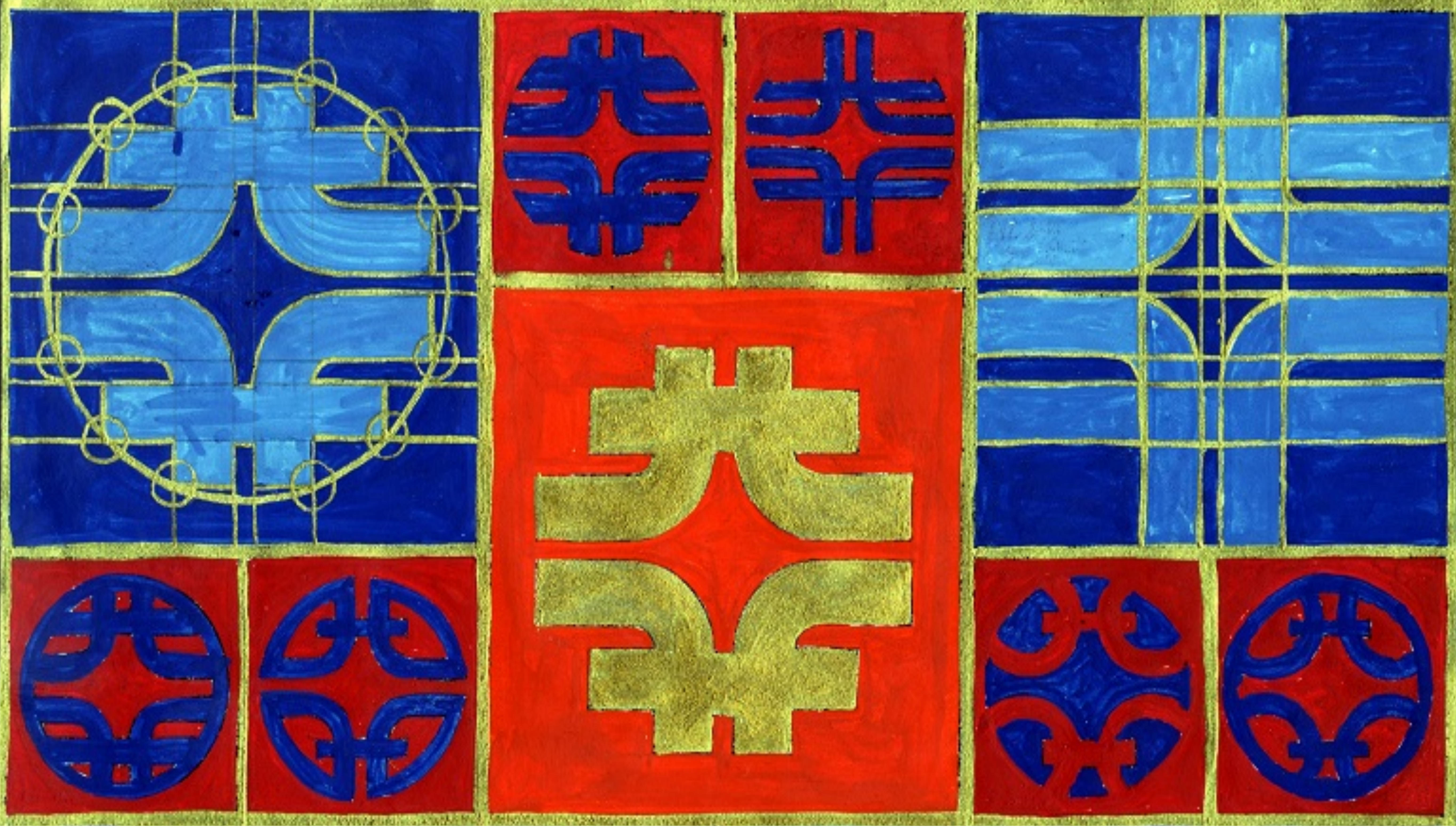
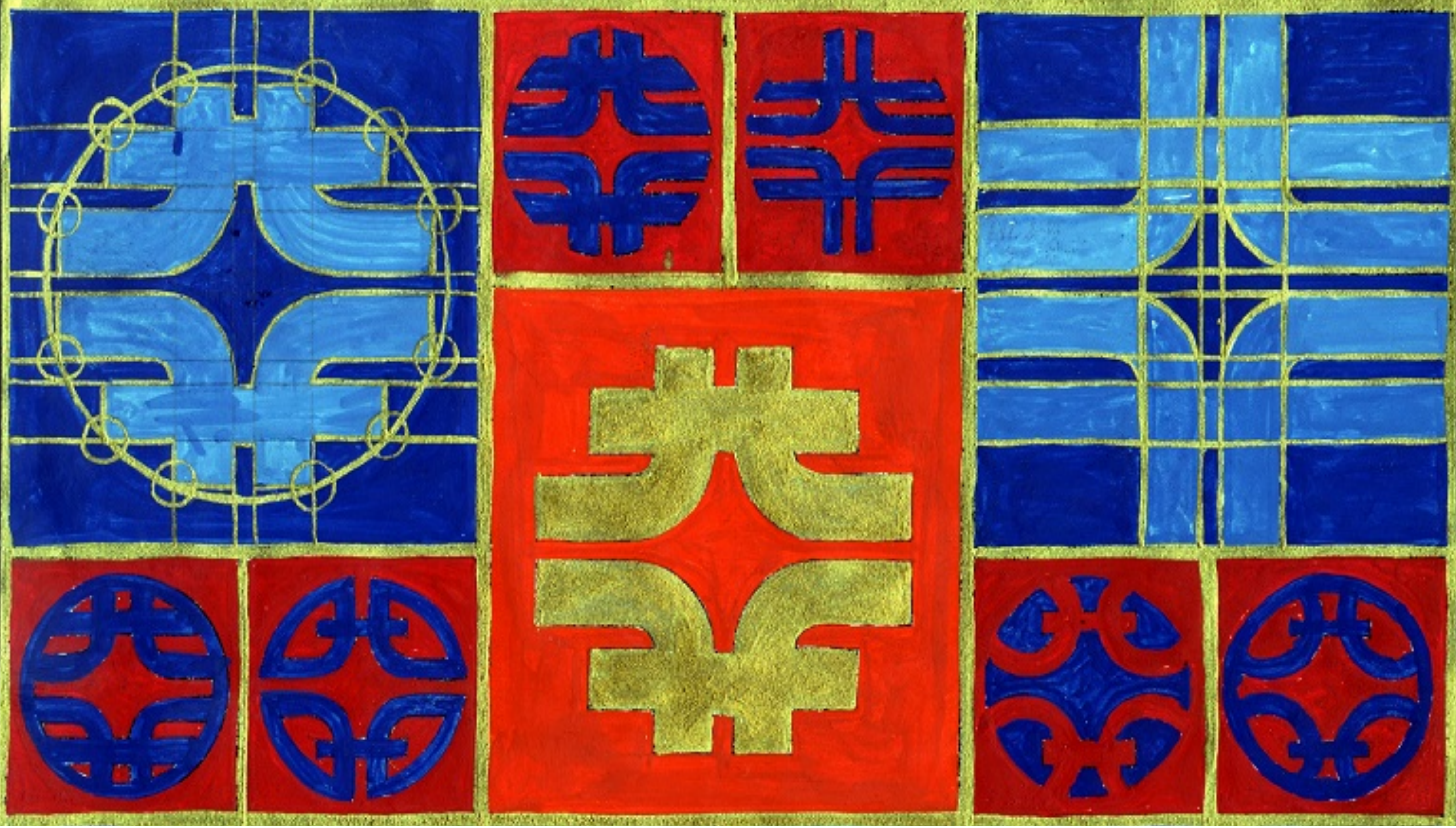


# Inclusive Semileptonic Decays from LQCD (overview)

William I. Jay (Fermilab)

Theory Meets Experiment on |Vub| and |Vcb|— 12 Jan 2021



# Inclusive (hadronic processes) from LQCD (overview)

William I. Jay (Fermilab)

Theory Meets Experiment on |Vub| and |Vcb|— 12 Jan 2021

# Outline

- Motivation
- How to get inclusive processes from 4pt functions
- What makes these calculations hard?
  - Algorithmic, theoretical, and practical challenges
- Review of some recent proposals
- My (biased) view of prospects

# Experimental Tension

Tension between inclusive/exclusive determinations of

- $|V_{cb}|$ from B → $D^{*}\ell\nu$ has $3.3\sigma$ tension
- $|V_{cb}|$ from B → $D\ell\nu$ has $2.0\sigma$ tension
- $|V_{ub}|$ from B → $\pi\ell\nu$ has $2.8\sigma$ tension

The most precise theoretical calculations employ different frameworks

- Inclusive decays: continuum heavy quark EFT + operator product expansion

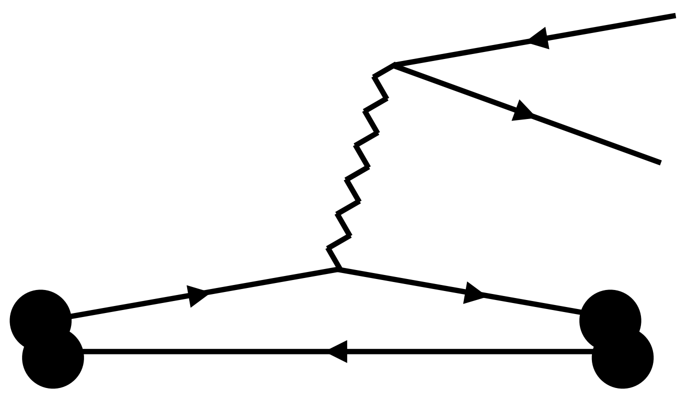
  → **See talks from T. Mannel, M. Steinhauser** for details

- Exclusive decays: numerical lattice gauge theory

  → **See previous talk from A. Vaquero** for details

# Exclusive semileptonic decays from LQCD

(See previous talk by A. Vaquero)



(form factors) ∝ (matrix elements)

$$f_J(p) \propto \langle \text{final} | \, J(p) \, | \text{initial} \rangle$$

- Methodology is well established
- Systematic effects are well understood
- Calculations are underway using physical quark masses: u, d, s, c, and b.
- Coming soon:

» B-meson decay form factors at the 1% level

» D-meson decay form factors at sub-percent level

# Frontier LQCD calculations

The physics of Euclidean correlation functions:

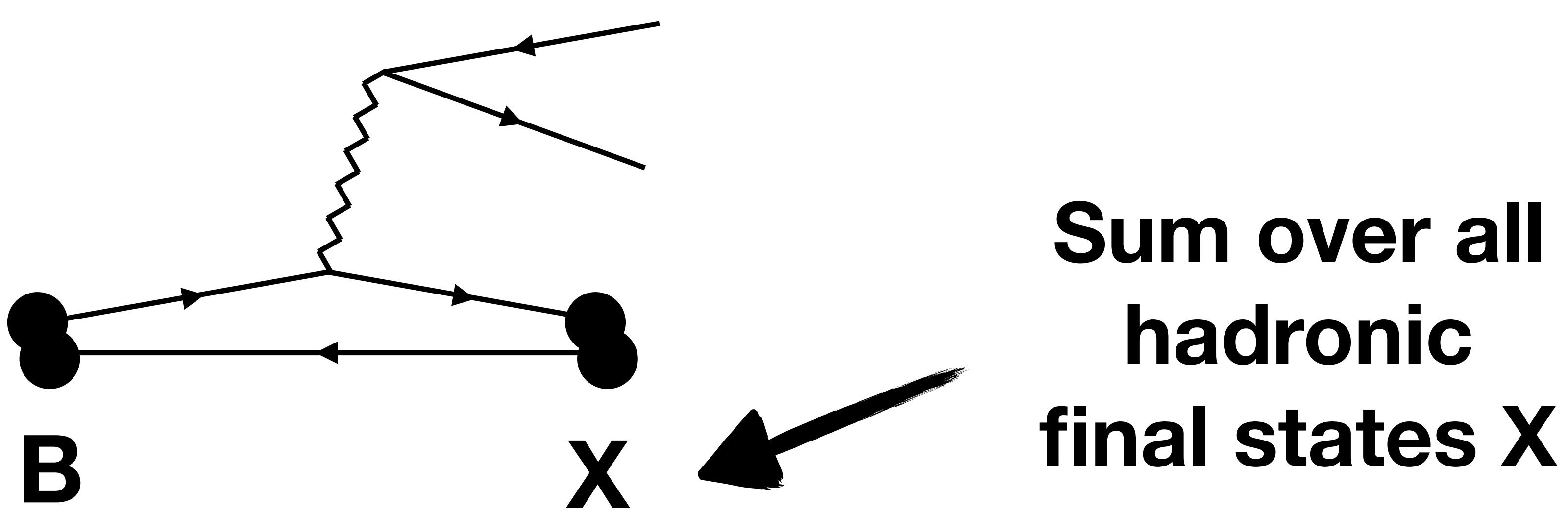
- 2-point functions: masses, decay constants ✓
- 3-point functions: form factors ✓

