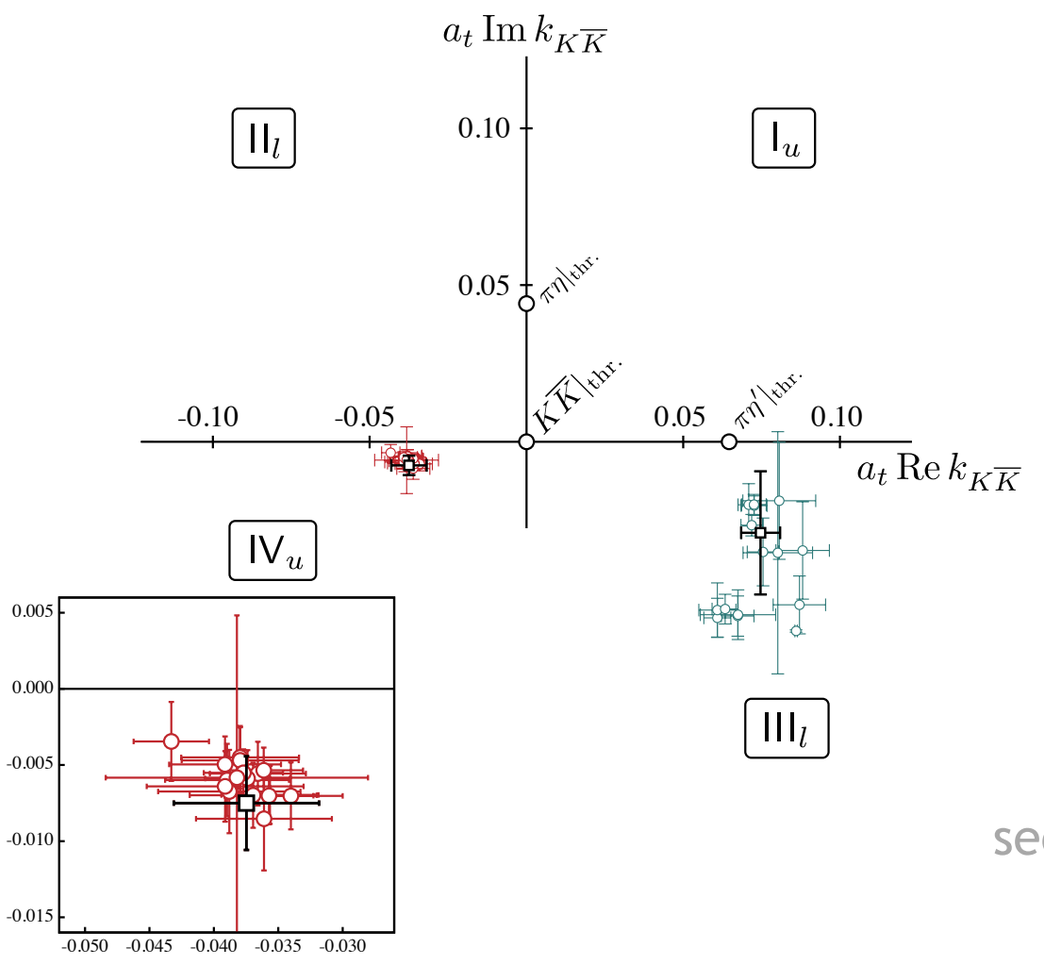


operator overlaps — $\pi\eta/K\bar{K}$

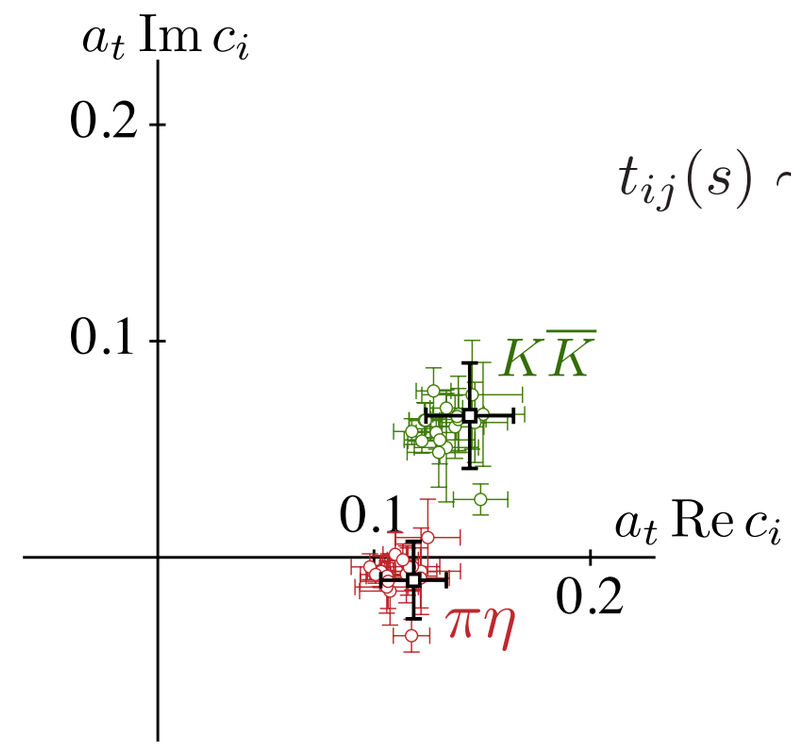


a_0 resonance at m_π~391 MeV

Riemann sheet structure



pole couplings



$$t_{ij}(s) \sim \frac{c_i\, c_j}{s_0 - s}$$

$$\left| \frac{c_{K\bar{K}}}{c_{\pi\eta}} \right| = 1.3(4)$$

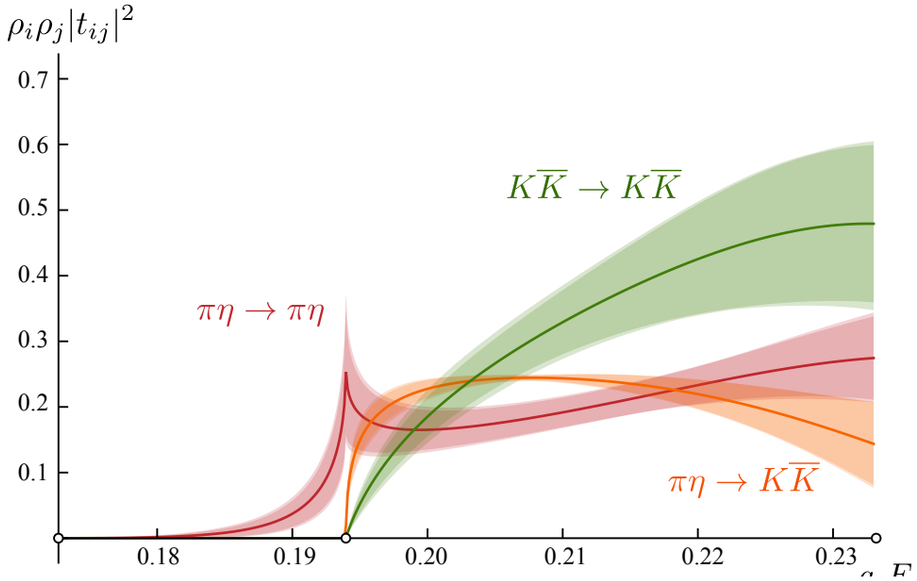
see also

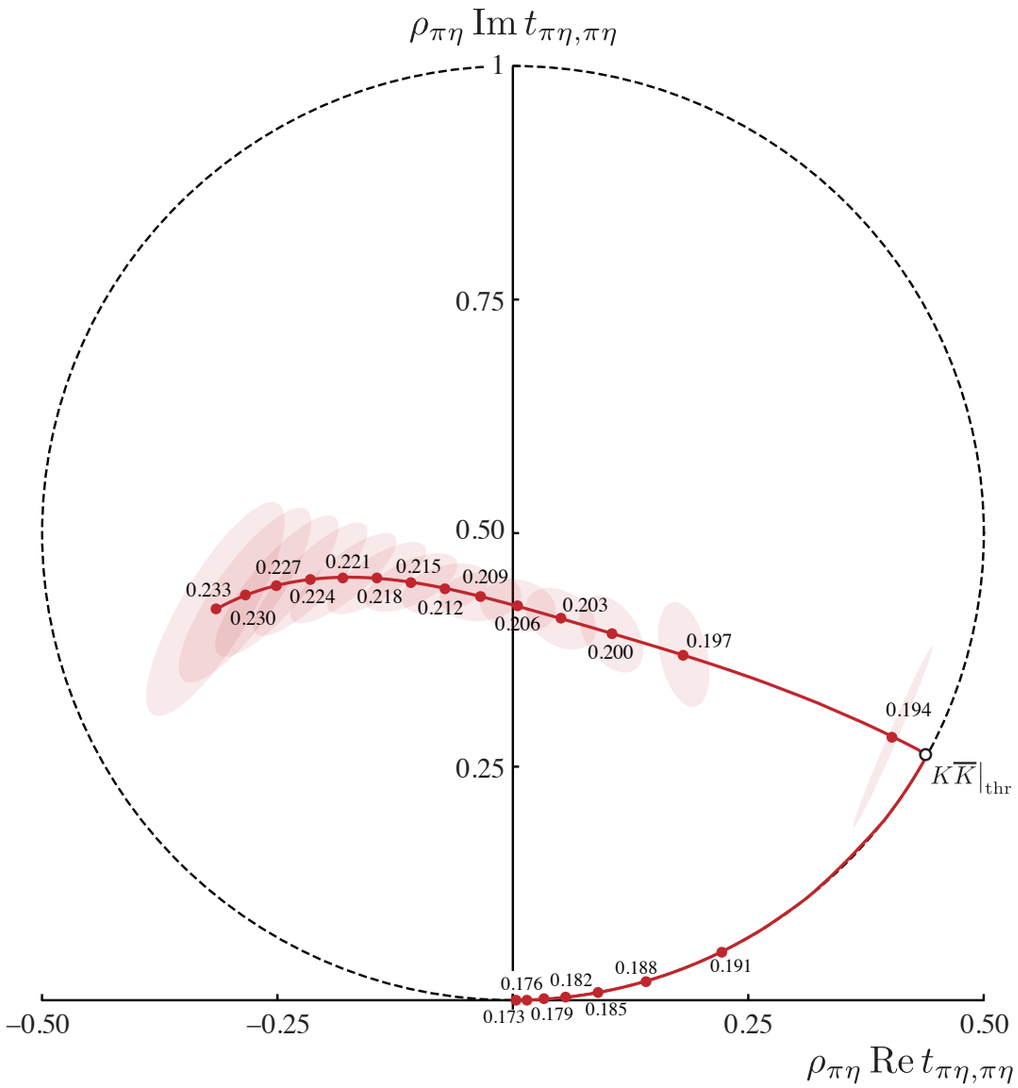
PHYSICAL REVIEW D **95**, 054004 (2017)

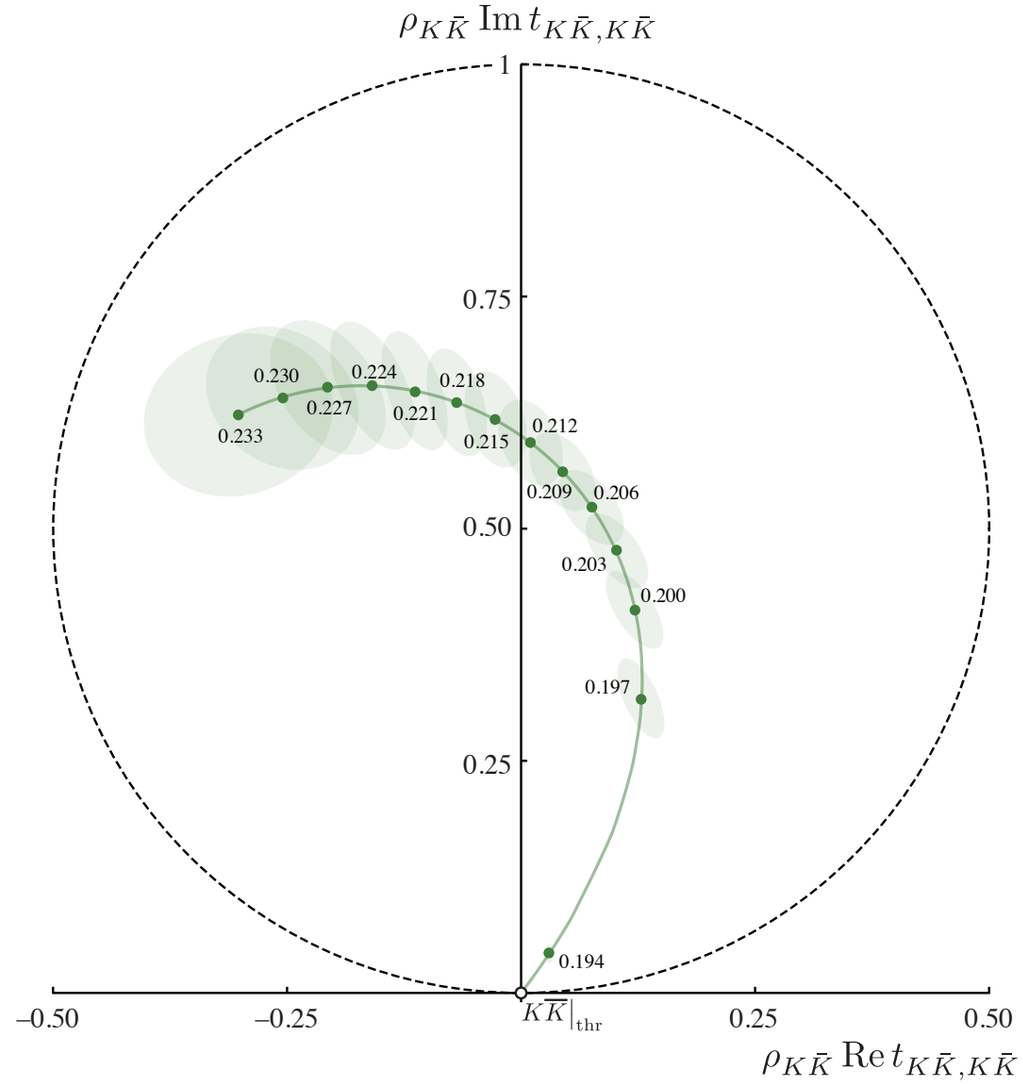
Chiral study of the $a_0(980)$ resonance and $\pi\eta$ scattering phase shifts in light of a recent lattice simulation

Zhi-Hui Guo,[1,2] Liuming Liu,[2] Ulf-G. Meißner,[2,3] J. A. Oller,[4] and A. Rusetsky[2]