- 4-point functions:
  - Flavor physics: Inclusive B-meson decays
  - Neutrino physics: νA-scattering, 0νββ-decay
  - Hadron structure: PDFs or hadronic tensor $H_{\mu\nu}$
  - Kaon physics: $K_L$-$K_S$ mixing, $\varepsilon_K$, rare kaon decays
  - … many others

# Frontier LQCD calculations

The physics of Euclidean correlation functions

- 2-point functions: masses, decay constants
- 3-point functions: form factors
- 4-point functions:



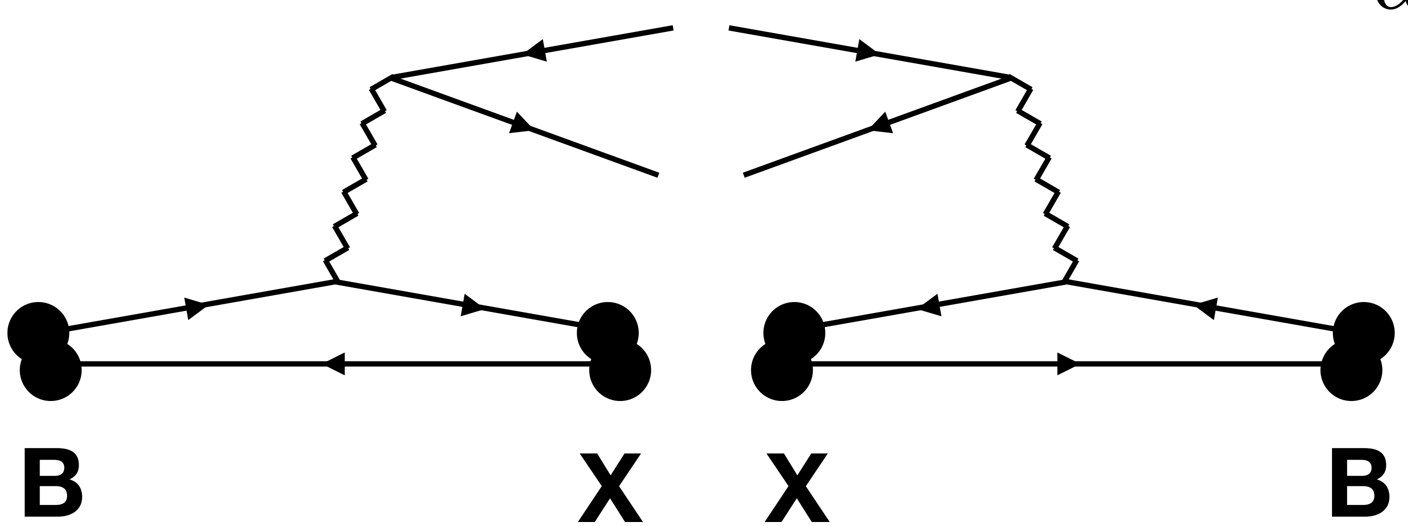
# Frontier LQCD calculations

The physics of Euclidean correlation functions

- 2-point functions: masses, decay constants
- 3-point functions: form factors
- 4-point functions:

$$d\sigma \propto \mathcal{M}\mathcal{M}^\dagger$$

$$\sum_X$$



# Frontier LQCD calculations

The physics of Euclidean correlation functions

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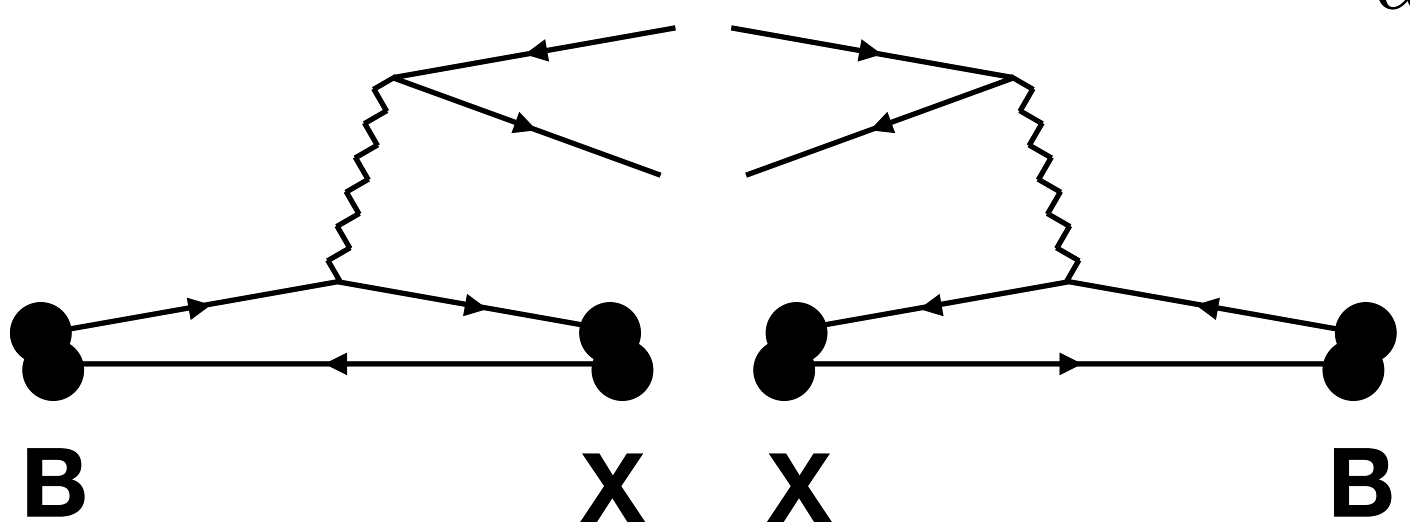
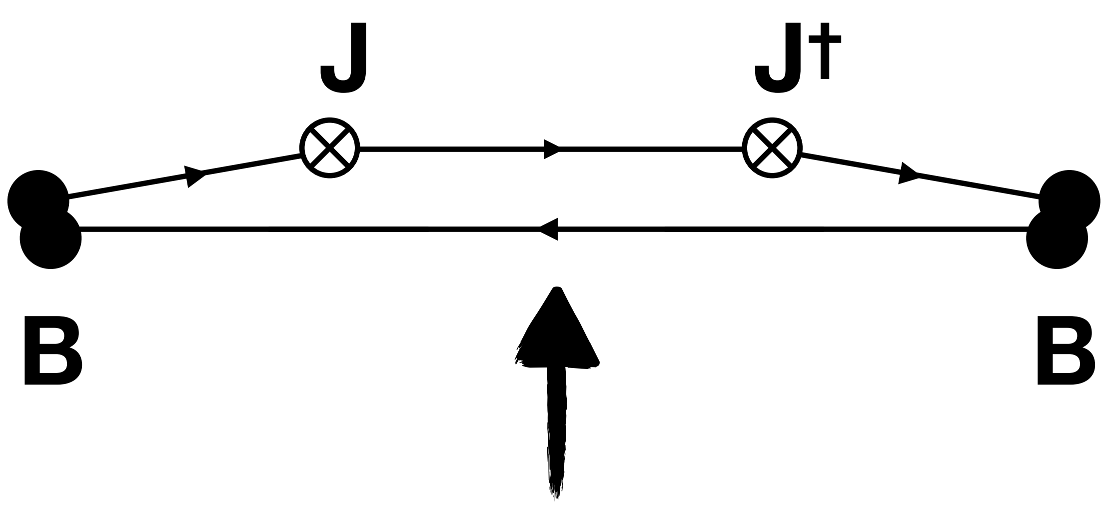



- In quantum mechanics: $\sum_X |X\rangle \langle X| = 1$

# Frontier LQCD calculations

The physics of Euclidean correlation functions

- 2-point functions: masses, decay constants
- 3-point functions: form factors
- 4-point functions:



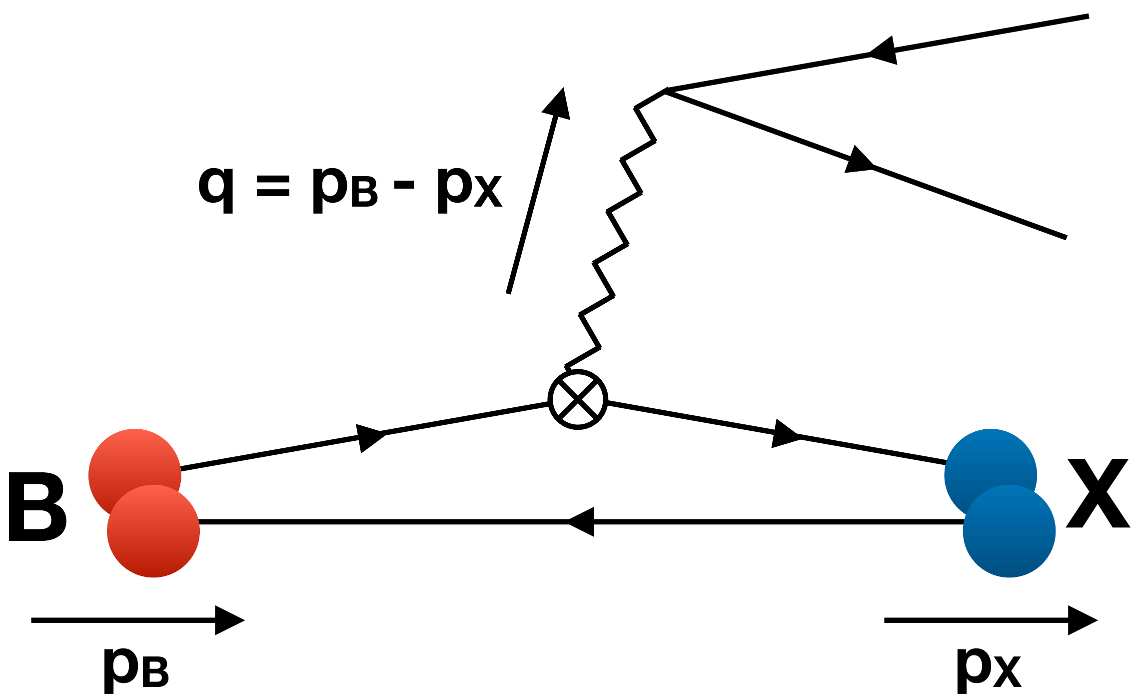

$\propto W_{\mu\nu}$

**Automatic sum over intermediate states X**

**“Hadronic tensor” ⟺ structure functions**

# Connection to observables

- Experiments measure rates and cross sections




**Leptonic tensor (know analytically)**

$$\frac{d^3\Gamma}{dE_\ell dq^2 dq^0} \propto G_F^2 |V_{cb}|^2 L_{\mu\nu} W^{\mu\nu}$$