a_0 resonance at m_π~391 MeV





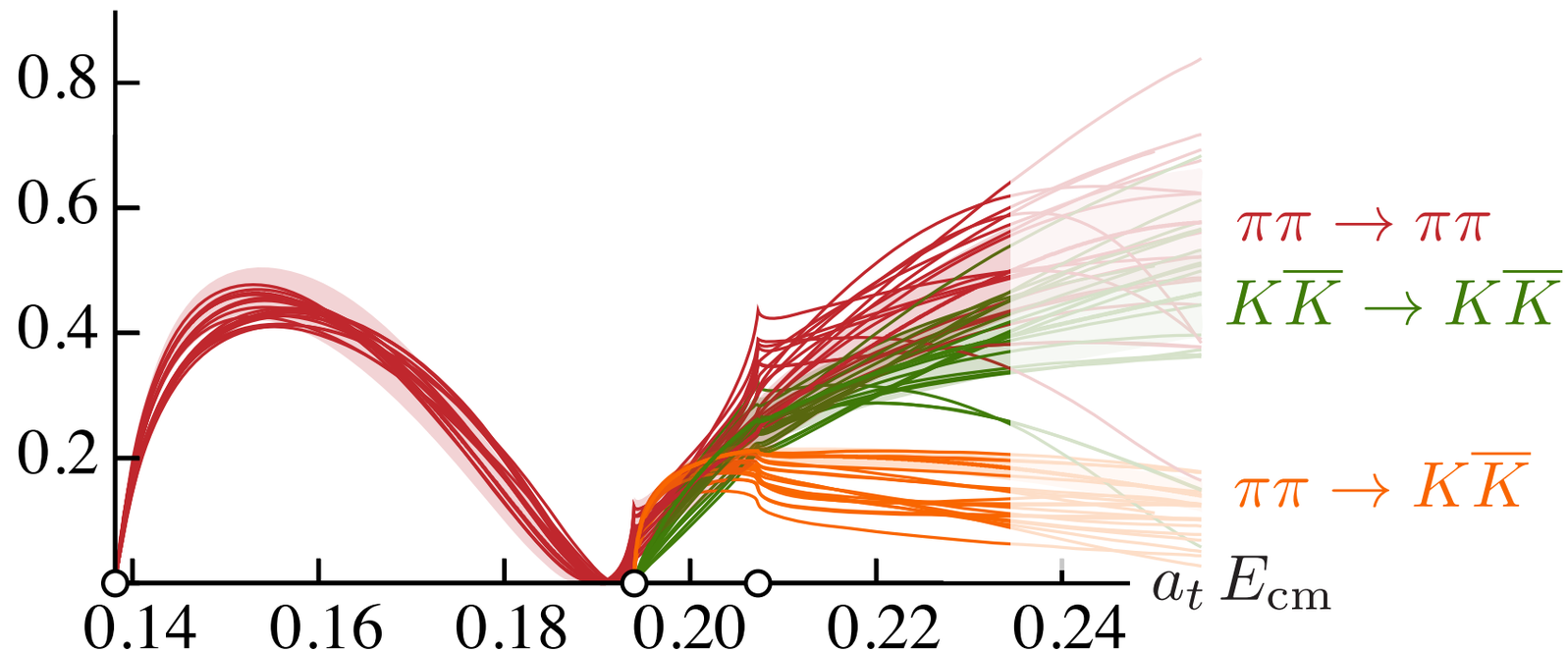


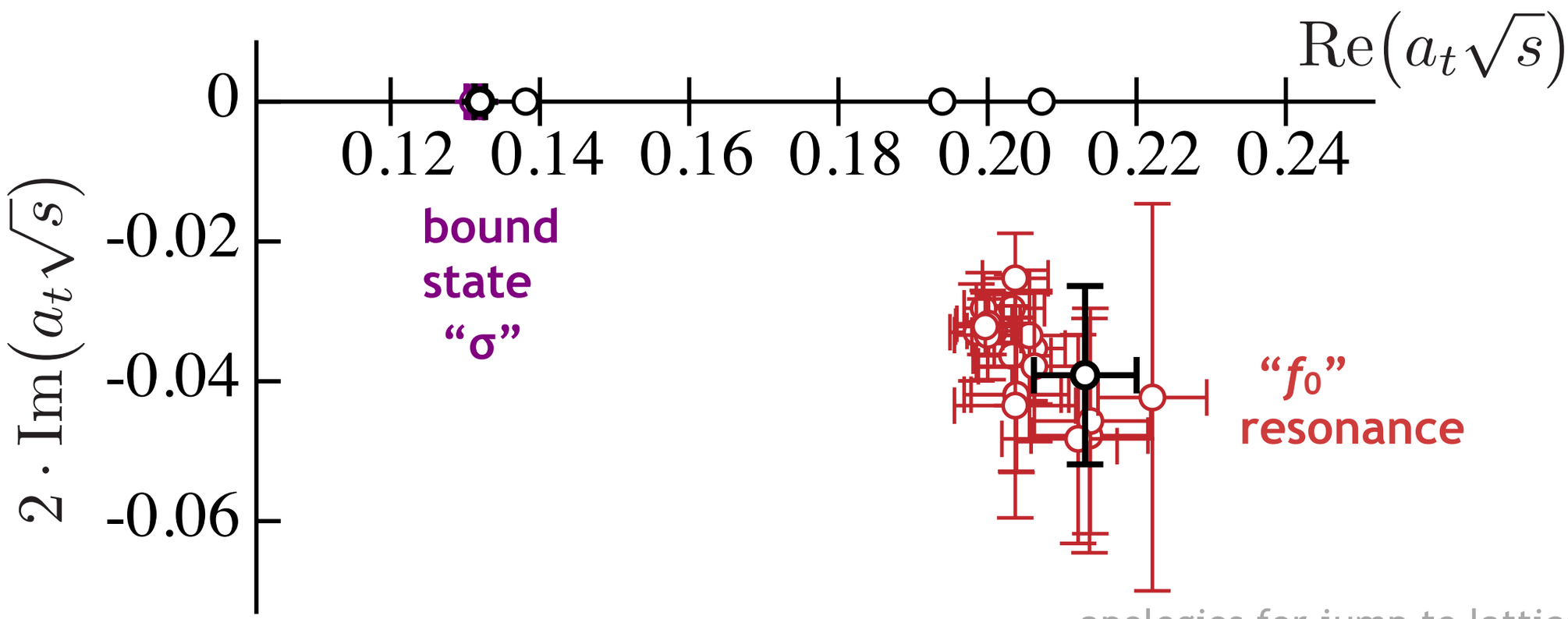
unique to that one parameterization ?

20 parameterization variations
(many more considered)

K^{-1} as matrix of polynomials,
K as matrix of polynomials,
K as pole plus matrix of polynomials,
simple versus Chew-Mandelstam phase-space …

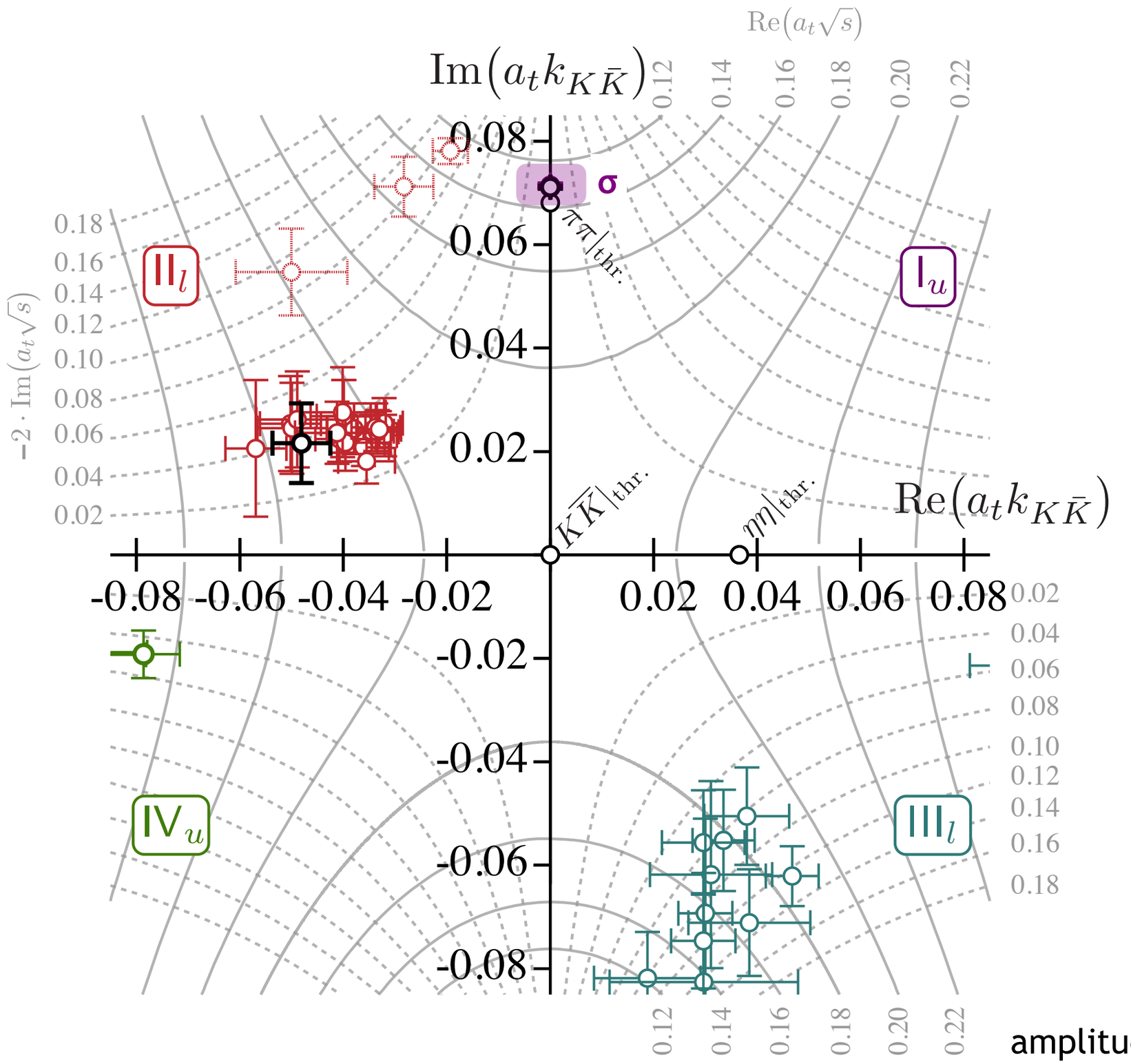
keep choices that can describe
spectra with good χ^2





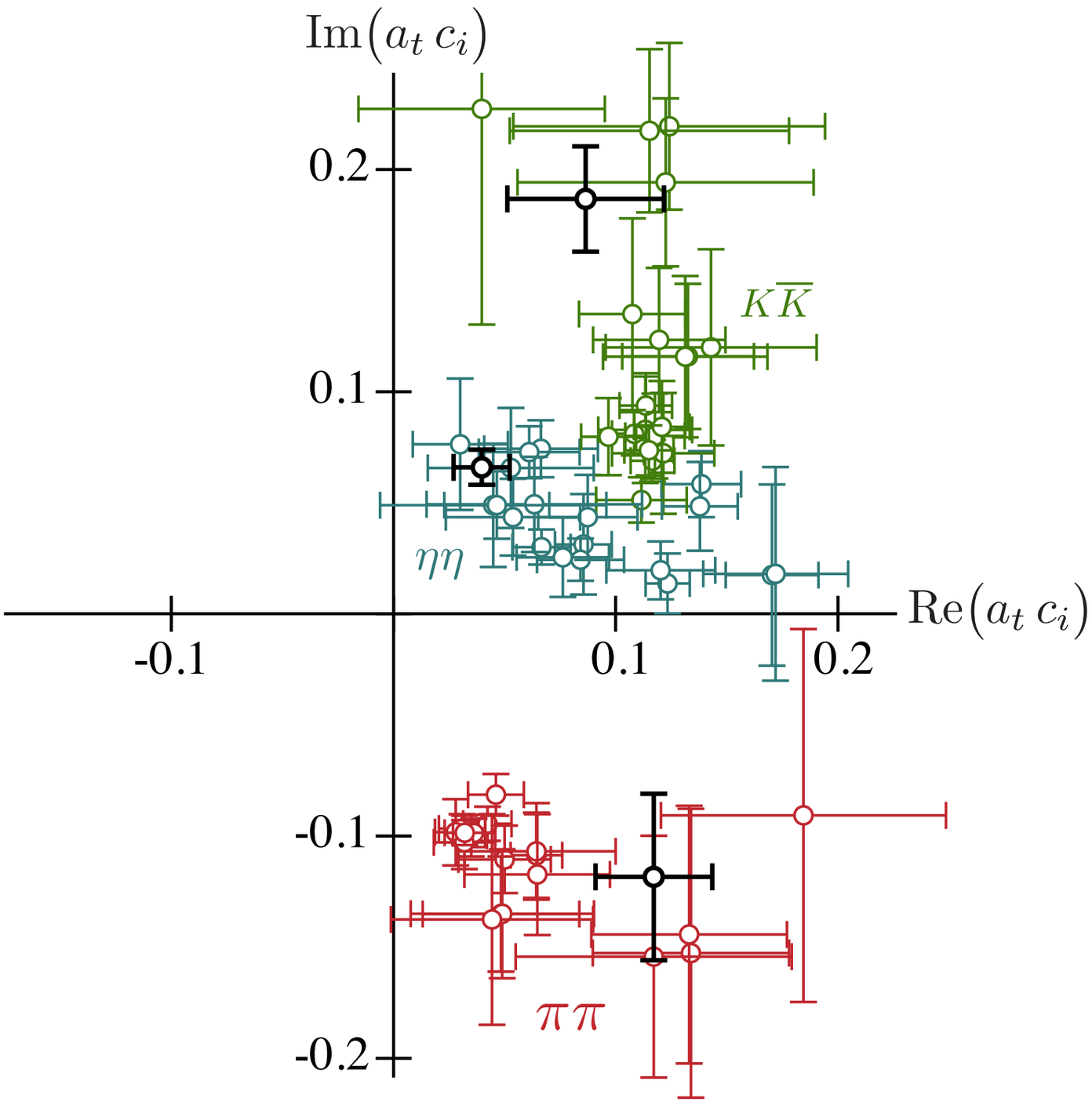
apologies for jump to lattice units

sheet structure

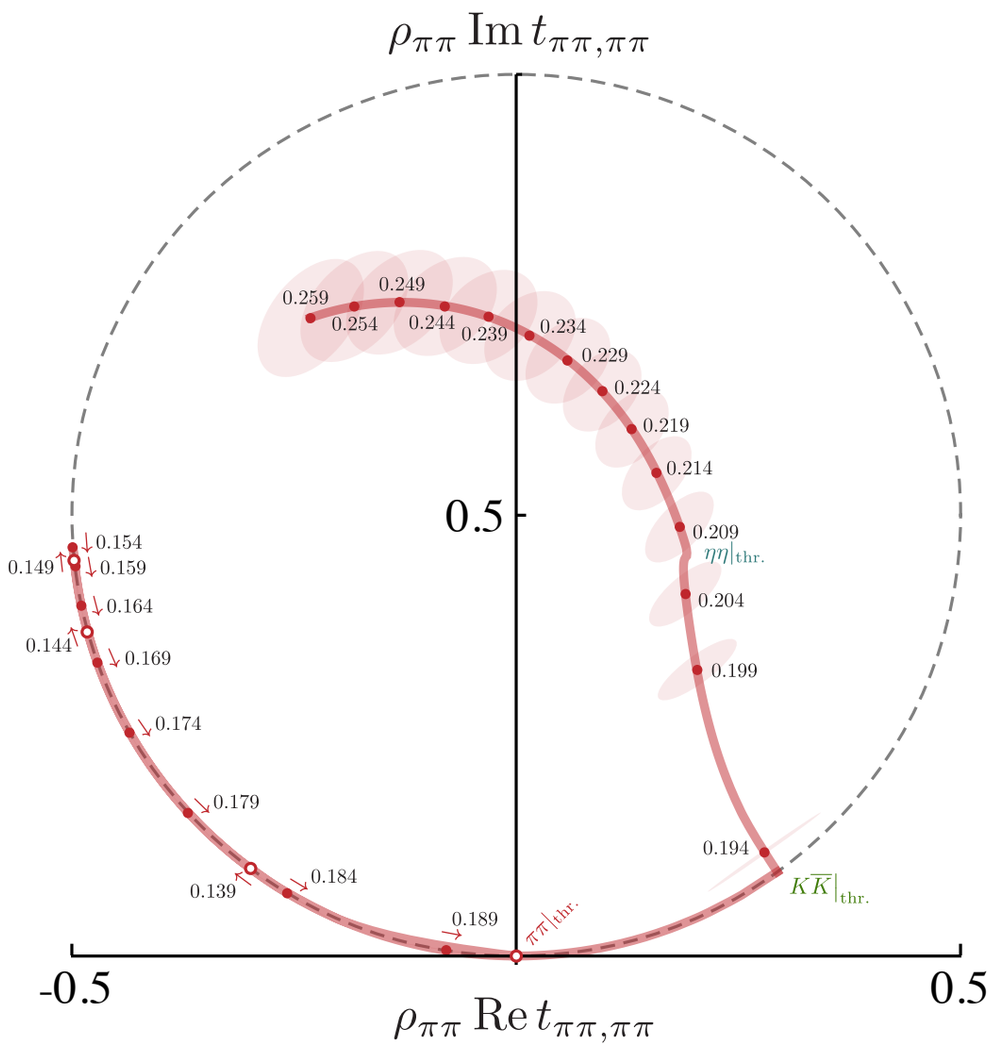


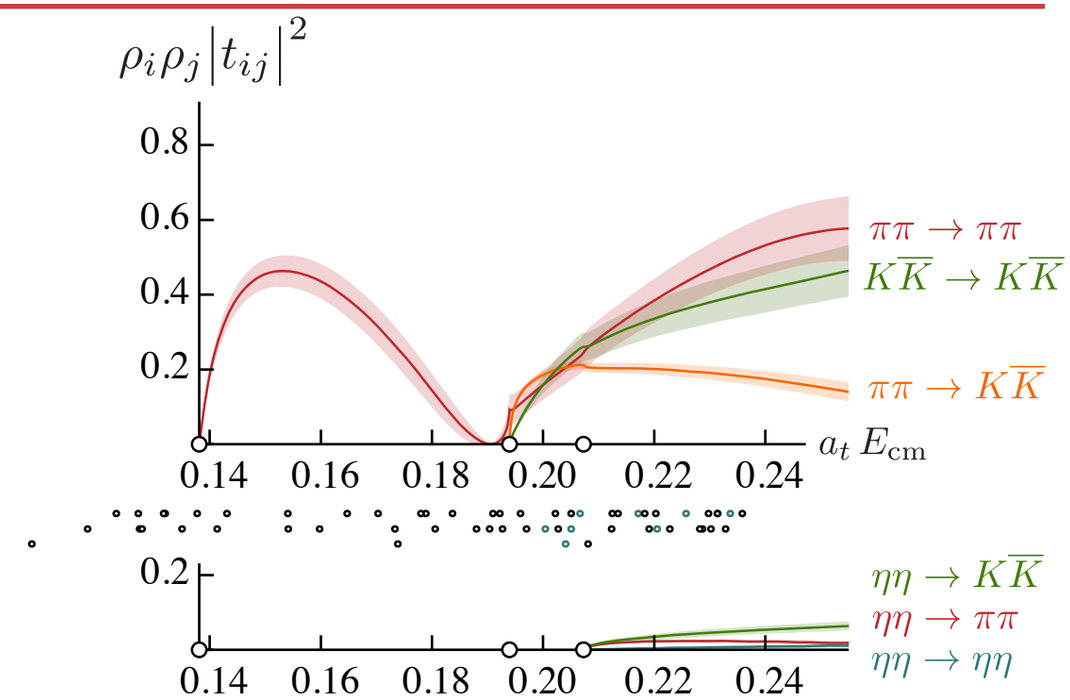
amplitude in the vicinity of the $K\bar{K}$ threshold appears to be dominated by a sheet II pole

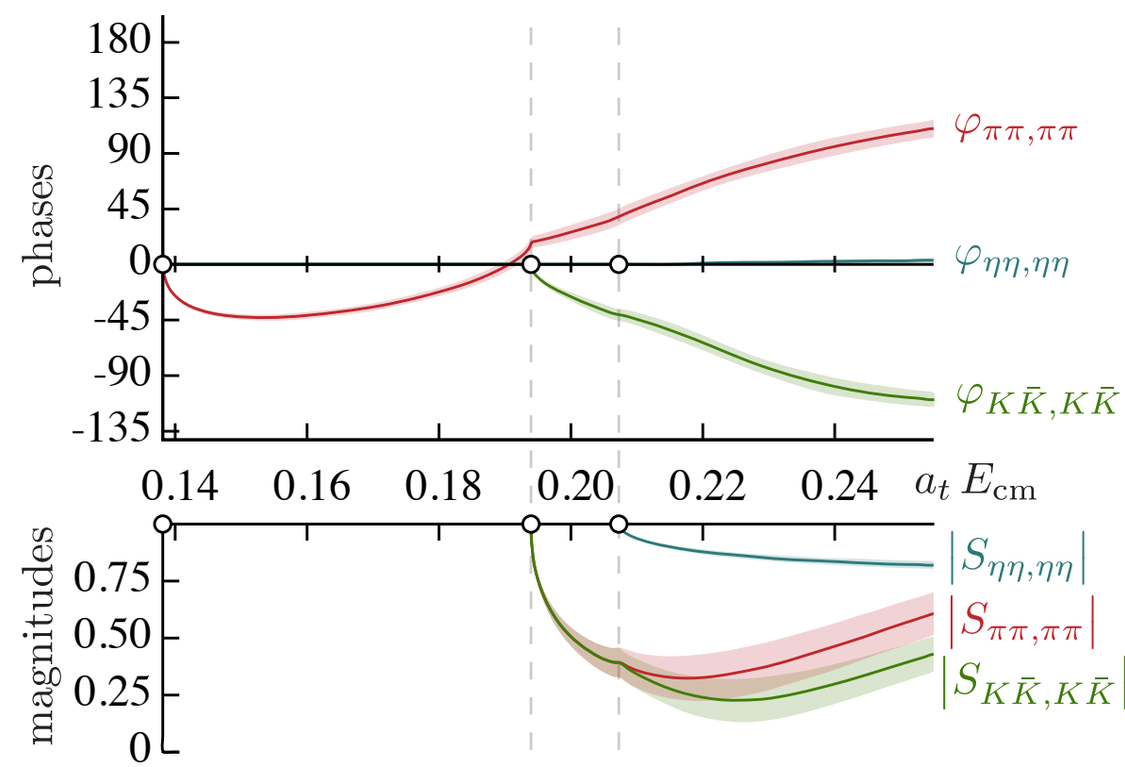
$\pi\pi/K\bar{K}$ extras — f_0 couplings



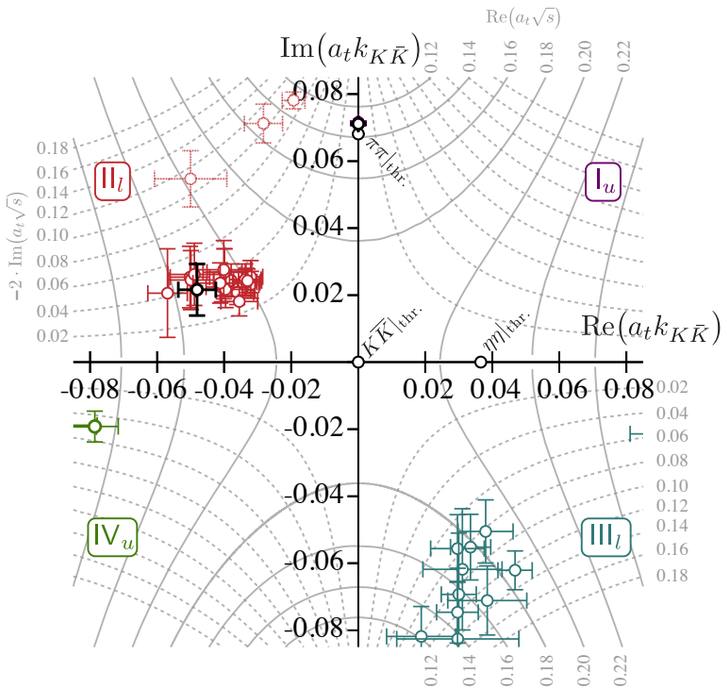
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering





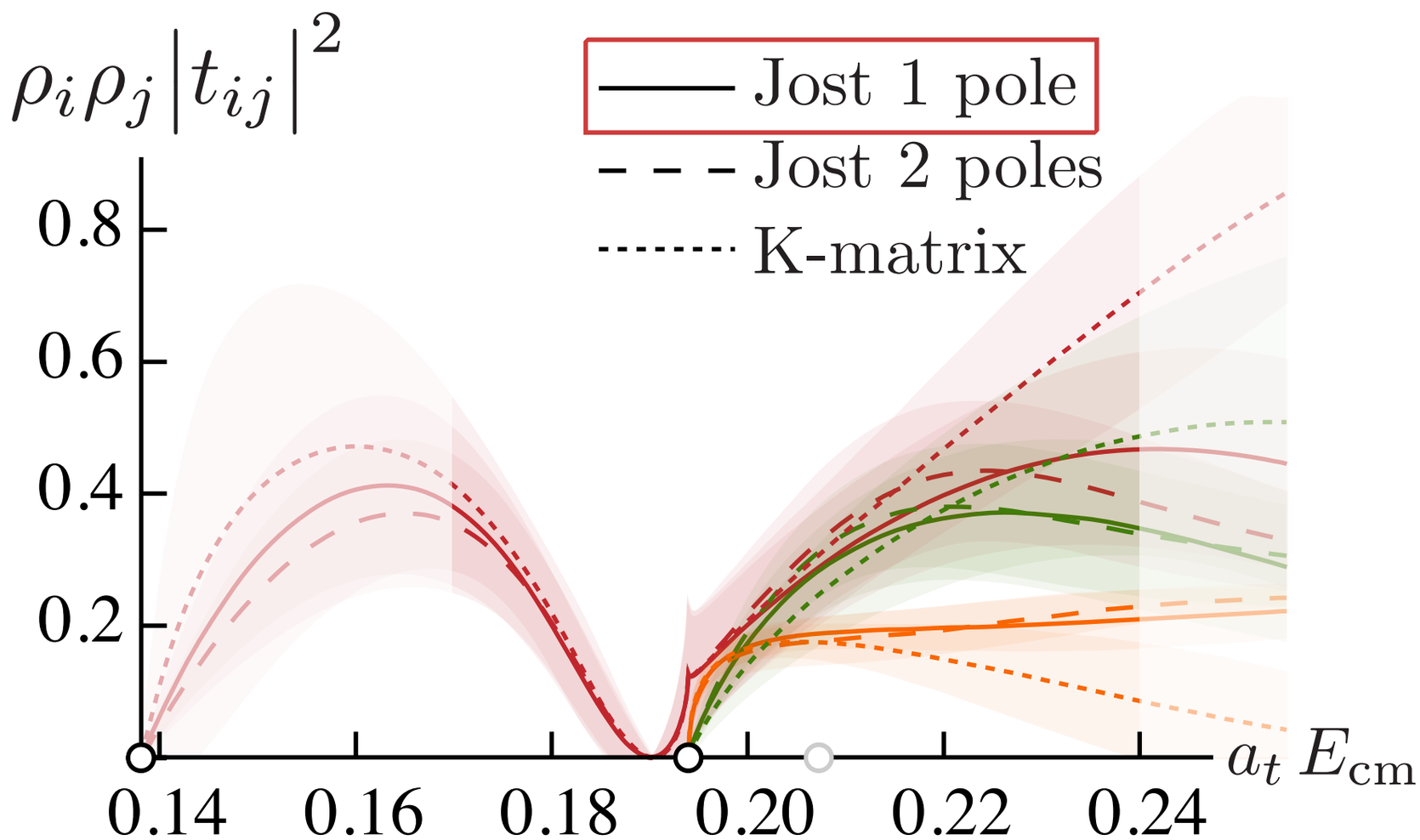


pole counting & Jost amplitudes



do we really need the **sheet III** pole ?

Jost functions allow us to directly control the pole distribution …



f_0, a_0 similarities ?

masses similar

$$m_R(f_0) = 1166(45)\,\mathrm{MeV},$$
$$m_R(a_0) = 1177(27)\,\mathrm{MeV},$$

widths a little different

$$\Gamma_R(f_0) = 181(68)\,\mathrm{MeV},$$
$$\Gamma_R(a_0) = 49(33)\,\mathrm{MeV}.$$

but channel couplings quite similar ?

$$\left|c(a_0 \to K\overline{K})\right| \approx \left|c(f_0 \to K\overline{K})\right| \quad \sim 850\,\mathrm{MeV}$$
$$\left|c(a_0 \to \pi\eta)\right| \approx \left|c(f_0 \to \pi\pi)\right| \quad \sim 700\,\mathrm{MeV}.$$

main difference is the **larger phase-space for $\pi\pi$ compared to $\pi\eta$**

can explore the effect using the simple Flatté amplitude

Flatté denominator $\quad D(s) = m_0^2 - s - i g_1^2\,\rho_1(s) - i g_2^2\,\rho_2(s)$

has zeros at

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[\left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} - 1\right] \quad \text{on sheet II, if} \quad \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} > 1, \text{ or,}$$

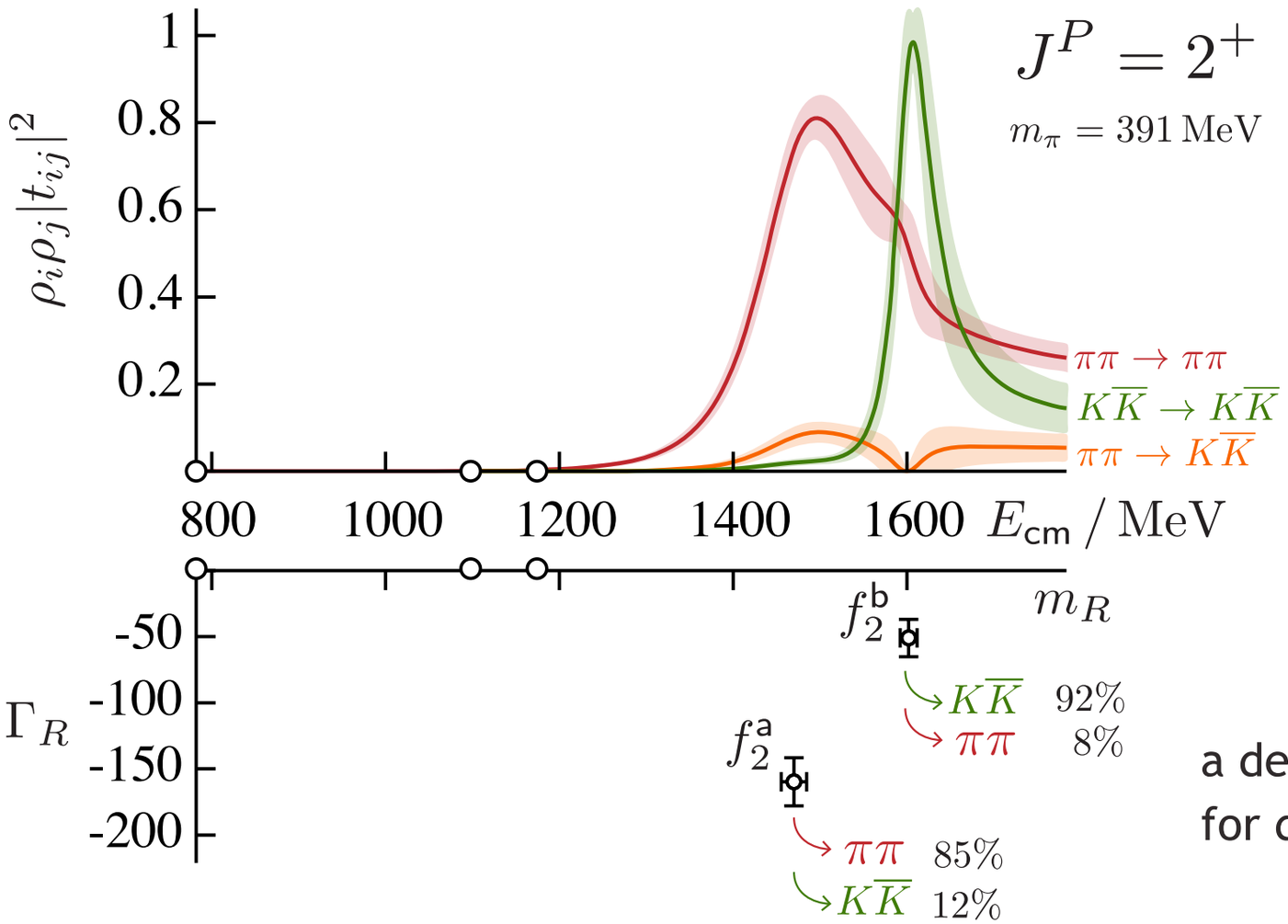
$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 - \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet IV, if} \quad \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} < 1, \text{ and,}$$

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 + \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet III, in all cases,}$$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering — $J^P = 2^+$

D-wave amplitudes & poles

m_π ~ 391 MeV
m_K ~ 549 MeV
m_η ~ 587 MeV

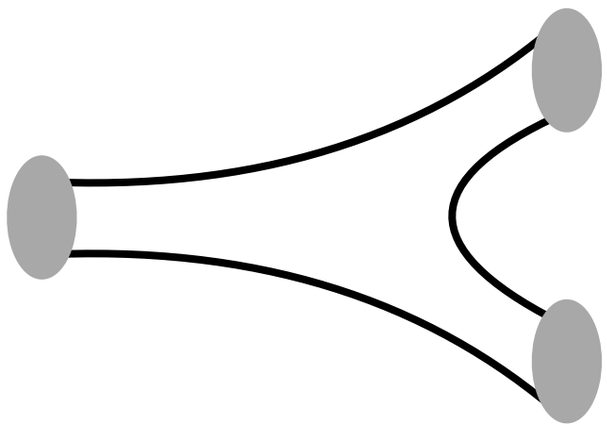


in this case uncontrolled (but justified?) approximation of neglecting $\pi\pi\eta$, $\pi\pi\pi\pi$

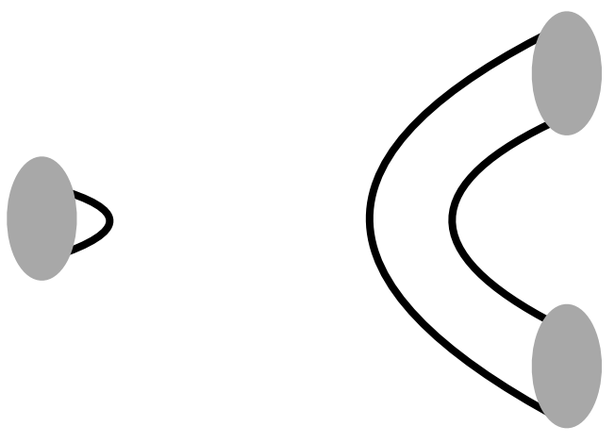
a demonstration of the OZI phenomenology for dominantly $u\bar{u}$+$d\bar{d}$, $s\bar{s}$ like states …

f_2 resonances — decay couplings & OZI

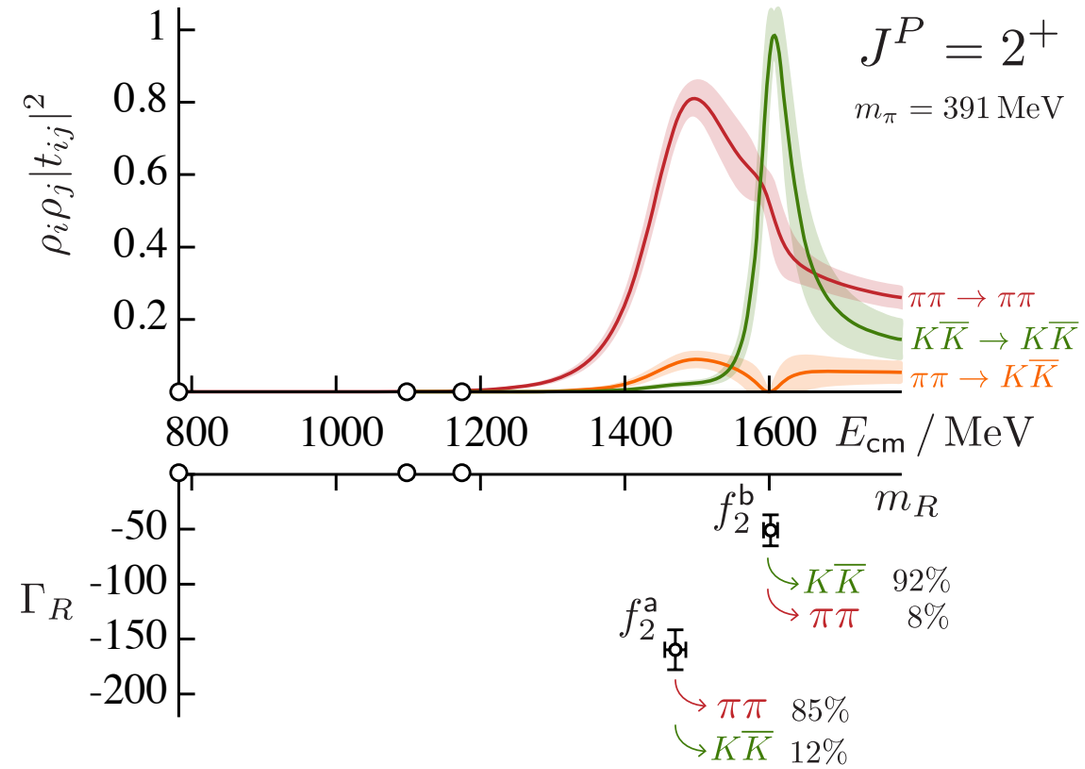
PRD97 054513 (2018)



OZI 'allowed'



OZI 'forbidden'