**Hadronic tensor: target for LQCD**

Basically**:**

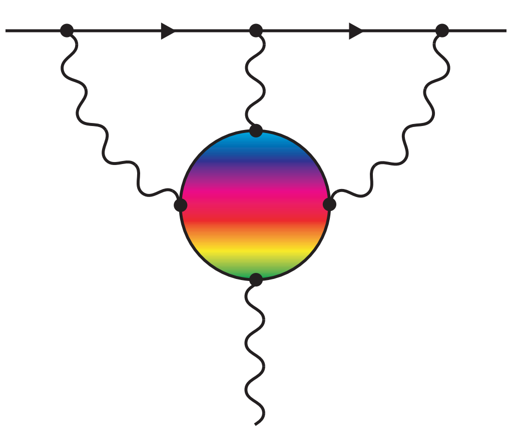
**“(Lorentz stuff) x (invariant functions)”**

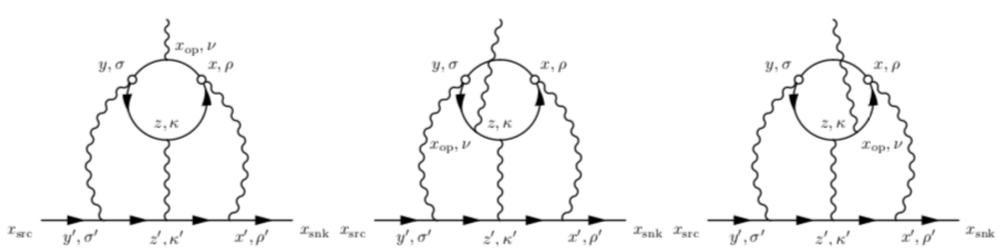
# Technical challenges

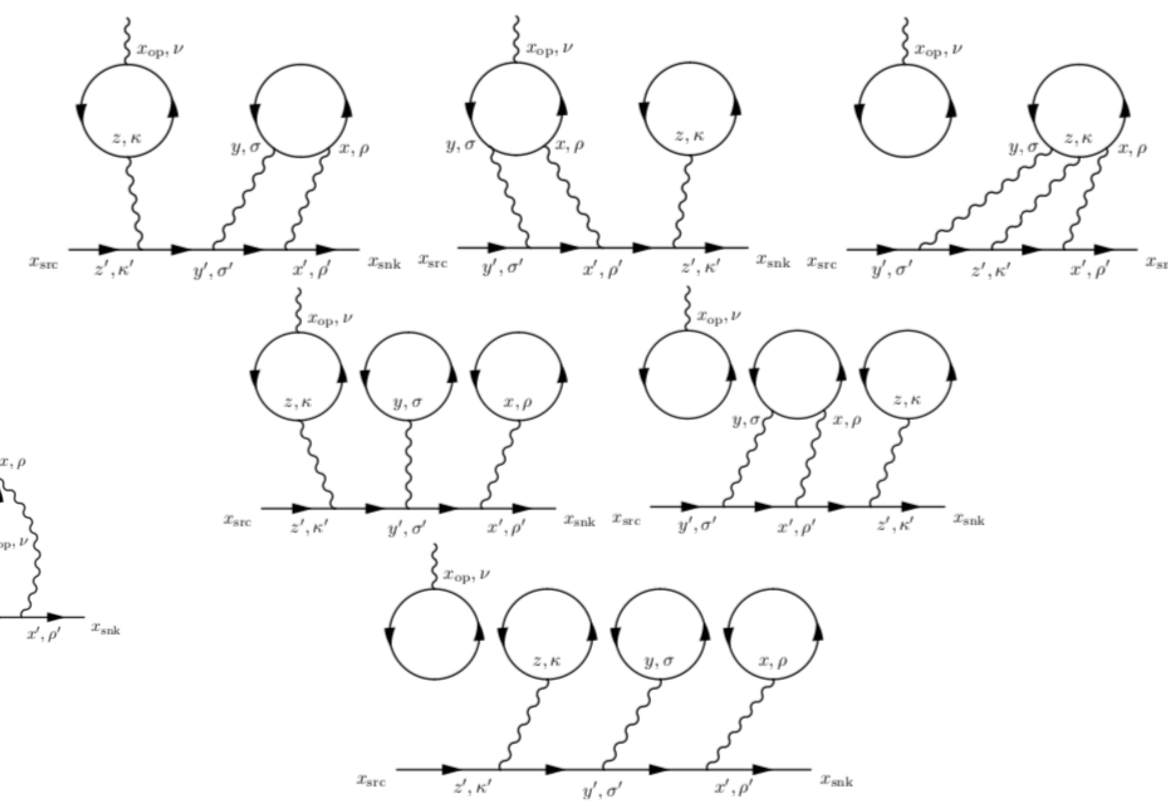
- **Algorithmic**: how to compute challenging 4pt functions efficiently using Monte Carlo techniques?
- **Theoretical**: how to relate Euclidean correlation functions to physical kinematic regime in Minkowski space?
- **Practical**: how to analyze finite, discrete simulation results for best precision?

# Technical challenges

- **Algorithmic**: how to compute challenging 4pt functions efficiently using Monte Carlo techniques?
  - 4pt functions require careful numerical treatment for all-to-all fermion propagators
  - Ex: State-of-the-art calculation of HLbL for (g-2)
  - T. Blum et al., PRL 124 (2020) 13, 132002







# Technical challenges

- **Theoretical**: how to relate Euclidean correlation functions to physical kinematic regime in Minkowski space?

- **Practical**: how to analyze finite, discrete simulation results for best precision?

# Back to Minkowski space

- "Wick rotation" requires an inverse Laplace transform
- For finite simulation data, the problem is ill-posed
- Progress requires good theoretical ideas

Euclidean correlator

Kernel function

Physical "spectral density"

$$G(\tau_i) = \int_0^\infty d\omega K(\tau_i, \omega)\rho(\omega)$$

$$G_i = T_{ij}\rho_j$$

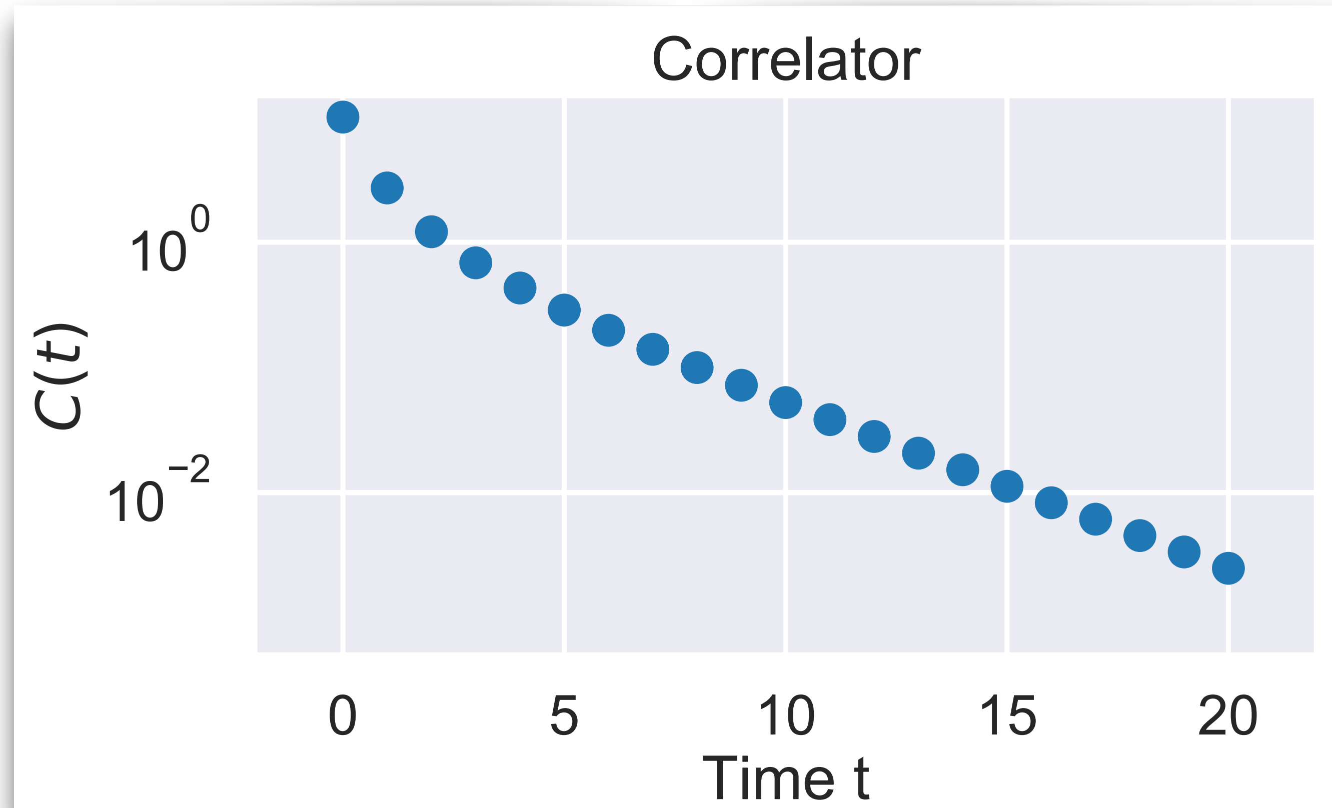
$$T_{ij} = \Delta\omega K_i(\omega_j)$$

Ill-posed, since $T_{ij}$ is not square ($i \ll j$)

# Quantum Mechanics in a Box

$$C(t) = \langle 0|\pi(t)\pi(0)|0\rangle$$

$$C(t) = \sum_n |Z_n|^2 e^{-E_n t}$$



# Quantum Mechanics in a Box

$$C(t) = \langle 0 | \pi(t) \pi(0) | 0 \rangle$$

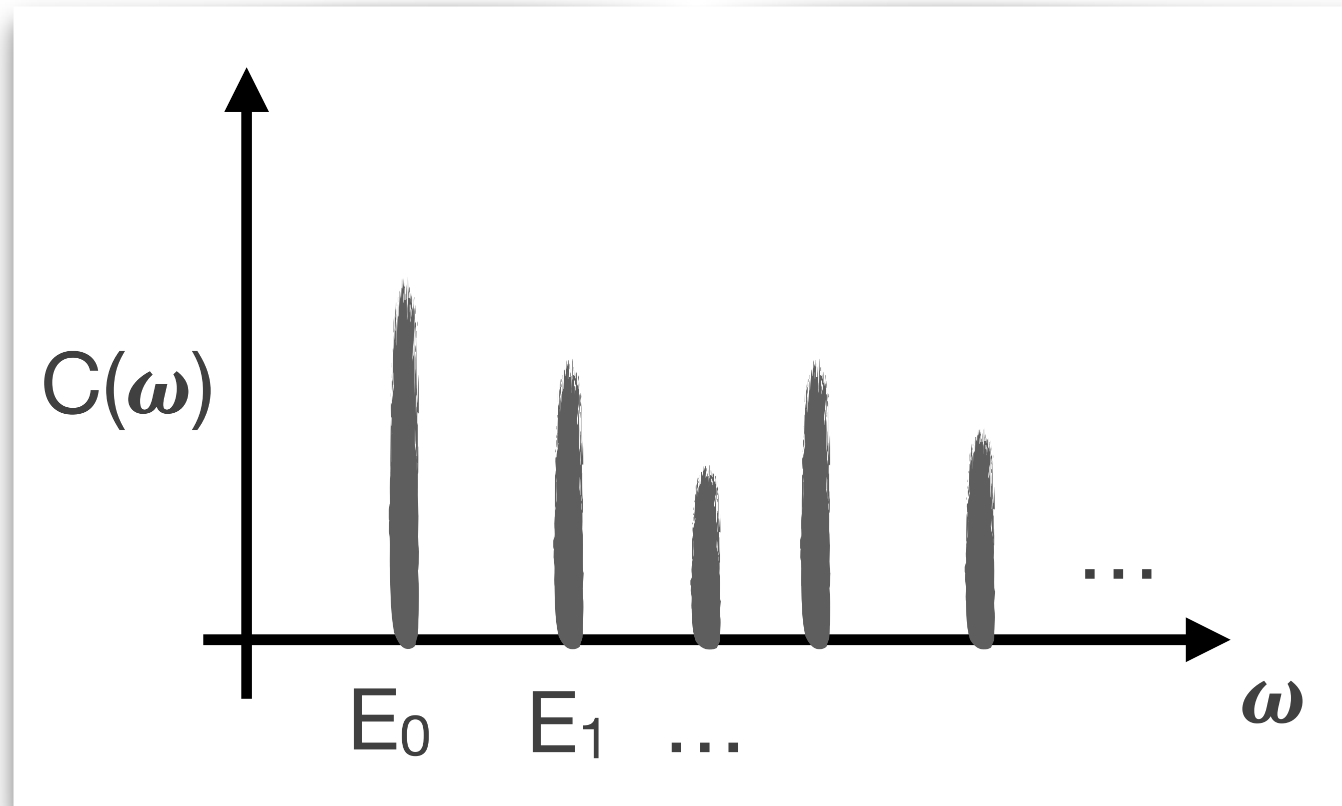
$$C(t) = \sum_n |Z_n|^2 e^{-E_n t}$$

Laplace transform

$$C(\omega) = \sum_n |Z_n|^2 \delta(\omega - E_n)$$

$$= \langle 0 | \mathcal{O} \, \delta(\hat{H} - \omega) \, \mathcal{O} \, | 0 \rangle$$

$$= \langle \psi | \, \delta(\hat{H} - \omega) \, | \psi \rangle$$




Or for inclusive B-decays:

$$|\psi\rangle = J \mathcal{O} \, |0\rangle$$

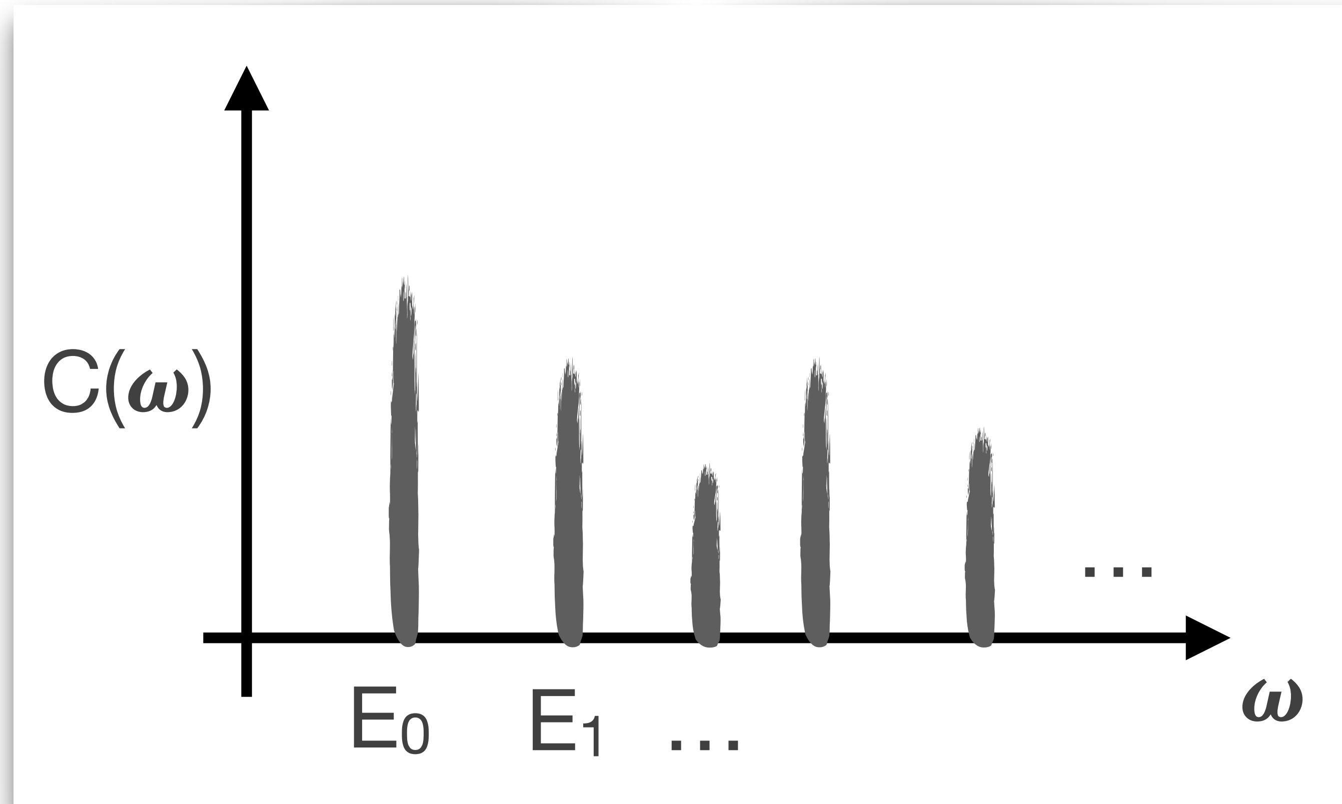
# Quantum Mechanics in a Box

$$C(t) = \langle 0|\pi(t)\pi(0)|0\rangle$$

$$C(t) = \sum_n |Z_n|^2 e^{-E_n t}$$

Laplace transform

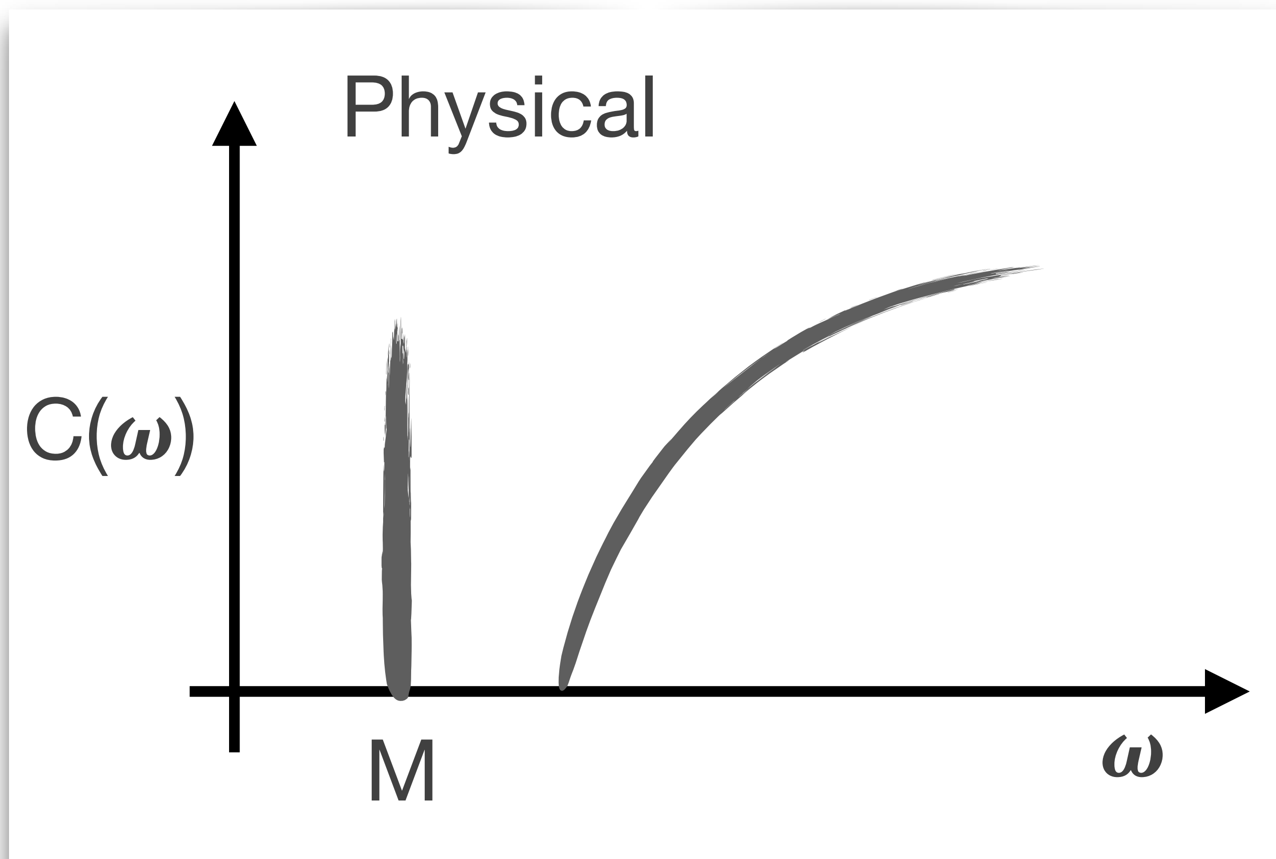
$$C(\omega) = \sum_n |Z_n|^2 \delta(\omega - E_n)$$

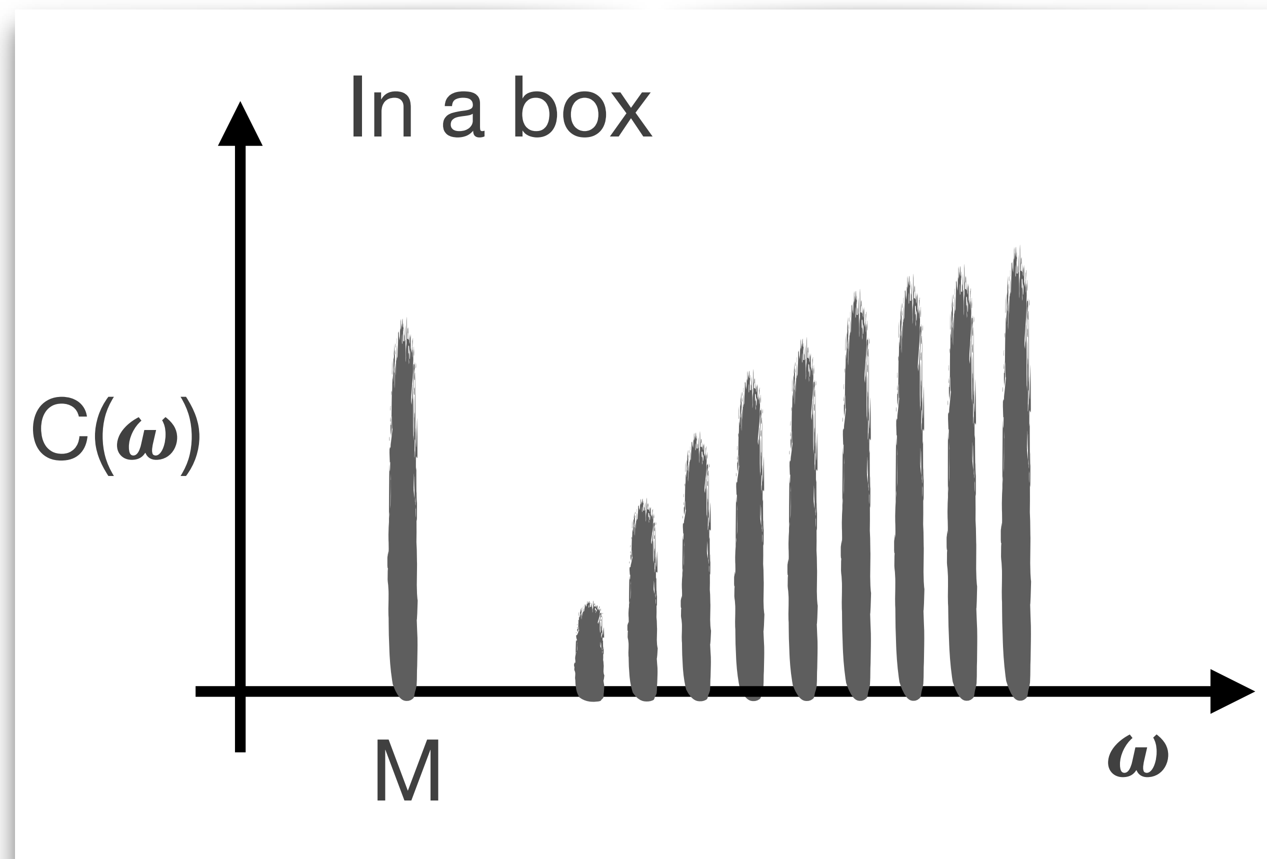



- This situation is completely generic
- Systems in a box have discrete energy levels