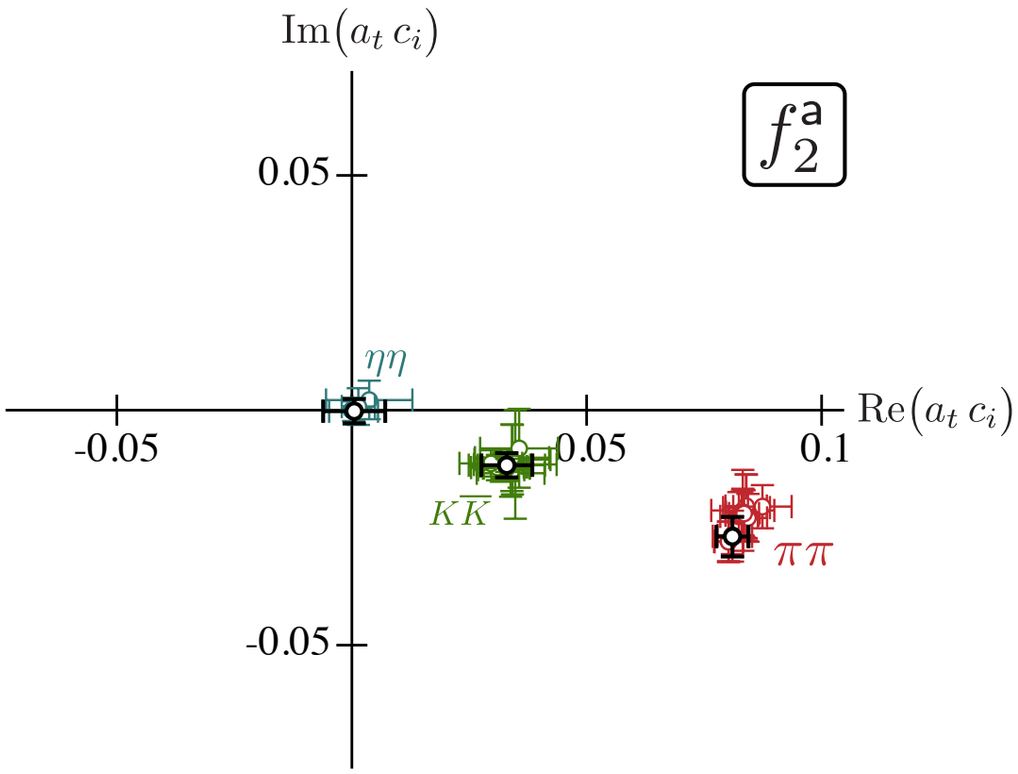
$$f_2^{\rm a} \sim u\bar{u} + d\bar{d} \qquad f_2^{\rm b} \sim s\bar{s}$$

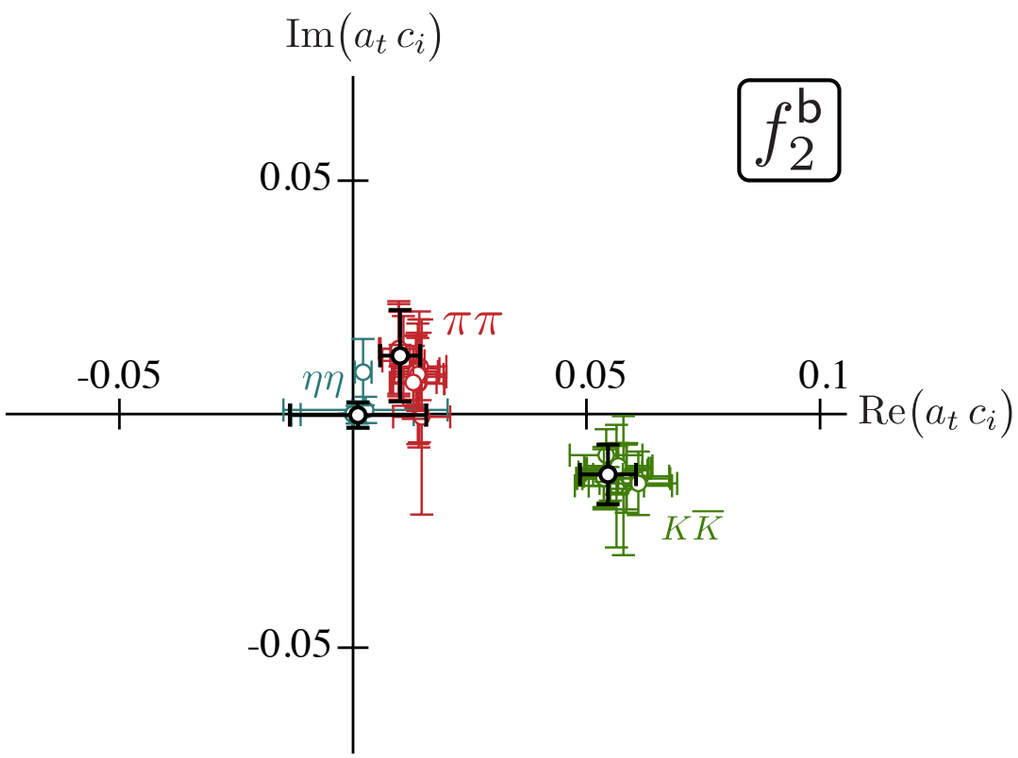
couplings from pole residue

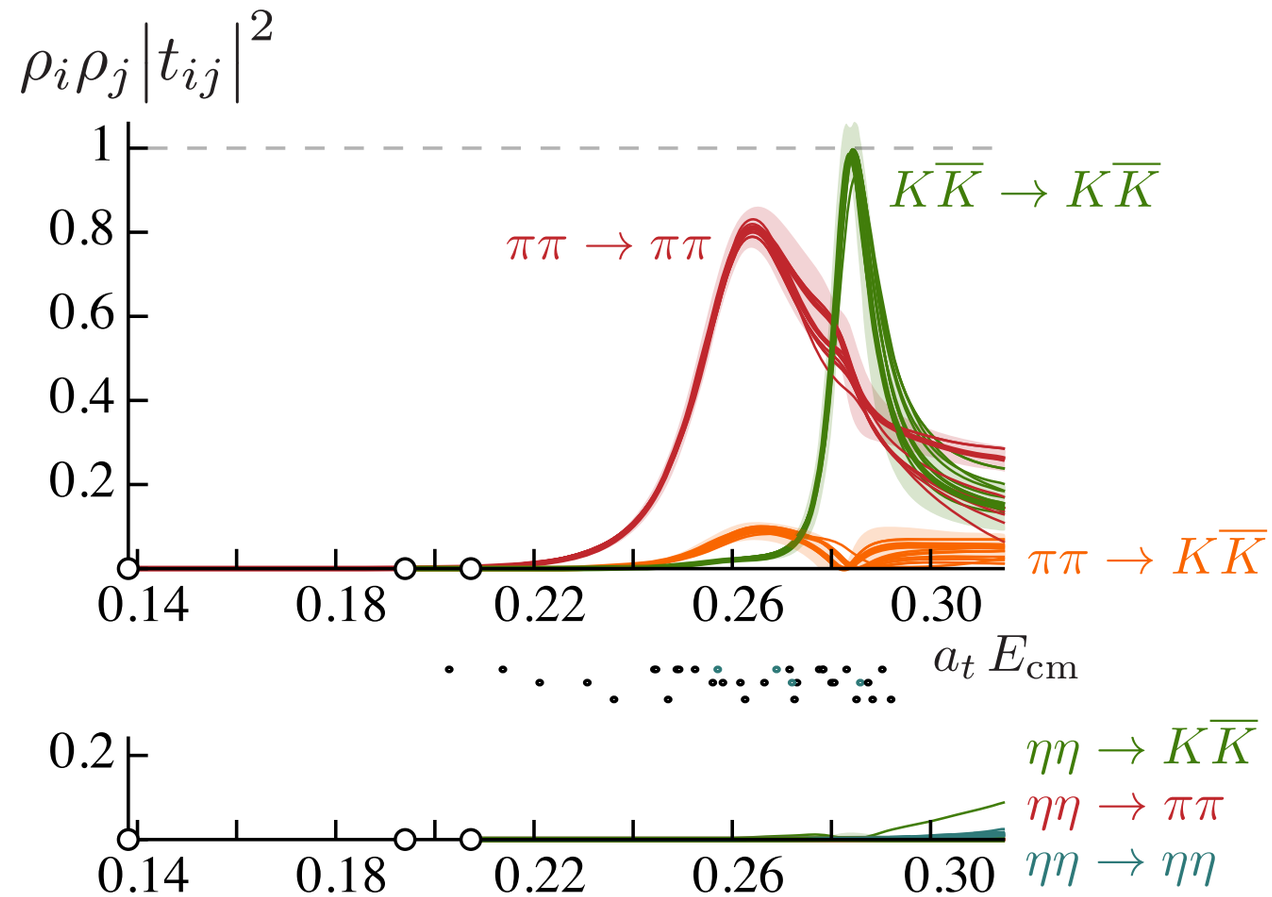
	$\frac{a_t\lvert c_{\pi\pi}\rvert}{(a_t k_{\pi\pi})^2}$	$\frac{a_t\lvert c_{K\bar{K}}\rvert}{(a_t k_{K\bar{K}})^2}$
$f_2^{\rm a}$	7.1(4)	4.8(9)
$f_2^{\rm b}$	1.0(3)	5.5(8)

zero in 'OZI' limit
— requires $s\bar{s}$ annihilation

$\pi\pi/K\overline{K}$ extras — f_2 couplings







Luescher finite-volume functions

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$$

$$\overline{\mathcal{M}}_{\ell Jm,\,\ell' J' m'} = \sum_{m_\ell, m'_\ell, m_S} \langle \ell m_\ell; 1 m_S | J m\rangle \langle \ell' m'_\ell; 1 m_S | J' m'\rangle$$

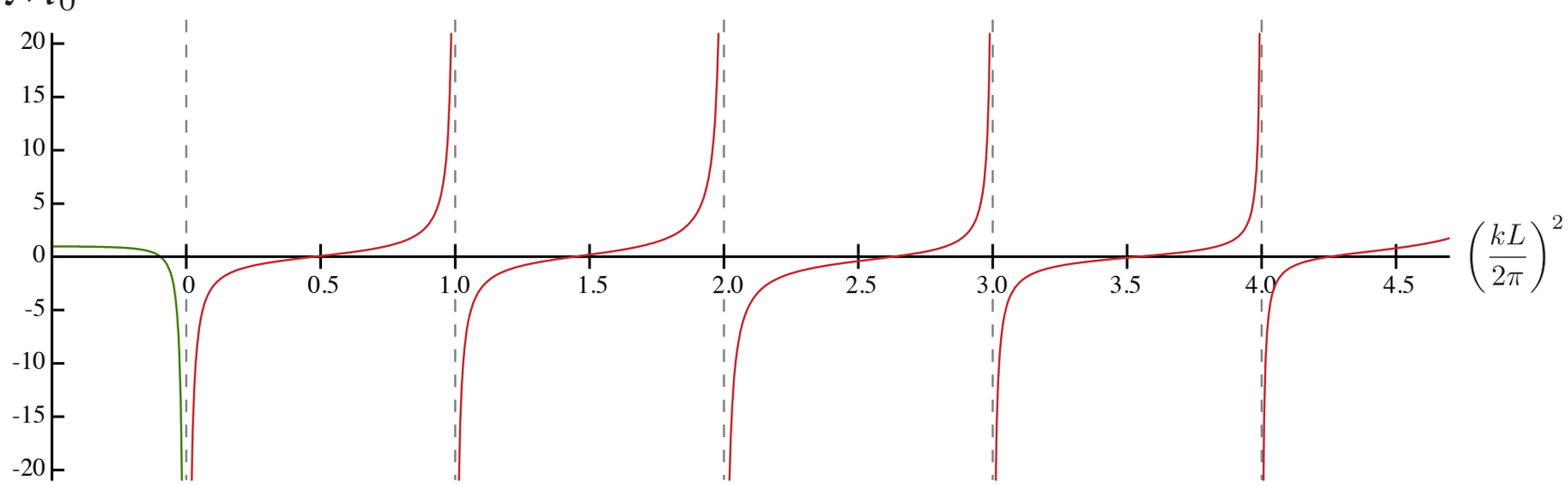
$$\times \sum_{\bar\ell, \bar m_\ell} \frac{(4\pi)^{3/2}}{k_{\mathsf{cm}}^{\bar\ell+1}}\, c^{\vec n}_{\bar\ell,\bar m_\ell}(k^2_{\mathsf{cm}}; L) \int d\Omega\, Y^*_{\ell m_\ell} Y^*_{\bar\ell \bar m_\ell} Y_{\ell' m'_\ell}$$

"spinless" Luescher functions

to respect the lattice symmetries,
need to subduce into irreducible representations

$$\overline{\mathcal{M}}^{\vec n,\Lambda}_{\ell Jn,\,\ell' J' n'}\,\delta_{\Lambda,\Lambda'}\delta_{\mu,\mu'} = \sum_{\substack{m,\lambda\\ m',\lambda'}} \mathcal{S}^{J\lambda *}_{\Lambda\mu n}\, D^{(J)*}_{m\lambda}(R)\;\; \overline{\mathcal{M}}^{\vec n}_{\ell Jm,\,\ell' J' m'}\;\; \mathcal{S}^{J'\lambda'}_{\Lambda'\mu' n'}\, D^{(J')}_{m'\lambda'}(R)$$

e.g. $\mathcal{M}_0$

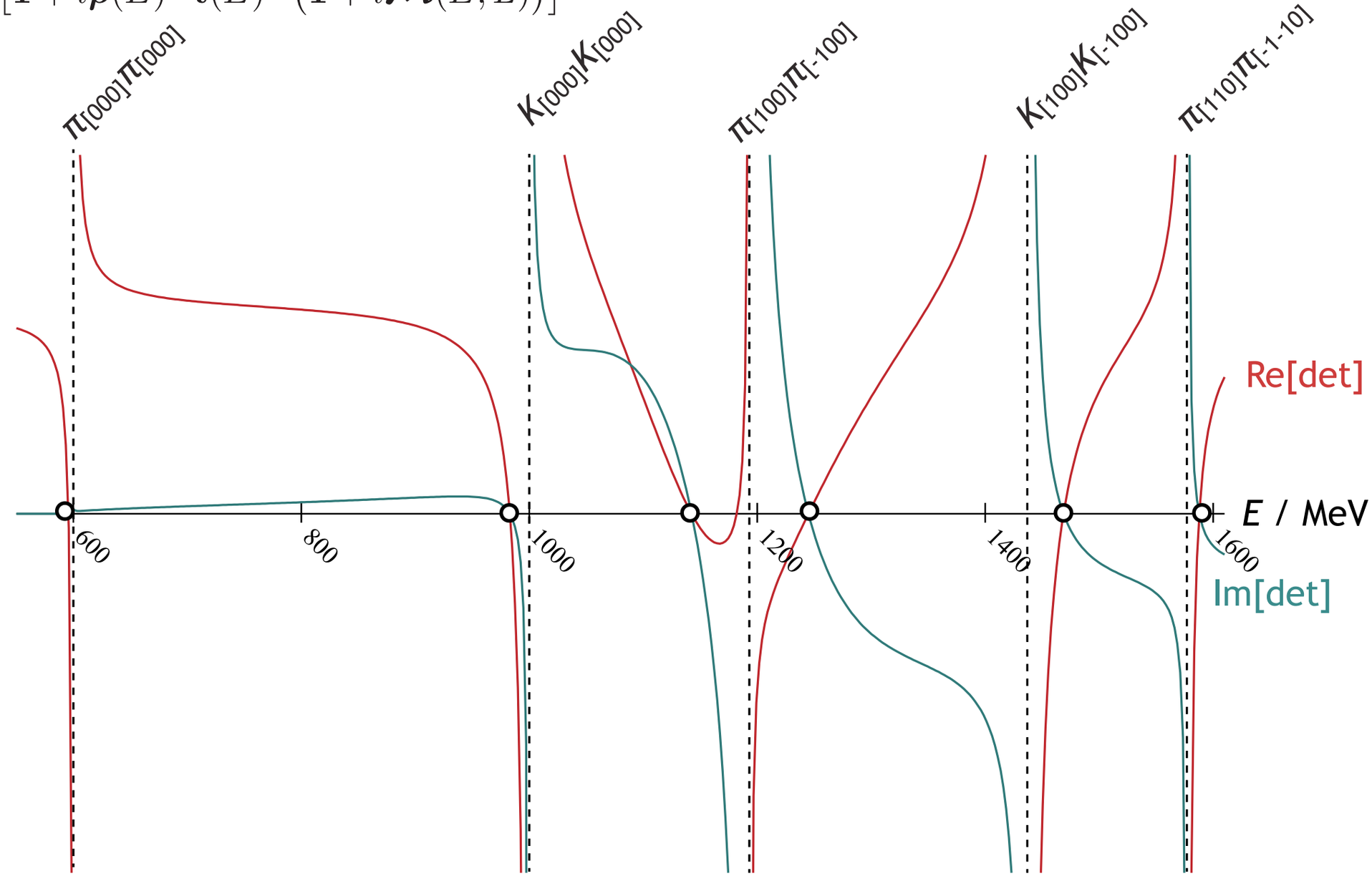


zeroes of the determinant

e.g. a two-channel Flatté form — [000] A_1^+ irrep in L=2.4 fm box

m_π = 300 MeV
m_K = 500 MeV

$$\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$$



some technical stuff — rotational symmetry

a finite cubic lattice has a smaller rotational symmetry group than an infinite continuum

simpler example of the problem: a rotationally symmetric two-dim system $\psi(r,\theta) = R_m(r)\, e^{im\theta}$

now considered on a square grid — minimum rotation is by $\pi/2$

m and m+4n transform the same !

back in 3D — **irreducible representations** of the reduced symmetry group contain multiple spins

cubic symmetry

$\Lambda(\dim)$	$A_1(1)$	$T_1(3)$	$T_2(3)$	$E(2)$	$A_2(1)$
J	$\mathbf{0}, 4\ldots$	$\mathbf{1}, 3, 4\ldots$	$\mathbf{2}, 3, 4\ldots$	$\mathbf{2}, 4\ldots$	$\mathbf{3}\ldots$

subduction $\left|\Lambda,\rho\right\rangle = \sum_m S^{\Lambda,\rho}_{J,m} \left|J,m\right\rangle$

for non-zero momentum it's even worse
— in continuum have **little group**, those rotations which don't change ***p***

⇒ label by **helicity**

can subduce helicity states into irreps of the reduced cubic symmetry

PRD85 014507 (2012)

some technical stuff — rotational symmetry

reduction of rotational symmetry is an important feature of the quantization condition too

for elastic scattering, what we previously presented as $\cot\delta_\ell(E) = \mathcal{M}_\ell\big(E(L), L\big)$

should actually be $0 = \det\Big[\cot\delta_\ell\ \delta_{\ell,\ell'}\ \delta_{m,m'} - \mathcal{M}_{\ell m;\ell' m'}\Big]$

which when subduced becomes $0 = \det\Big[\cot\delta_\ell\ \delta_{\ell,\ell'}\ \delta_{n,n'} - \mathcal{M}^{\Lambda}_{\ell n;\ell' n}\Big]$

features **all ℓ** subduced into irrep Λ

n = embedding of ℓ into Λ

what allows us to make progress is that $\delta_\ell(E) \sim k^{2\ell+1}$ at energies not too far from threshold

so higher angular momenta are naturally suppressed

in practice, truncate at some ℓ_{max} ...