# Quantum Mechanics in a Box

- What about hadronic tensor W(***ω***, **q**)?
- Elastic channel: $\propto \delta(\omega - M)$
- Inelastic thesholds: $\propto \Theta(\omega - E_{\text{thresh}}) \times (\text{phase space})$



# Quantum Mechanics in a Box

- Somehow must connect these two pictures
- Try smearing
- (Actual methods quite different, but a classic idea)

PHYSICAL REVIEW D VOLUME 13, NUMBER 7 1 APRIL 1976

**Smearing method in the quark model***

E. C. Poggio, H. R. Quinn,† and S. Weinberg
*Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138*
(Received 8 December 1975)

We propose that perturbation theory in a gauge field theory of quarks and gluons can be used to calculate physical mass-shell processes, provided that both the data and the theory are smeared over a suitable energy range. This procedure is explored in detail for the process of electron-positron annihilation into hadrons. We show that a smearing range of 3 $GeV^2$ in the squared center-of-mass energy should be adequate to allow the use of lowest-order perturbation theory. The smeared data are compared with theory for a variety of models. It appears to be difficult to fit the present data with theory, unless there is a new charged lepton or quark with mass in the region 2 to 3 GeV.

# Some recent proposals

**Inclusive decays, specifically**

- M.T. Hansen, Meyer, Robaina: PRD 96 (2017) 9, 094513. arXiv: 1704.08993
- Hashimoto PTEP (2017) 5, 053B03, arXiv:1703.01881
- Gambino and Hashimoto: PRL 125 (2020) 3, 032001. arXiv: 2005.13730 → **See next talk from P. Gambino**

**General aspects of the inverse problem**

- M. Hansen, Lupo, and Tantalo PRD 99 (2019) 9, 094508. arXiv: 903.06476
- M. Bruno and M.T. Hansen: arXiv:2012.11488

# Backus-Gilbert reconstruction

PHYSICAL REVIEW D **96,** 094513 (2017)

**From deep inelastic scattering to heavy-flavor semileptonic decays: Total rates into multihadron final states from lattice QCD**

Maxwell T. Hansen,[1,*] Harvey B. Meyer,[1,2,†] and Daniel Robaina[3,‡]

[1]*Helmholtz Institut Mainz, D-55099 Mainz, Germany*

[2]*PRISMA Cluster of Excellence and Institut für Kernphysik, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany*

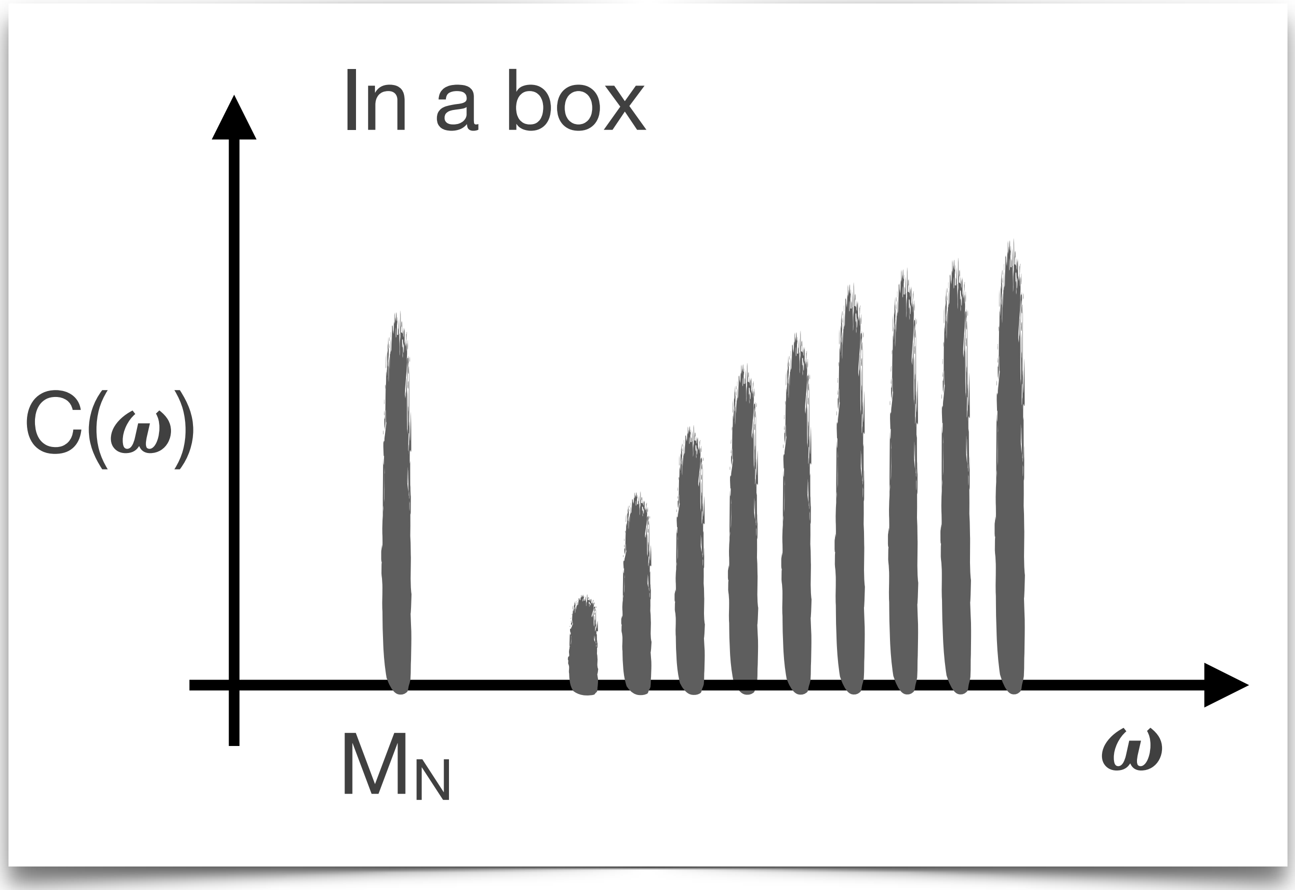
[3]*Institut für Kernphysik, Technische Universität Darmstadt, Schlossgartenstrasse 2, D-64289 Darmstadt, Germany*

(Received 23 May 2017; published 29 November 2017)

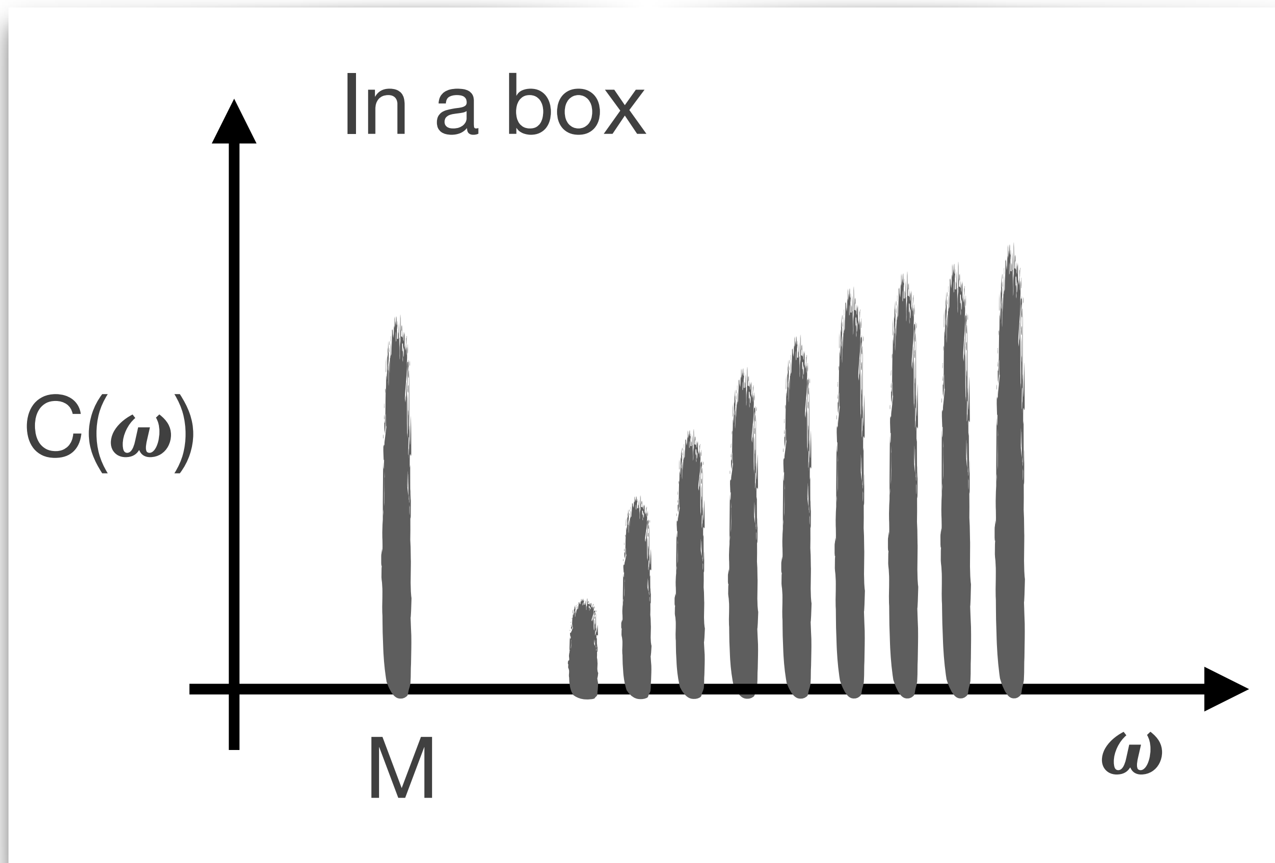
We present a new technique for extracting decay and transition rates into final states with any number of hadrons. The approach is only sensitive to total rates, in which all out-states with a given set of QCD quantum numbers are included. For processes involving photons or leptons, differential rates with respect to the nonhadronic kinematics may also be extracted. Our method involves constructing a finite-volume Euclidean four-point function, with a corresponding spectral function that measures the decay and transition rates in the infinite-volume limit. This requires solving the inverse problem of extracting the spectral function from the correlator and also necessitates a smoothing procedure so that a well-defined infinite-volume limit exists. Both of these steps are accomplished by the Backus-Gilbert method, and, as we show with a numerical example, reasonable precision can be expected in cases with multiple open decay channels. Potential applications include nucleon structure functions and the onset of the deep-inelastic scattering regime, as well as semileptonic $D$ and $B$ decay rates.

DOI: 10.1103/PhysRevD.96.094513

# Backus-Gilbert reconstruction



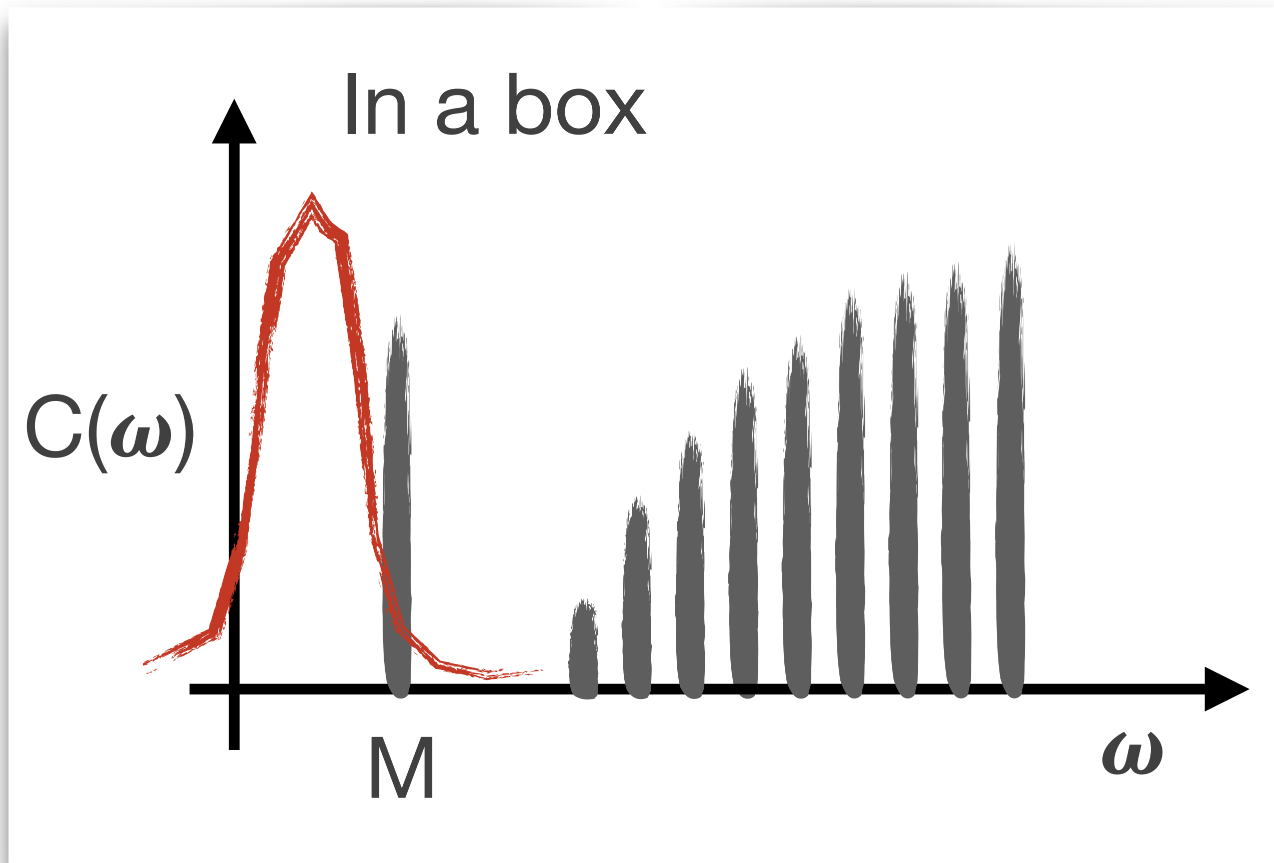
# Backus-Gilbert reconstruction




$$= \delta_\Delta(\omega, \omega')$$

Smearing kernel = Regulated **δ**-function

# Backus-Gilbert reconstruction

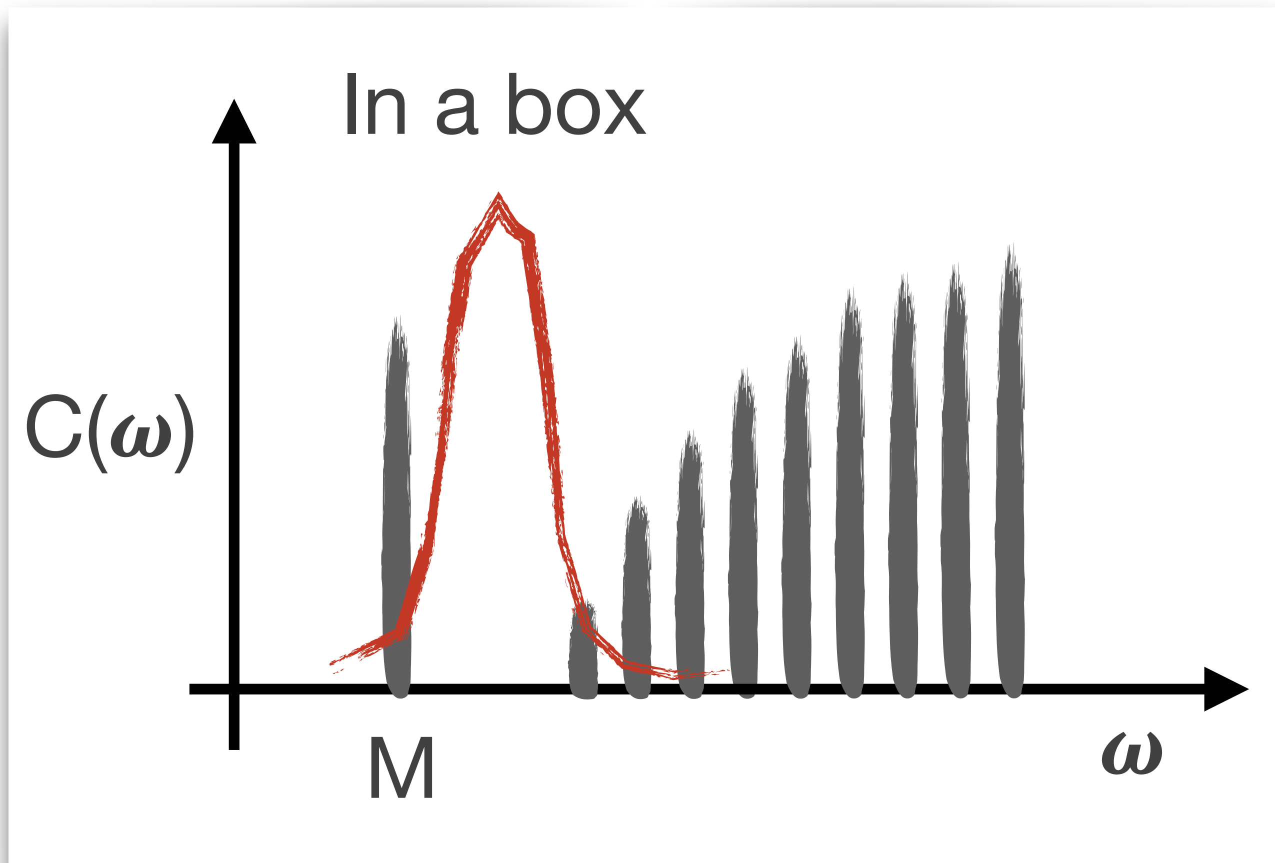



$$= \delta_\Delta(\omega, \omega')$$

$$C_\Delta(\omega) = \int d\omega' \, \delta_\Delta(\omega, \omega') C(\omega')$$

Smearing kernel = Regulated $\boldsymbol{\delta}$-function

# Backus-Gilbert reconstruction




$$= \delta_\Delta(\omega, \omega')$$

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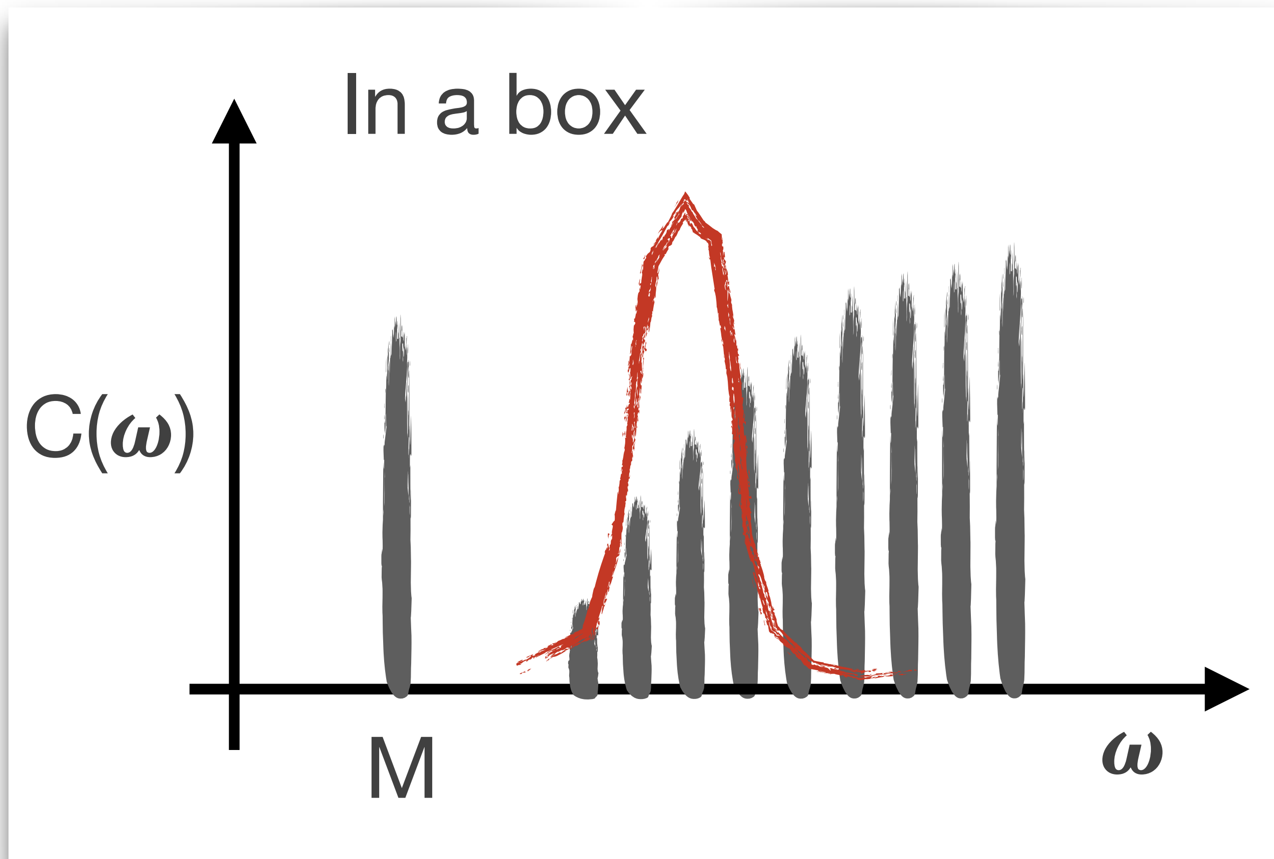
# Backus-Gilbert reconstruction