coupled-channel scattering in a finite-volume

the quantization condition generalizes to

$$\boxed{0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]}$$

coupled-channel scattering in a finite-volume

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$$\boxed{0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]}$$

e.g. in A_1^+ irrep (ℓ = 0, 4 ...)

$$\mathbf{t} = \begin{pmatrix} \begin{pmatrix} t_{11}^{(0)} & t_{12}^{(0)} \\ t_{12}^{(0)} & t_{22}^{(0)} \end{pmatrix} & \mathbf{0} & \cdots \\ \mathbf{0} & \begin{pmatrix} t_{11}^{(4)} & t_{12}^{(4)} \\ t_{12}^{(4)} & t_{22}^{(4)} \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

dense in channel space
— infinite-volume dynamics mixes channels

diagonal in angular momentum space
— ℓ good q.n. in infinite-volume

coupled-channel scattering in a finite-volume

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dense in channel space
— infinite-volume dynamics mixes channels

diagonal in angular momentum space
— ℓ good q.n. in infinite-volume

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \begin{pmatrix} \mathcal{M}_{00}^{A_1^+}(k_1) & 0 \\ 0 & \mathcal{M}_{00}^{A_1^+}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}_{04}^{A_1^+}(k_1) & 0 \\ 0 & \mathcal{M}_{04}^{A_1^+}(k_2) \end{pmatrix} & \cdots \\ \begin{pmatrix} \mathcal{M}_{40}^{A_1^+}(k_1) & 0 \\ 0 & \mathcal{M}_{40}^{A_1^+}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}_{44}^{A_1^+}(k_1) & 0 \\ 0 & \mathcal{M}_{44}^{A_1^+}(k_2) \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

diagonal in channel space
— no dynamics

dense in angular momentum
— cubic symmetry lives here

$$k_1 = \tfrac{1}{2}\sqrt{E^2 - 4m_1^2}$$

$$k_2 = \tfrac{1}{2}\sqrt{E^2 - 4m_2^2}$$

below closed thresholds

- multiple possible scattering channels (still just spin-0—spin-0)
  - the quantisation condition is

$$0 = \det\left[\delta_{\ell n;\ell' n'}\delta_{\alpha\beta} + i\rho_\alpha t^{[\ell]}_{\alpha\beta}\left(\delta_{\ell n;\ell' n'} + i\mathcal{M}^{\vec{d};\Lambda}_{\ell n,\ell' n'}(k_\alpha)\right)\right]$$

  - must we include all possible channels ?

    no, only channels which are kinematically open, or close to opening

    e.g. $E < E_{\text{thr}}$, $k = i\kappa$ $\qquad \mathcal{M}^{\vec{0};A_1}_{01,01}(i\kappa) = i - \frac{i}{\kappa}\sum_{\vec{n}\neq 0}\frac{e^{-\kappa|\vec{n}|L}}{|\vec{n}|L}$

    e.g. two-channels, *S*-wave

$$\begin{vmatrix} 1 + i\rho_1 t_{11}(1 + i\mathcal{M}_1) & i\rho_1 t_{12}(1 + i\mathcal{M}_1) \\ i\rho_2 t_{12}(1 + i\mathcal{M}_2) & 1 + i\rho_1 t_{11}(1 + i\mathcal{M}_2) \end{vmatrix} \xrightarrow[\text{threshold 2}]{\text{well below}} 1 + i\rho_1 t_{11}(1 + i\mathcal{M}_1)$$

elastic condition

coupled-channel scattering

now need a partial-wave **scattering matrix**

e.g. $\mathbf{t} = \begin{array}{c} \begin{array}{ccc} \pi\pi & K\bar{K} & \eta\eta \end{array} \\ \begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ & \blacksquare & \blacksquare \\ & & \blacksquare \end{pmatrix} \end{array} \begin{array}{c} \\ \pi\pi \\ K\bar{K} \\ \eta\eta \end{array}$

time-reversal symmetrizes

now multi-channel unitarity

$$\mathrm{Im}\left[t^{-1}(E)\right]_{ij} = \begin{cases} 0 & E < E_i^{\mathrm{thr.}} \\ -\rho_i(E)\,\delta_{ij} & E > E_i^{\mathrm{thr.}} \end{cases}$$

K-matrix approach guarantees unitarity

$$\left[t^{-1}(E)\right]_{ij} = \left[K^{-1}(E)\right]_{ij} + \delta_{ij} I_i(E)$$

$$\mathrm{Im}\, I_i(E) = -\rho_i(E)$$

e.g. $K_{ij}(E) = \dfrac{g_i\, g_j}{M^2 - s}$ *Flatté, or multichannel BW*

e.g. *polynomial in each element*

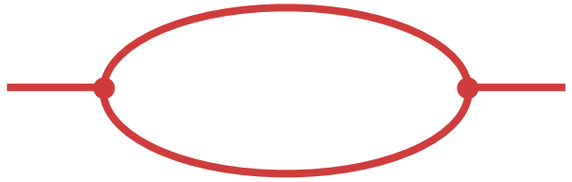
$\vdots$

Chew-Mandelstam phase-space

- equal mass case

$$I(s) = -C(s)$$

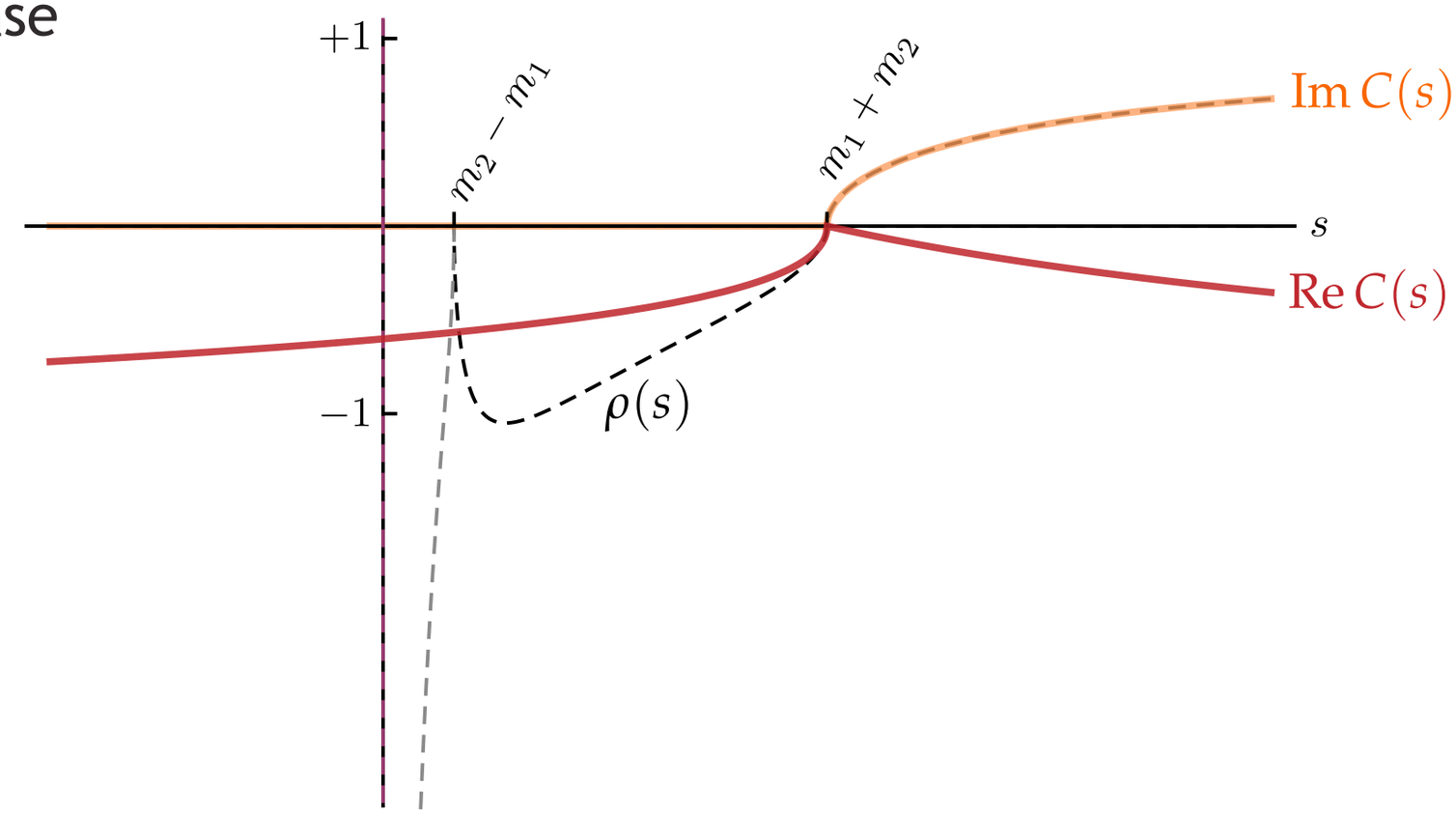
$$C(s) = C(0) + \frac{s}{\pi}\int_{s_{\rm th}}^{\infty} ds' \sqrt{1-\frac{s_{\rm th}}{s'}}\,\frac{1}{s'(s'-s)}$$



$$C(s) = \frac{\rho(s)}{\pi}\log\left[\frac{\rho(s)-1}{\rho(s)+1}\right]$$

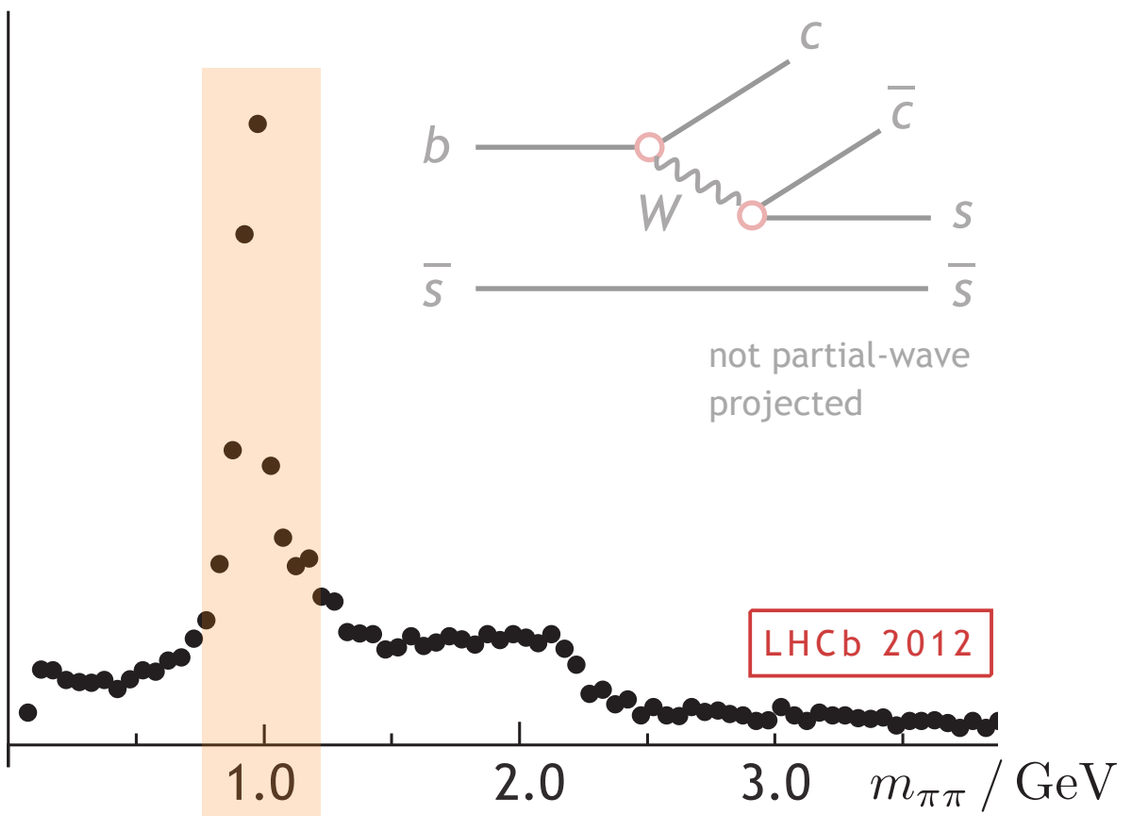
subtracting at threshold $C(s_{\rm th}) = 0$

- unequal mass case

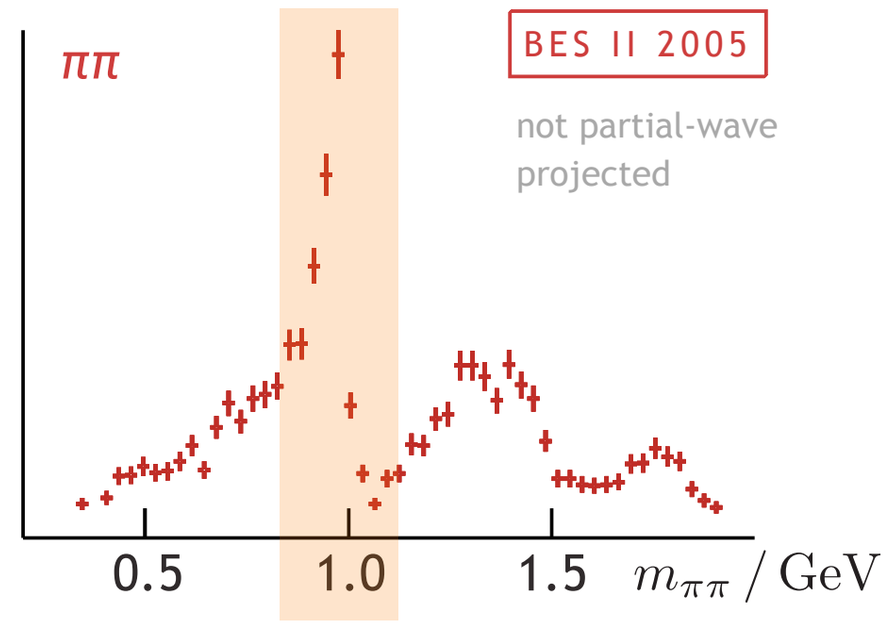


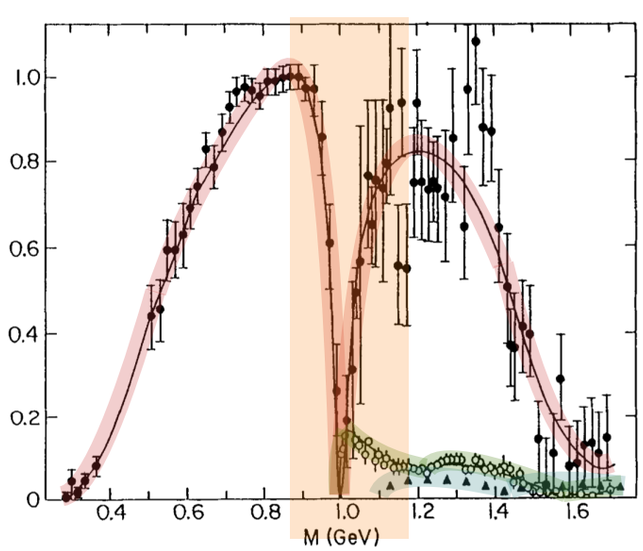
“production” of $\pi\pi$ (as opposed to scattering)

$\bar{B}_s^0 \to J/\psi \;\; \pi^+\pi^-$



$J/\psi \to \varphi \;\; \pi\pi$

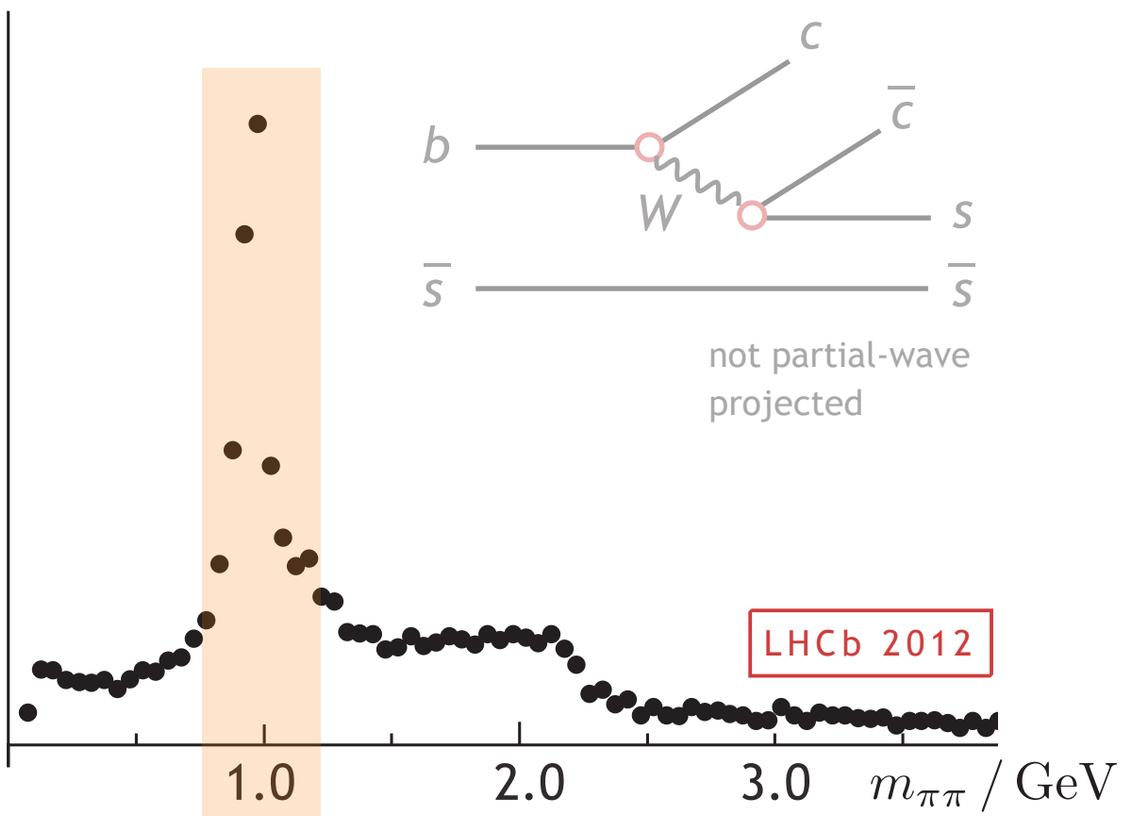




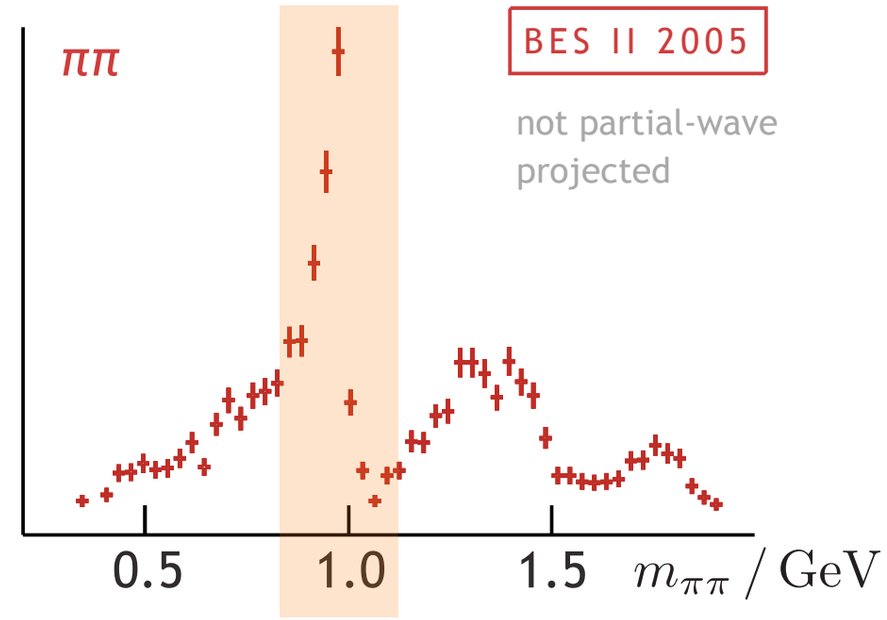
can ‘look’ drastically different to scattering !