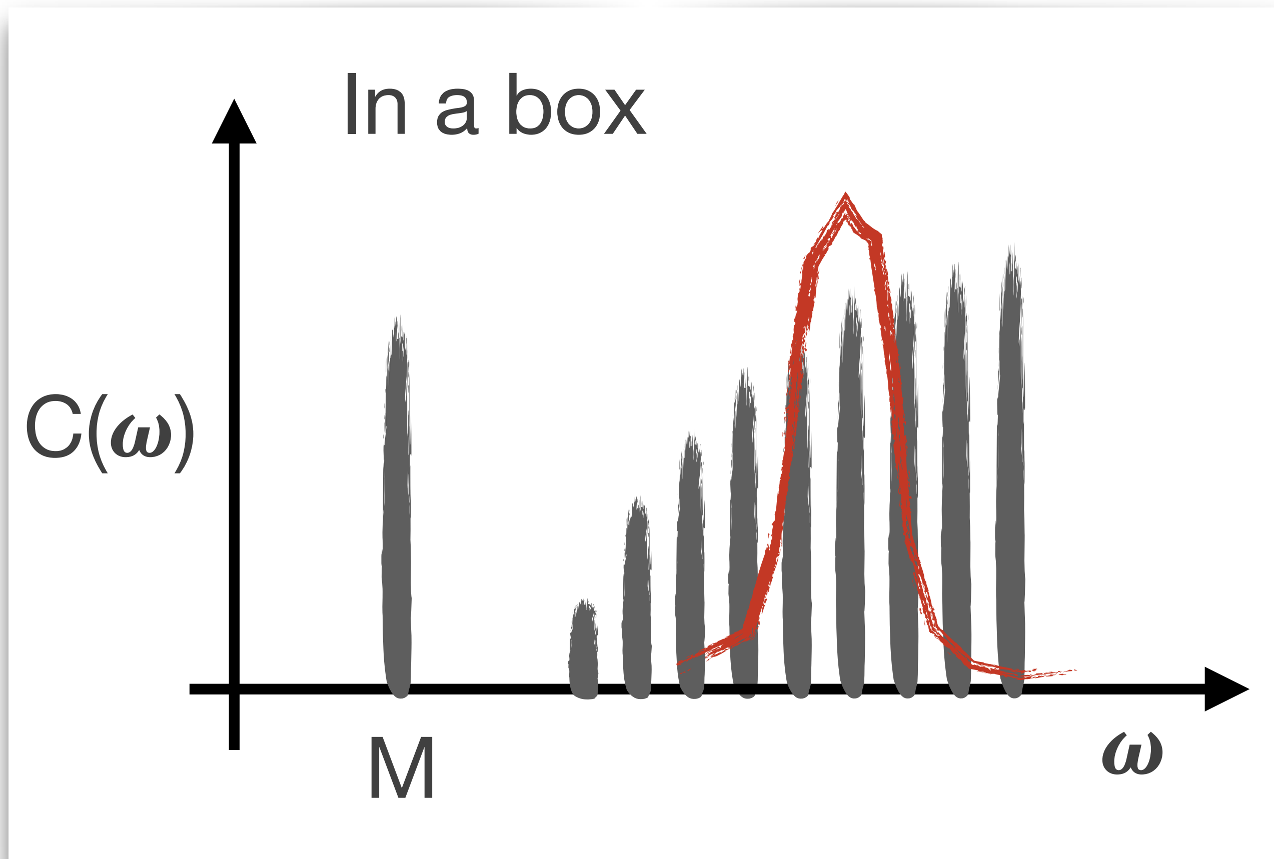



 $= \delta_{\Delta}(\omega, \omega')$

$$C_{\Delta}(\omega) = \int d\omega' \, \delta_{\Delta}(\omega, \omega') C(\omega')$$

Smearing kernel = Regulated **δ**-function

# Backus-Gilbert reconstruction

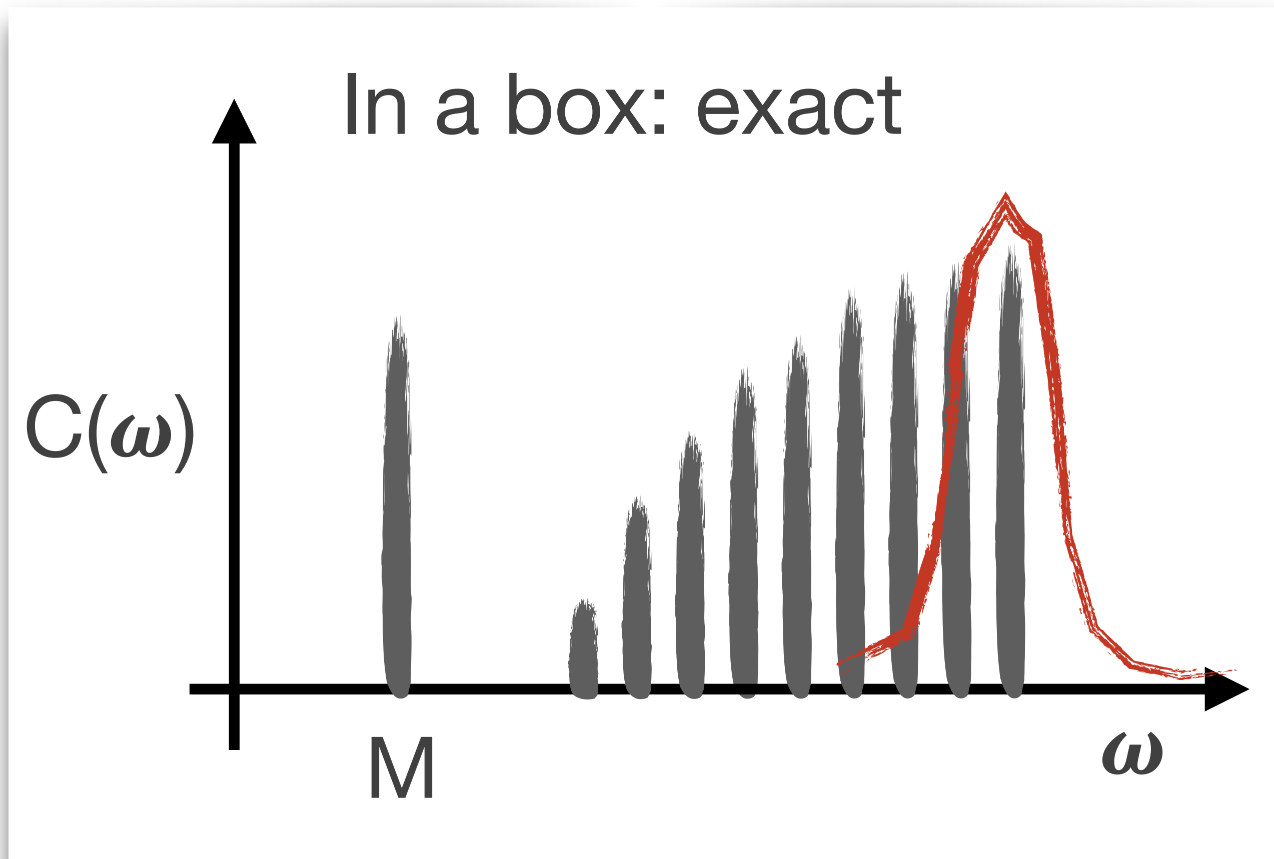


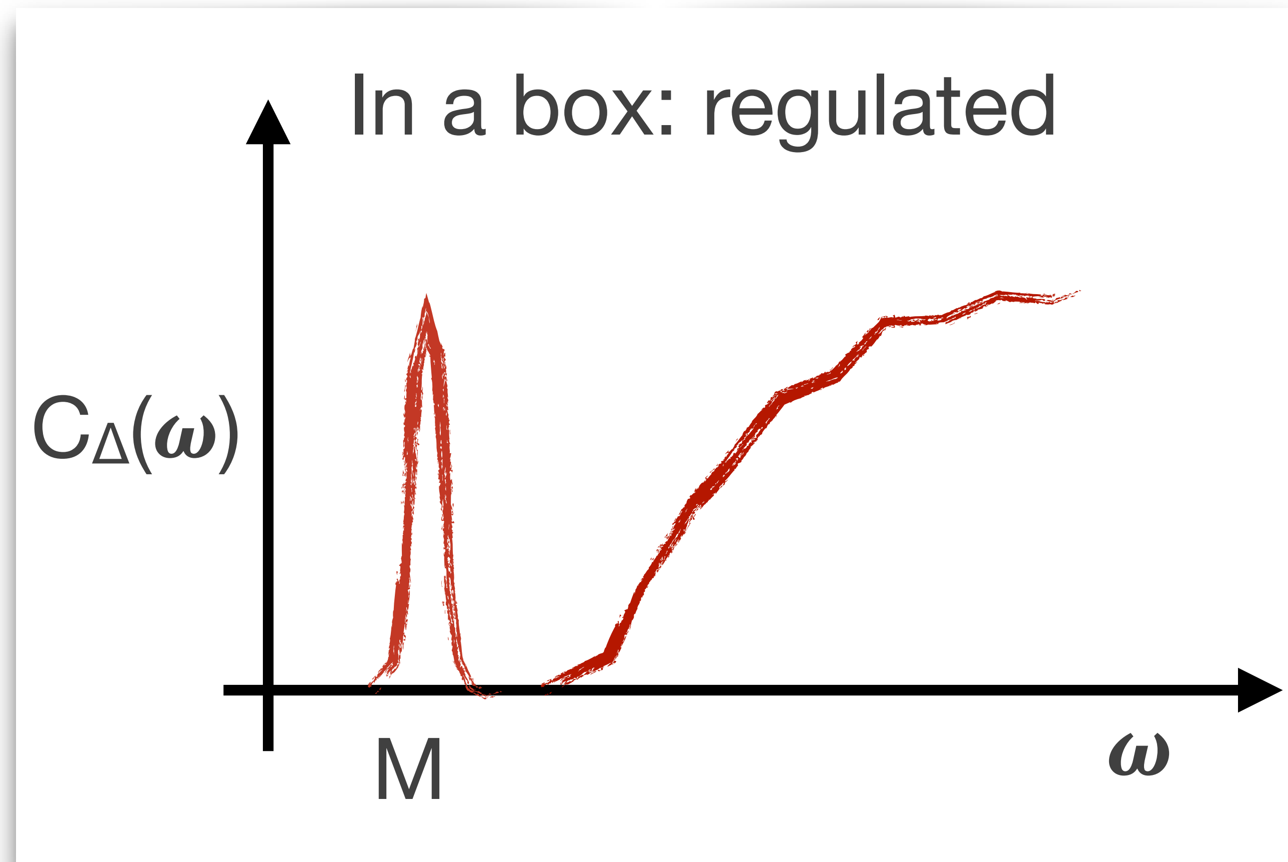

 $= \delta_\Delta(\omega, \omega')$

$$C_\Delta(\omega) = \int d\omega' \, \delta_\Delta(\omega, \omega') C(\omega')$$

Smearing kernel = Regulated $\boldsymbol{\delta}$-function

# Backus-Gilbert reconstruction




$$= \delta_\Delta(\omega, \omega')$$

$$C_\Delta(\omega) = \int d\omega' \, \delta_\Delta(\omega, \omega') C(\omega')$$