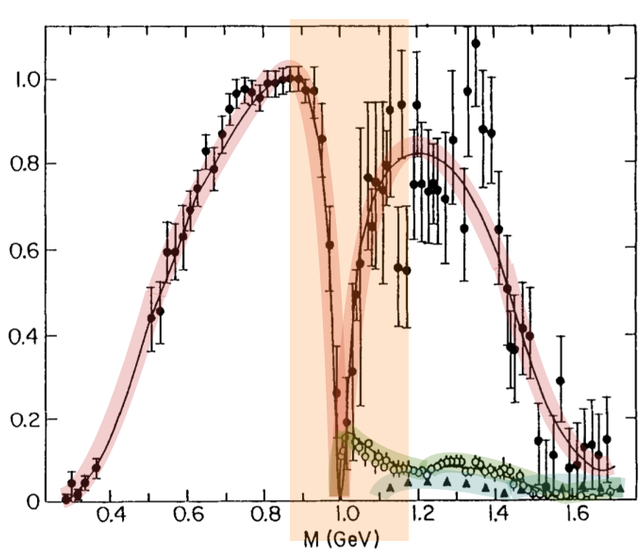
“production” of $\pi\pi$ (as opposed to scattering)

$\bar{B}_s^0 \to J/\psi \;\; \pi^+\pi^-$

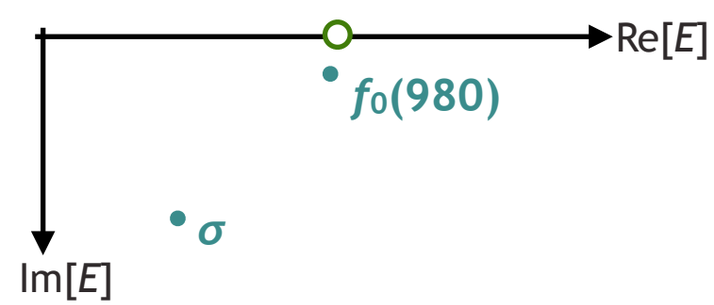


$J/\psi \to \varphi \;\; \pi\pi$



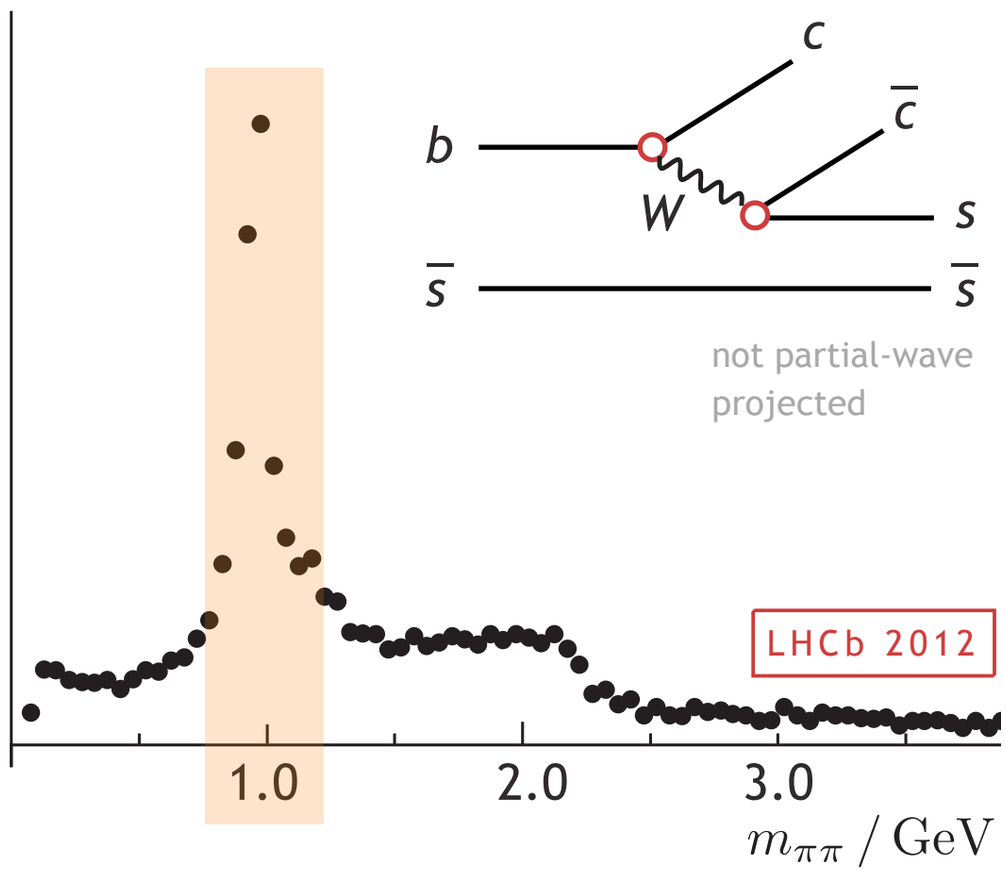


… **same poles** (σ, $f_0(980)$) — different couplings …

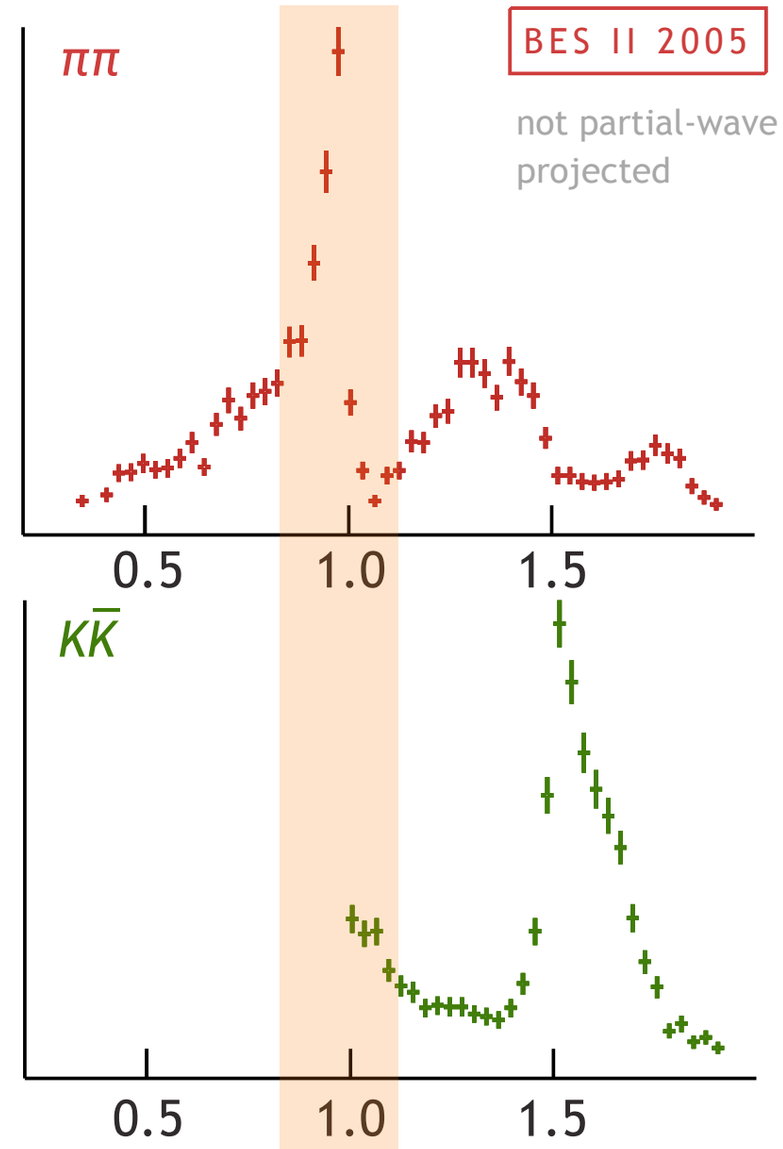


$f_0(980)$ as a peak in "$s\bar{s}$" production

$\bar{B}_s^0 \to J/\psi \ \pi^+\pi^-$



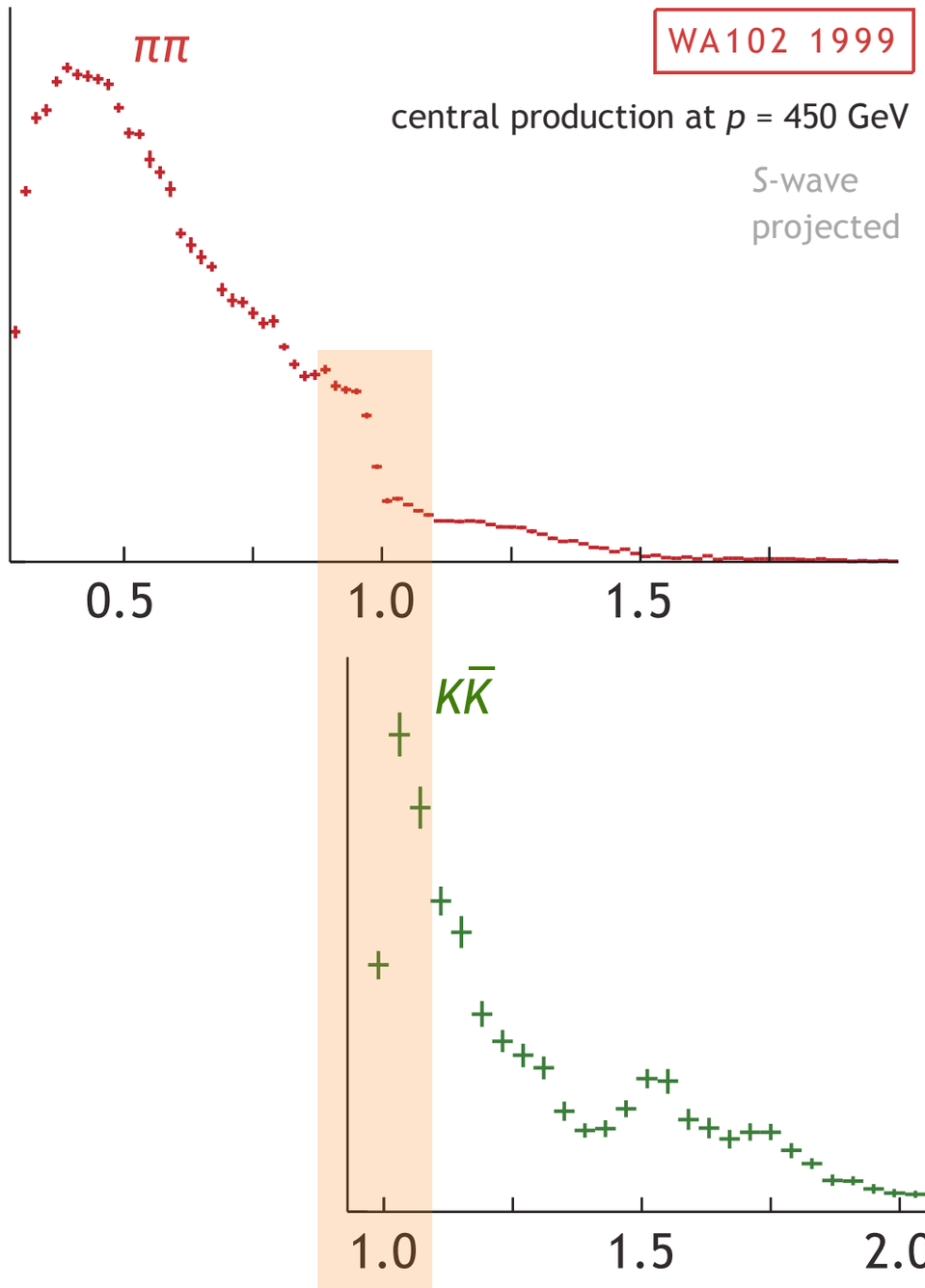
$J/\psi \to \varphi(\pi\pi, K\bar{K})$



note the rapid turn-on of $K\bar{K}$ at threshold

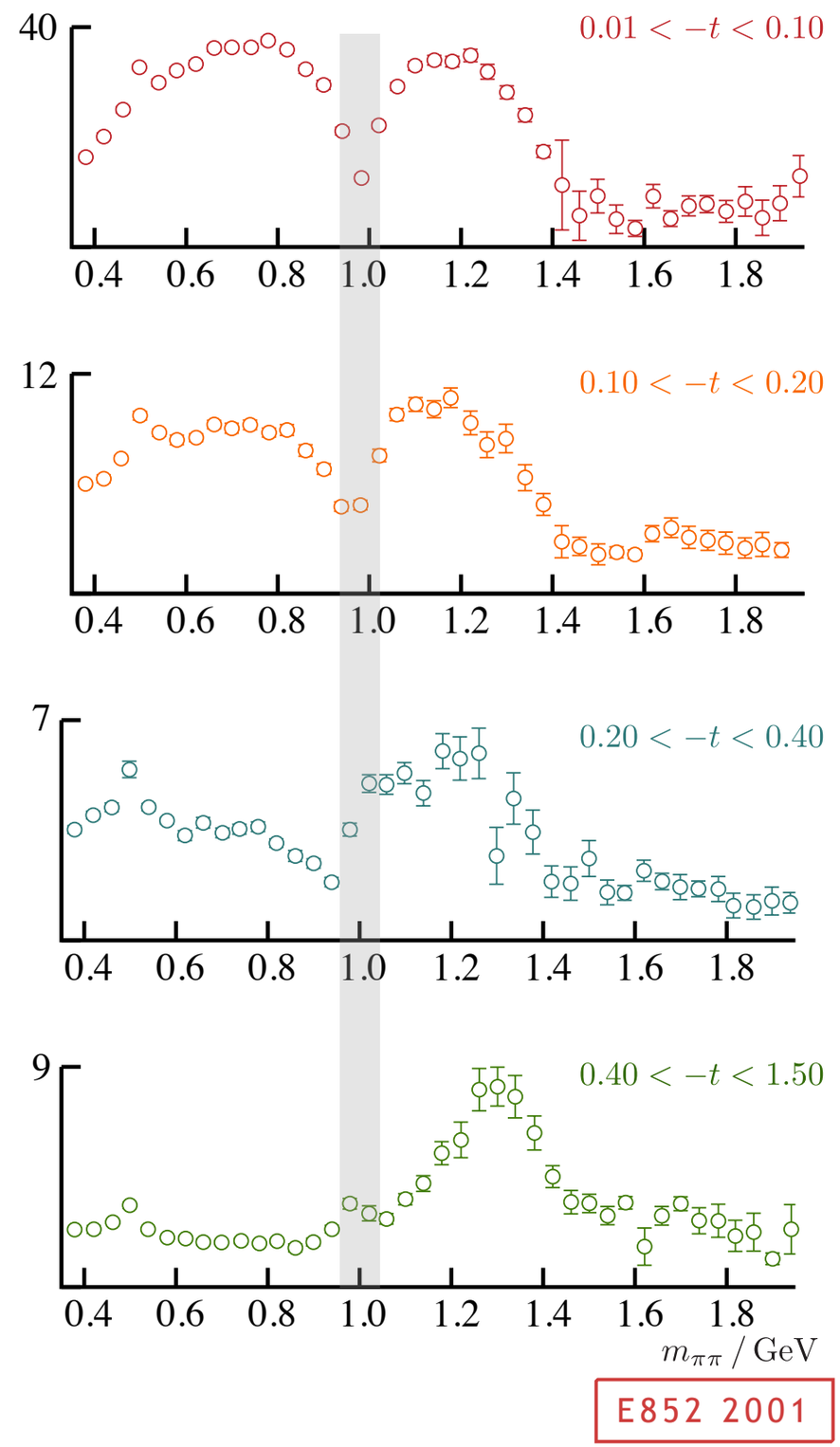
$f_0(980)$ as ?

$pp \to p(\pi\pi, K\bar{K})p$

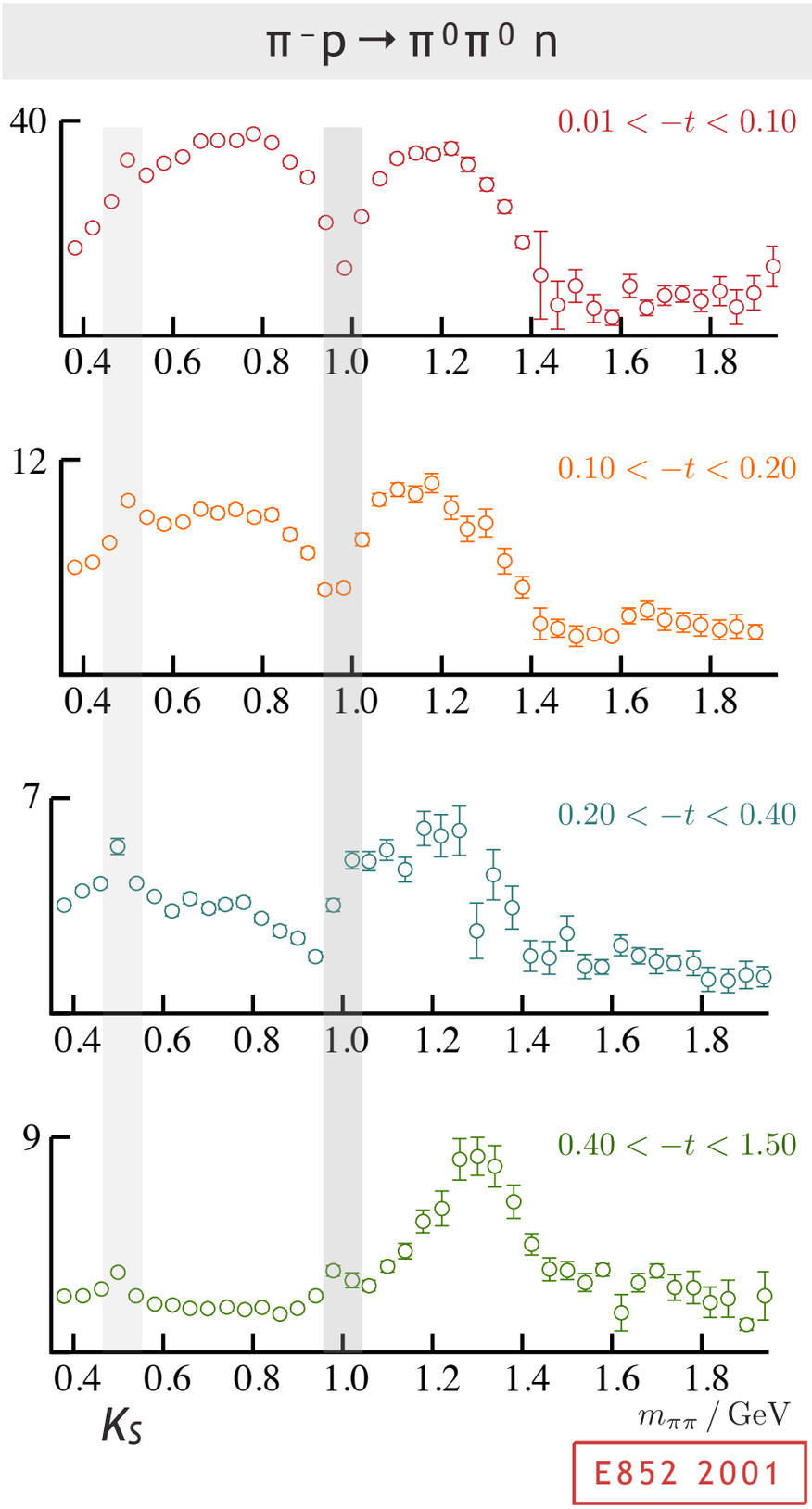


$f_0(980)$ as a shoulder on a large σ 'background'

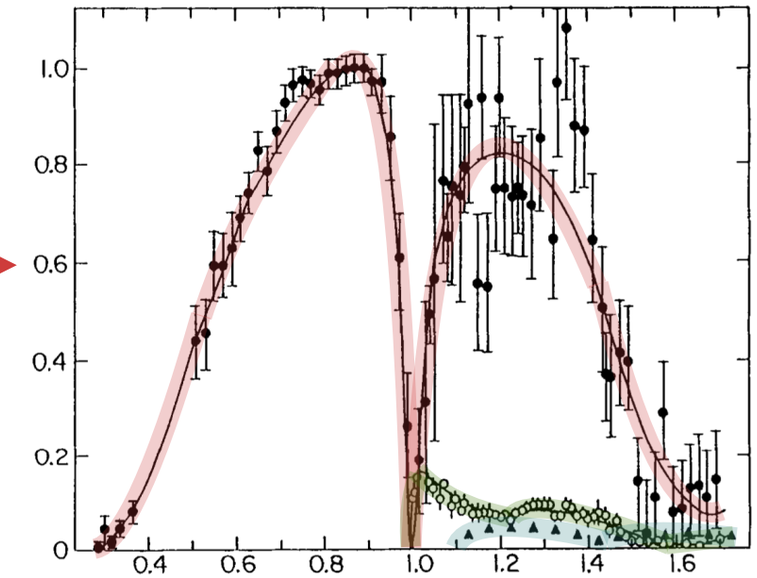
$\pi^- p \to \pi^0 \pi^0 n$



S—wave $\pi\pi$ production



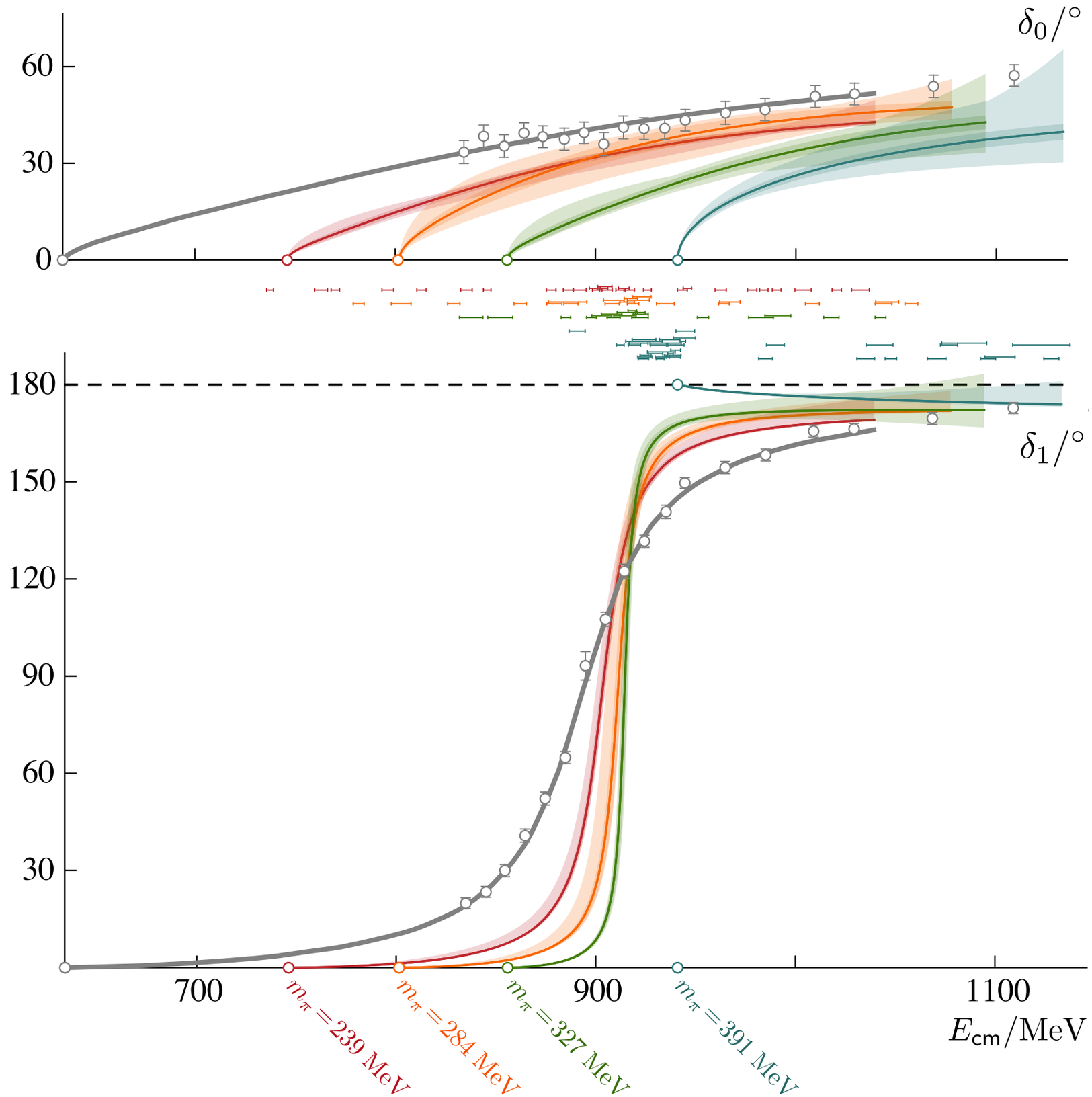
dominated by π exchange
— looks like the the 1970s
elastic phase-shift data



other (non-π) exchanges
becoming significant,
$f_0(980)$ dip less pronounced

σ no longer large,
$f_0(980)$ starting to be a peak ?

πK elastic scattering at four light quark masses



m_π ~

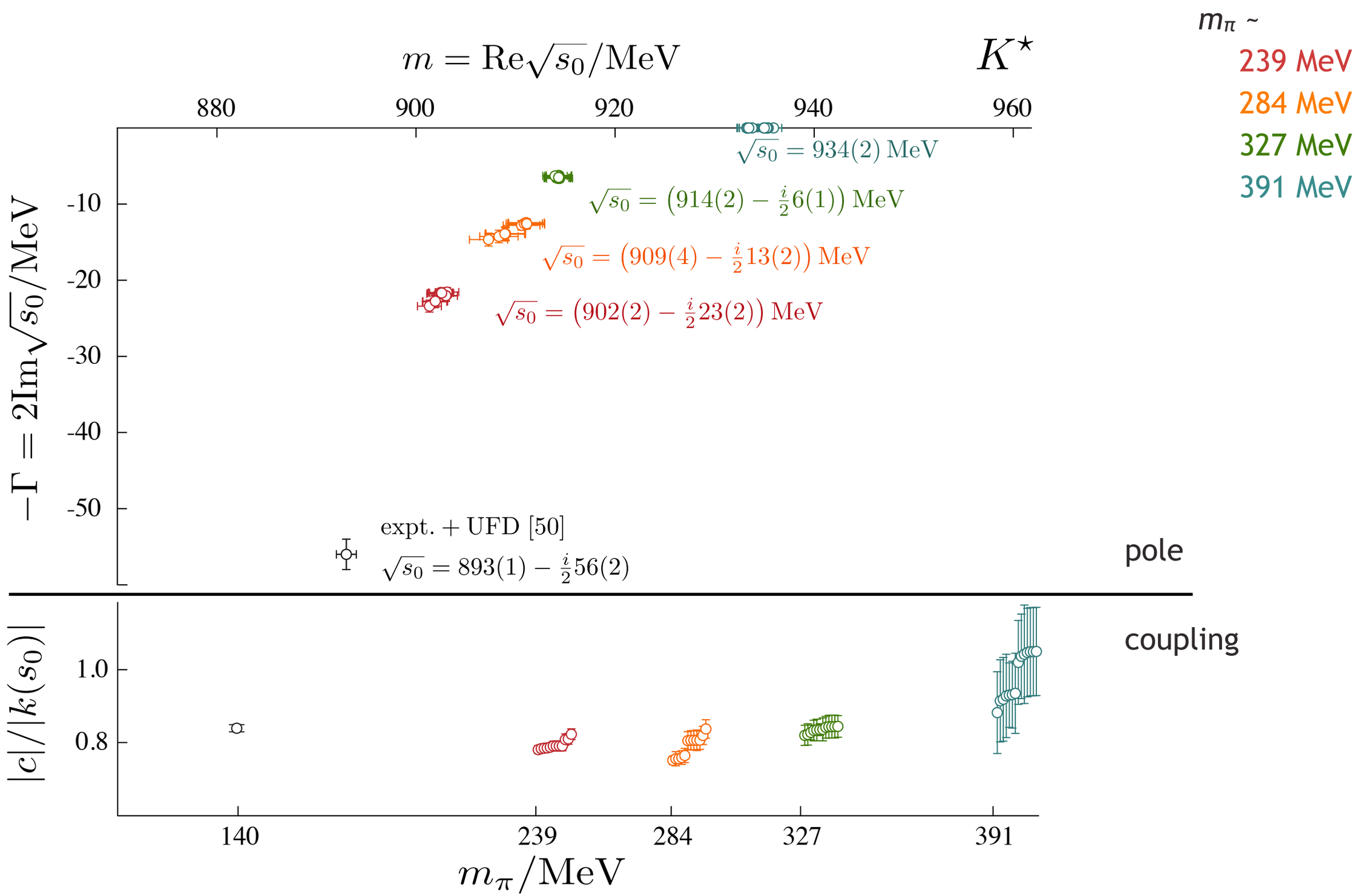
239 MeV

284 MeV

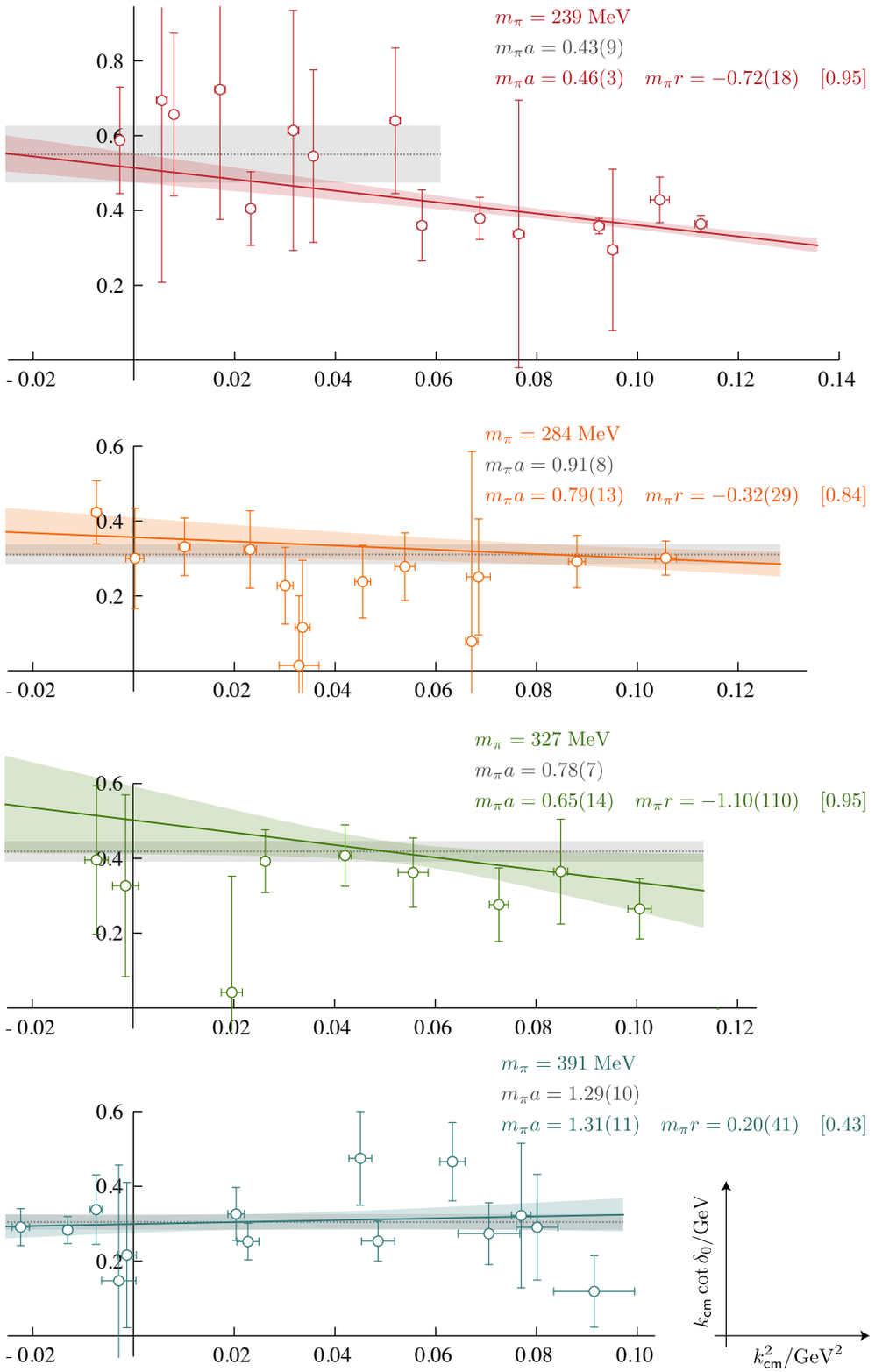
327 MeV

391 MeV

πK elastic scattering at four light quark masses



πK elastic scattering at four light quark masses



m_π ~

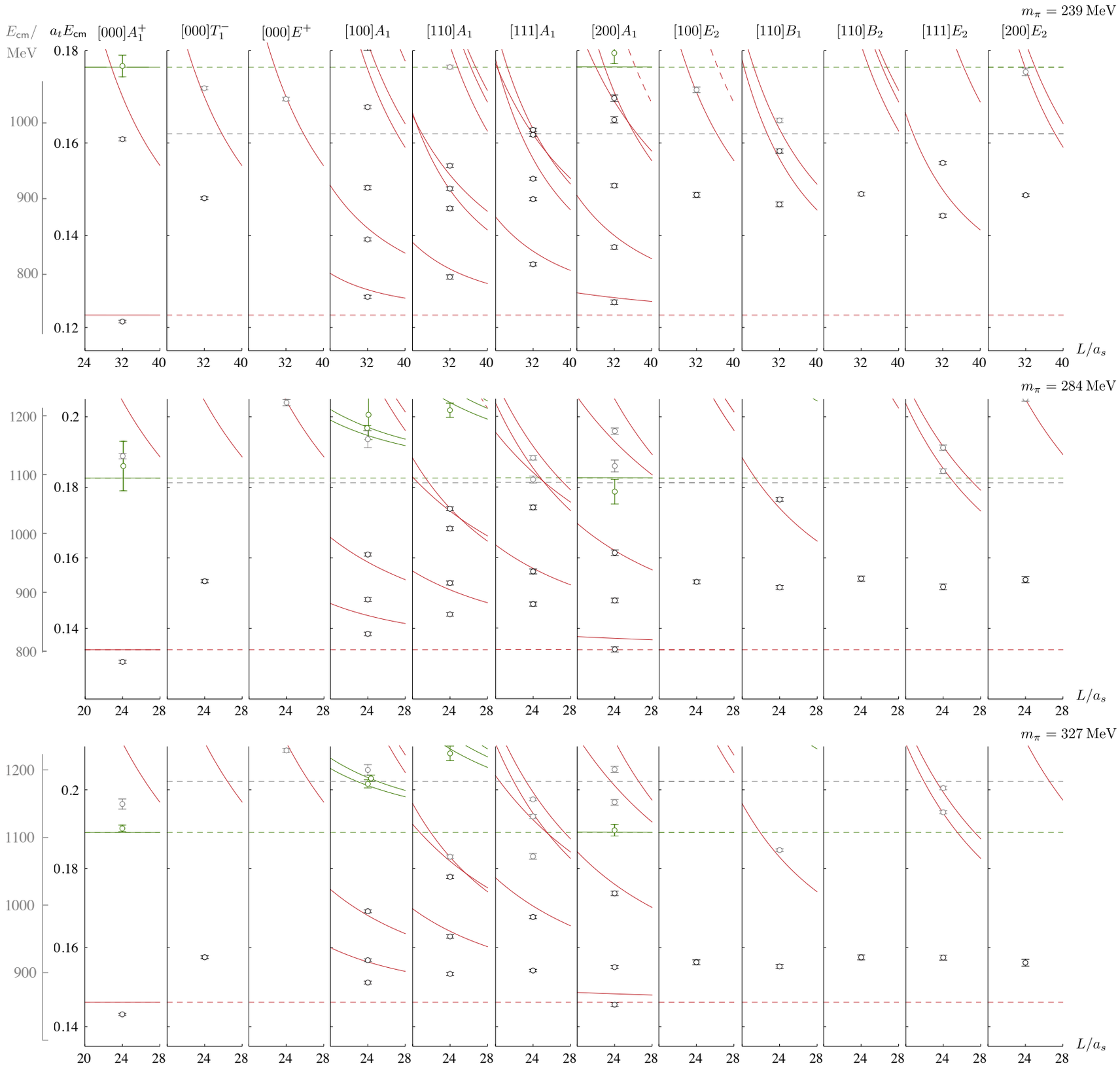
239 MeV

284 MeV

327 MeV

391 MeV

πK elastic scattering at four light quark masses



tetraquarks ?

m_π ~ 391 MeV

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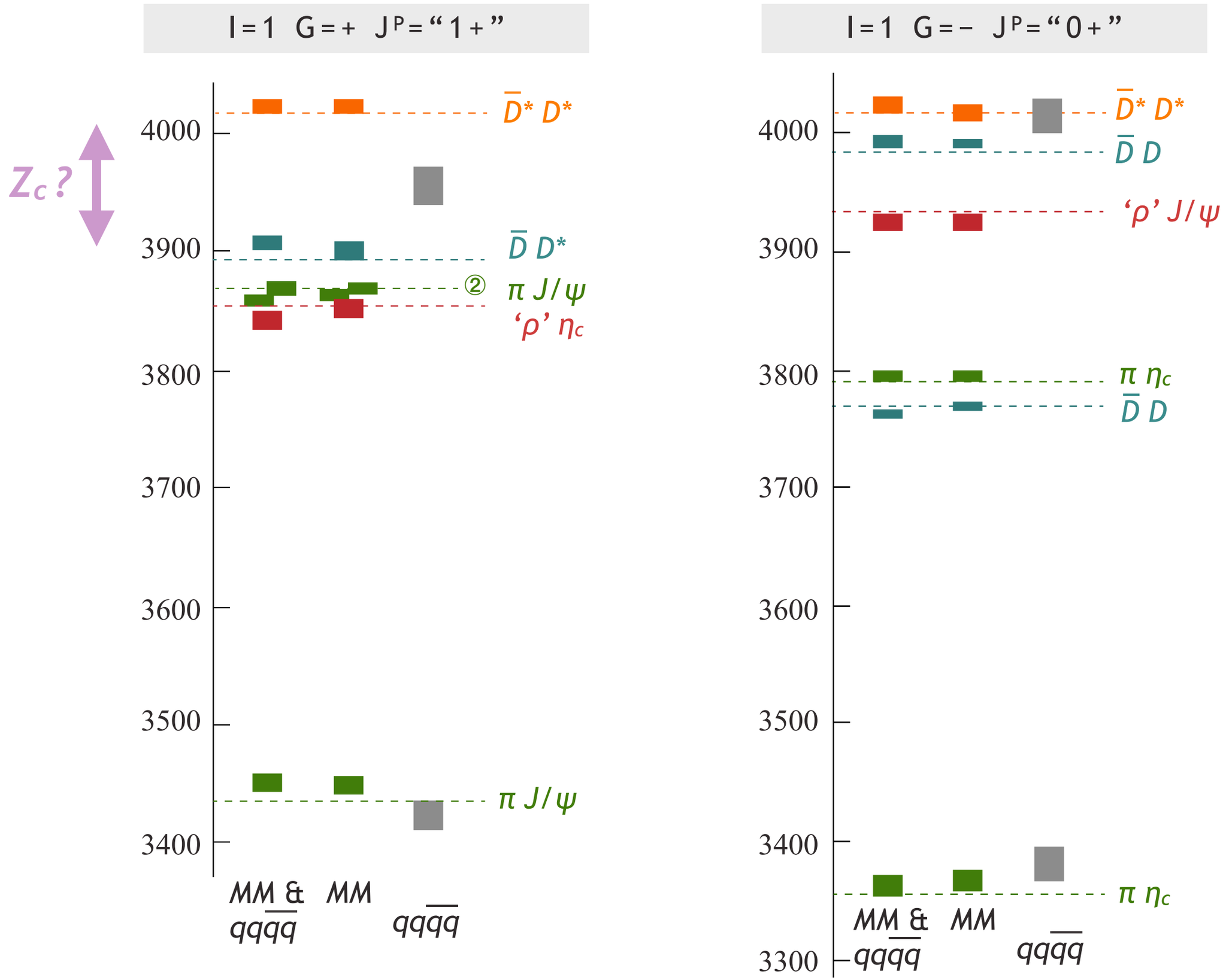
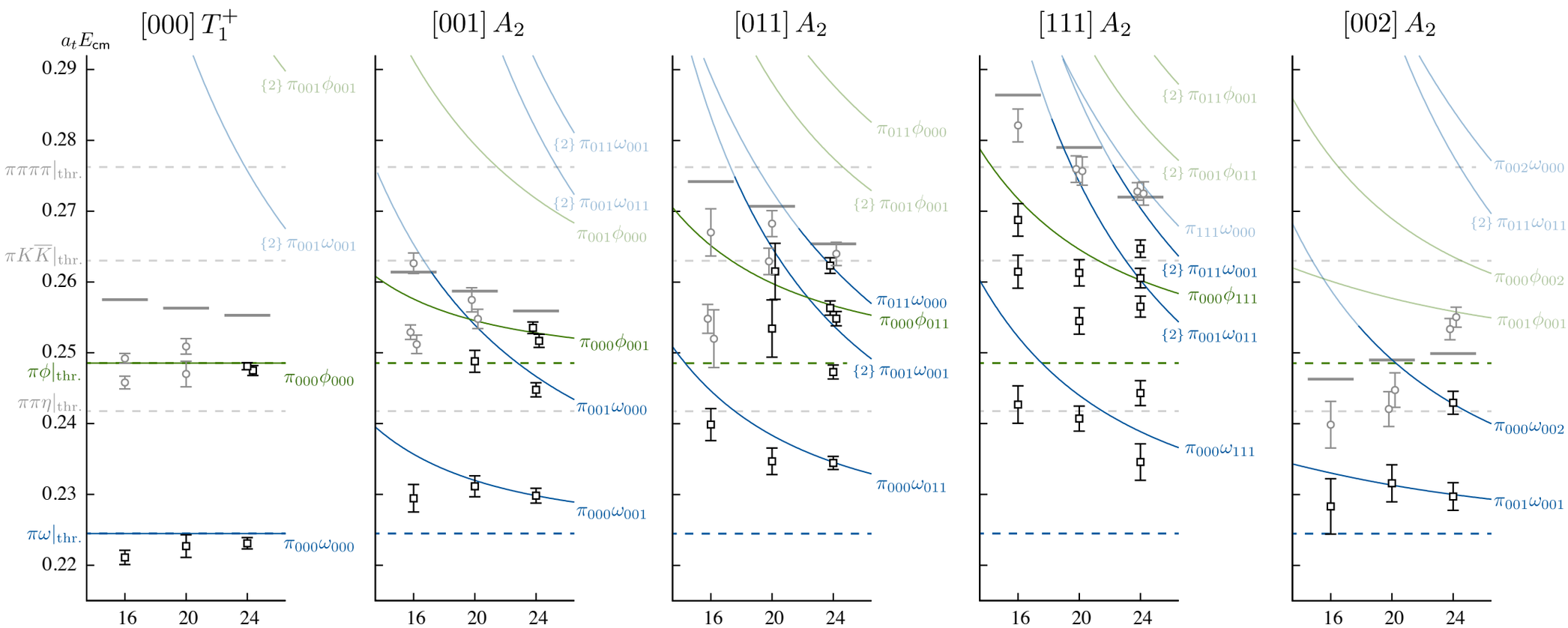


$\pi\omega$ scattering

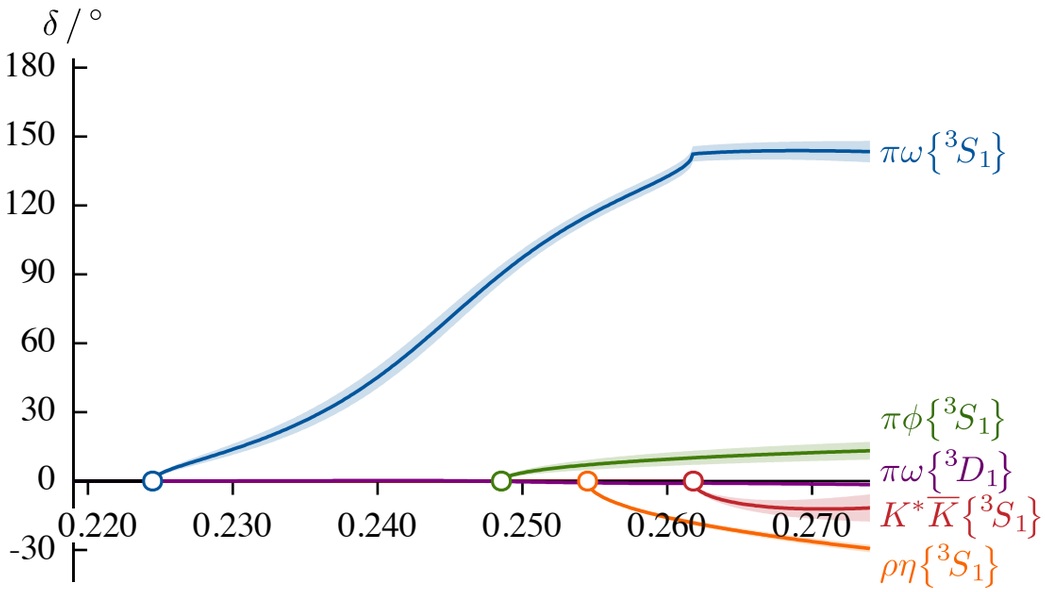
at m_π~391 MeV, the ω is a stable meson

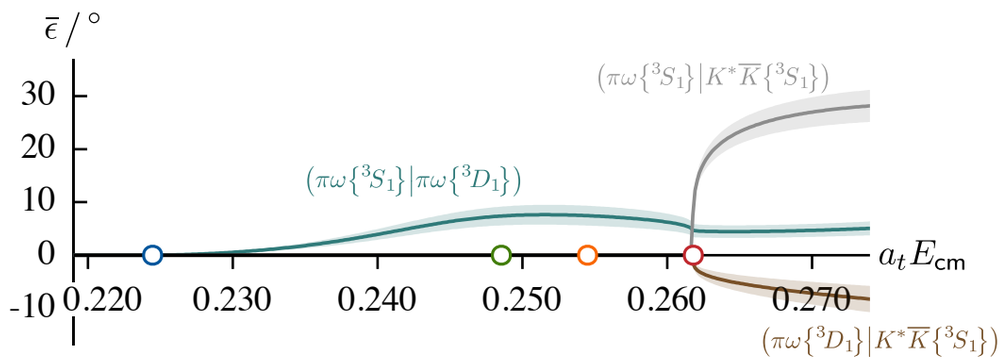
m_π~391 MeV finite-volume spectrum

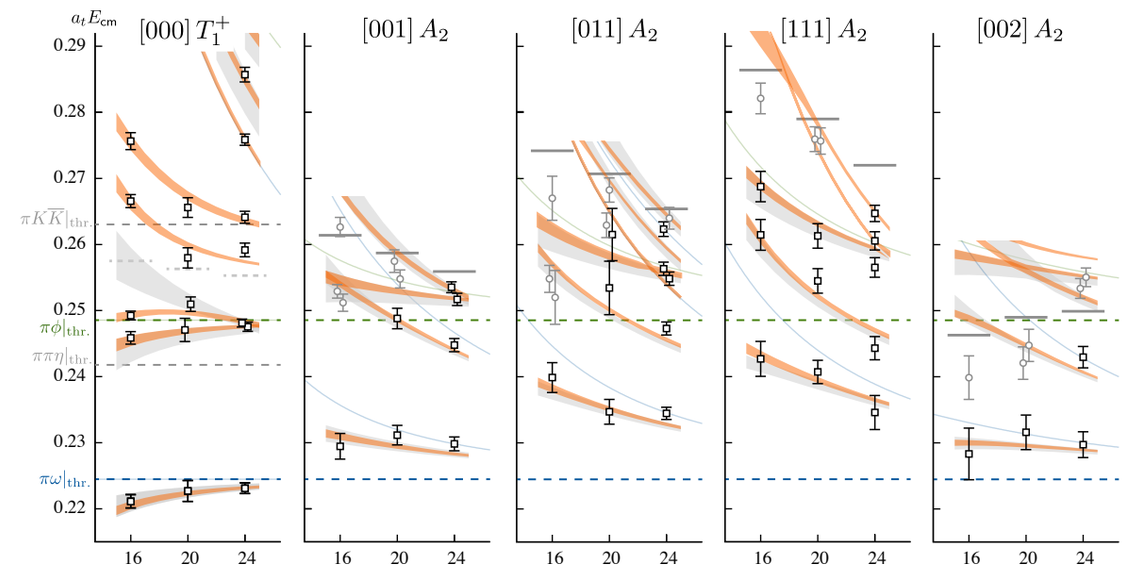


at low energies, a coupled $\pi\omega$, $\pi\varphi$ scattering system …

$\pi\omega$ scattering — with '$\rho\eta$', $K^*\overline{K}$







$b_1(1235)$ $\quad I^G(J^{PC}) = 1^+(1^{+-})$

Mass $m = 1229.5 \pm 3.2$ MeV (S = 1.6)
Full width $\Gamma = 142 \pm 9$ MeV (S = 1.2)

$b_1(1235)$ DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\omega\pi$	seen		348
[D/S amplitude ratio $= 0.277 \pm 0.027$]			
$\pi^\pm\gamma$	$(1.6\pm0.4)\times10^{-3}$		607
$\eta\rho$	seen		†
$\pi^+\pi^+\pi^-\pi^0$	< 50 %	84%	535
$K^*(892)^\pm K^\mp$	seen		†
$(K\overline{K})^\pm\pi^0$	< 8 %	90%	248
$K^0_S K^0_L \pi^\pm$	< 6 %	90%	235
$K^0_S K^0_S \pi^\pm$	< 2 %	90%	235
$\phi\pi$	< 1.5 %	84%	147

OMEGA yp

actually f1(1285) ?

As a final test of the effects of the $\rho\eta\{^3S_1\}$ and $K^*\overline{K}\{^3S_1\}$ channels, we find the resonance pole and corresponding couplings. There are 16 Riemann sheets and several 'mirror poles', but the closest pole is located at

$$a_t\sqrt{s_{0\,\mathrm{II}}} = 0.2448(12) - \tfrac{i}{2}\,0.0215(21), \qquad (28)$$

which agrees within uncertainties with the pole position found in Section VII. The corresponding couplings are,

$$\begin{aligned} a_t c(\pi\omega\{^3S_1\})_{\mathrm{II}} &= 0.117(7)\exp[-i\,\pi\,0.084(20)] \\ a_t c(\pi\omega\{^3D_1\})_{\mathrm{II}} &= 0.016(4)\exp[-i\,\pi\,0.182(22)] \\ a_t c(\rho\eta\{^3S_1\})_{\mathrm{II}} &= 0.003(52) \\ a_t c(K^*\overline{K}\{^3S_1\})_{\mathrm{II}} &= 0.166(8)\exp[-i\,\pi\,0.043(12)], \end{aligned} \qquad (29)$$

where we exclude the meaningless phase on $a_t c(\rho\eta\{^3S_1\})_{\mathrm{II}}$ as the magnitude is consistent with zero and where $c(\pi\phi\{^3S_1\})_{\mathrm{II}} = 0$ by choice of amplitude. The coupling to $\rho\eta\{^3S_1\}$ is small but has a large uncertainty, while the coupling to $K^*\overline{K}\{^3S_1\}$ is larger.[13]