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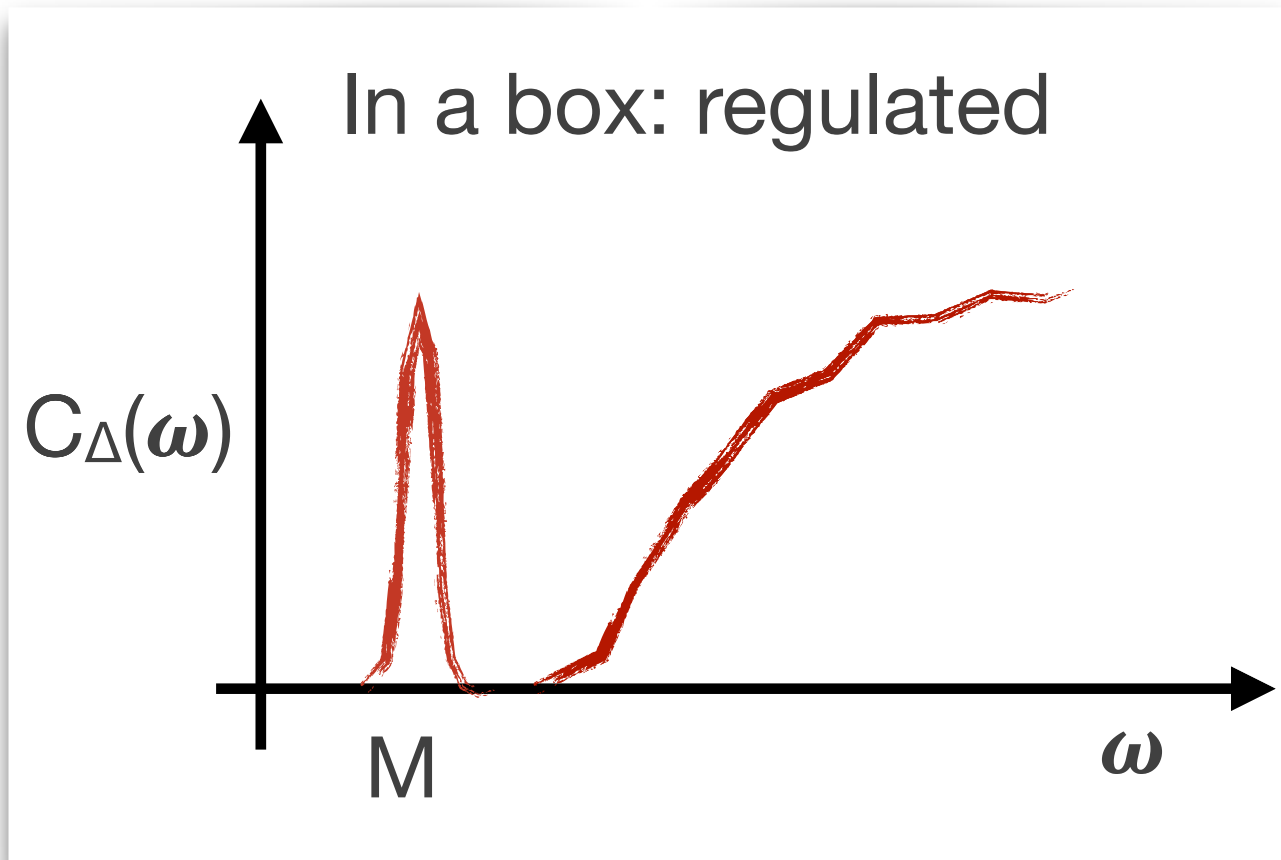
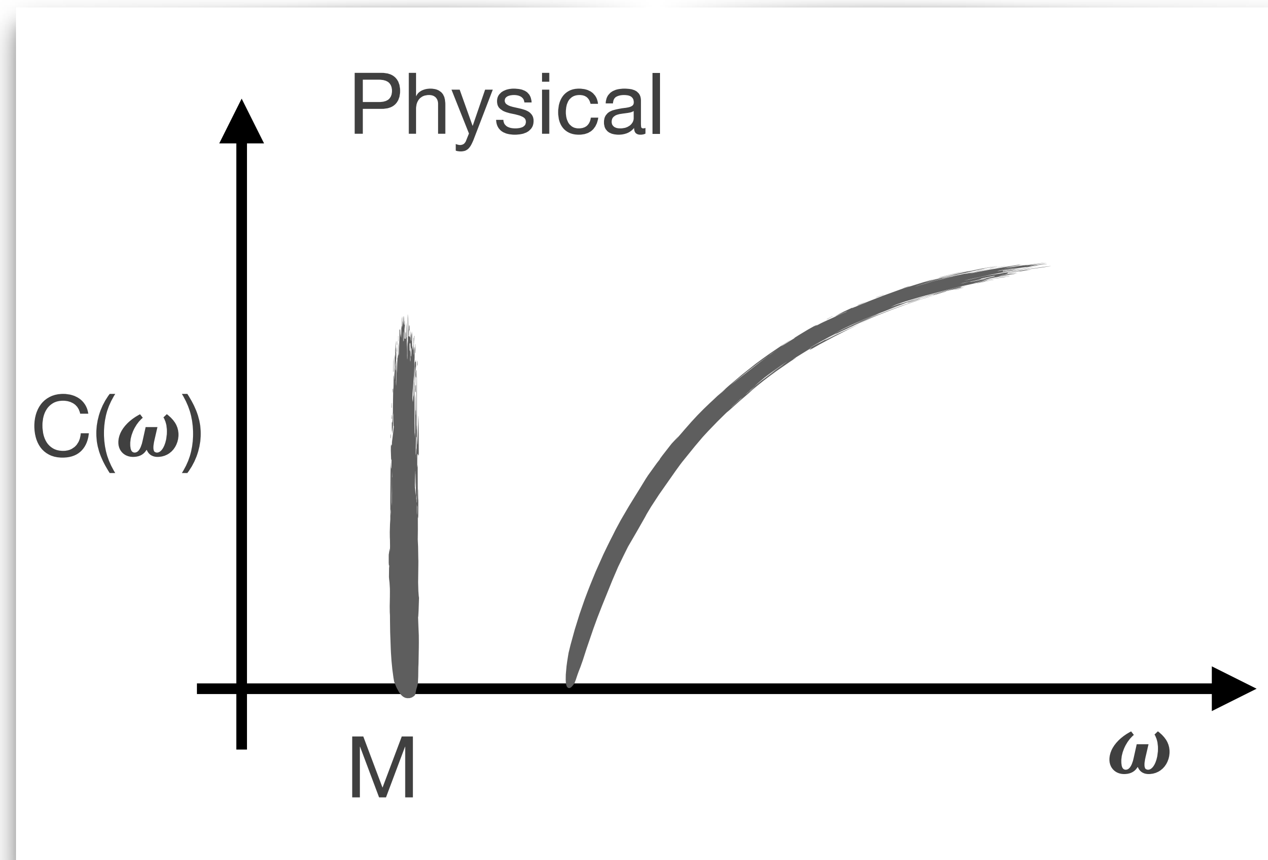
# Backus-Gilbert reconstruction




- Take limit to reach physical result

$$\lim_{\Delta \to 0} \lim_{\text{box} \to \infty} C_\Delta(\omega) \longrightarrow C(\omega)$$

# Backus-Gilbert reconstruction

1. Start with basic problem $$C(\tau_i) = \int d\omega e^{-\omega\tau_i}\rho(\omega)$$

2. Ansatz for solution $$\hat{\rho}(\omega) = \sum_j q(\tau_j, \omega) C(\tau_j)$$

3. Constraints? Put 1→2

$$\begin{aligned}
\hat{\rho}(\omega) &= \sum_j q(\tau_j, \omega) \times \int d\omega'\, e^{-\omega'\tau_j}\rho(\omega') \\
&= \int d\omega' \sum_j q(\tau_j, \omega') e^{-\omega'\tau_j}\rho(\omega) \\
&\equiv \int d\omega\, \hat{\delta}_\Delta(\omega, \omega')\rho(\omega)
\end{aligned}$$

# Backus-Gilbert reconstruction

1. Start with basic problem $$C(\tau_i) = \int d\omega e^{-\omega \tau_i} \rho(\omega)$$
2. Ansatz for solution $$\hat{\rho}(\omega) = \sum_j q(\tau_j, \omega) C(\tau_j)$$
3. Constraints? Put 1→2
4. Identify a delta function

$$\sum_j q(\tau_j, \omega') e^{-\omega' \tau_j} \equiv \hat{\delta}_\Delta(\omega, \omega')$$

5. If suitable q are found, Ansatz gives the solution:

$$\hat{\rho}(\omega) = \sum_j q(\tau_j, \omega) C(\tau_j)$$

# Backus-Gilbert reconstruction

- Backus Gilbert: “Choose q to minimize width of $\boldsymbol{\delta}_{\Delta}$”

$$\Delta(\omega) = \int d\omega' (\omega - \omega')^2 \hat{\delta}_{\Delta}(\omega, \omega')^2$$
$$\equiv q(\tau_i; \omega) W_{ij}(\omega) q(\tau_j, \omega)$$

- Regulate problem with the covariance matrix “S”:

$$\mathbf{q}(\omega) W(\omega) \mathbf{q}(\omega) \longrightarrow \mathbf{q}(\omega) \left[ W(\omega) + \lambda\, S \right] \mathbf{q}(\omega)$$

- Backus-Gilbert gives a *prescription* for defining the smearing kernel. The data determine its shape.
- $\boldsymbol{\lambda}$ is a free parameter. The shape of $\boldsymbol{\delta}$ is not.

# Backus-Gilbert reconstruction

- ■/■/■/■: regulated $\boldsymbol{\delta_\Delta(\omega)}$ (= ⋀) for different $\boldsymbol{\lambda}$

**(Using mock data from exactly treatable toy model)**

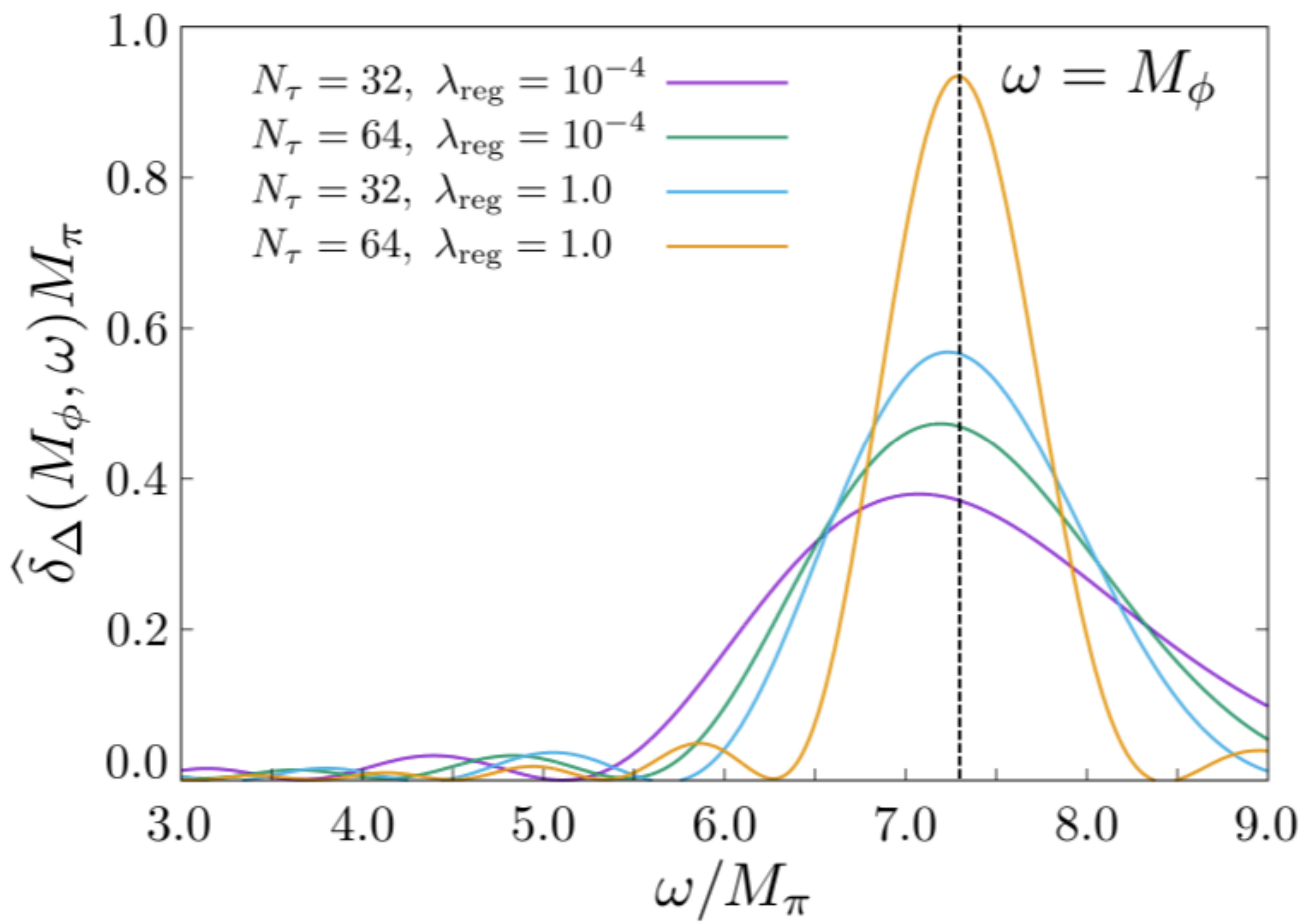



**Fig. 5 of M.T. Hansen, Meyer, Robaina arXiv:1704.08993**

# Backus-Gilbert reconstruction

- **Black line:** Exact result for toy problem
- **Backus-Gilbert smearing**

"To this end we reemphasize that one may also smear experimental or model data with the same resolution function to perform a fully controlled comparison."

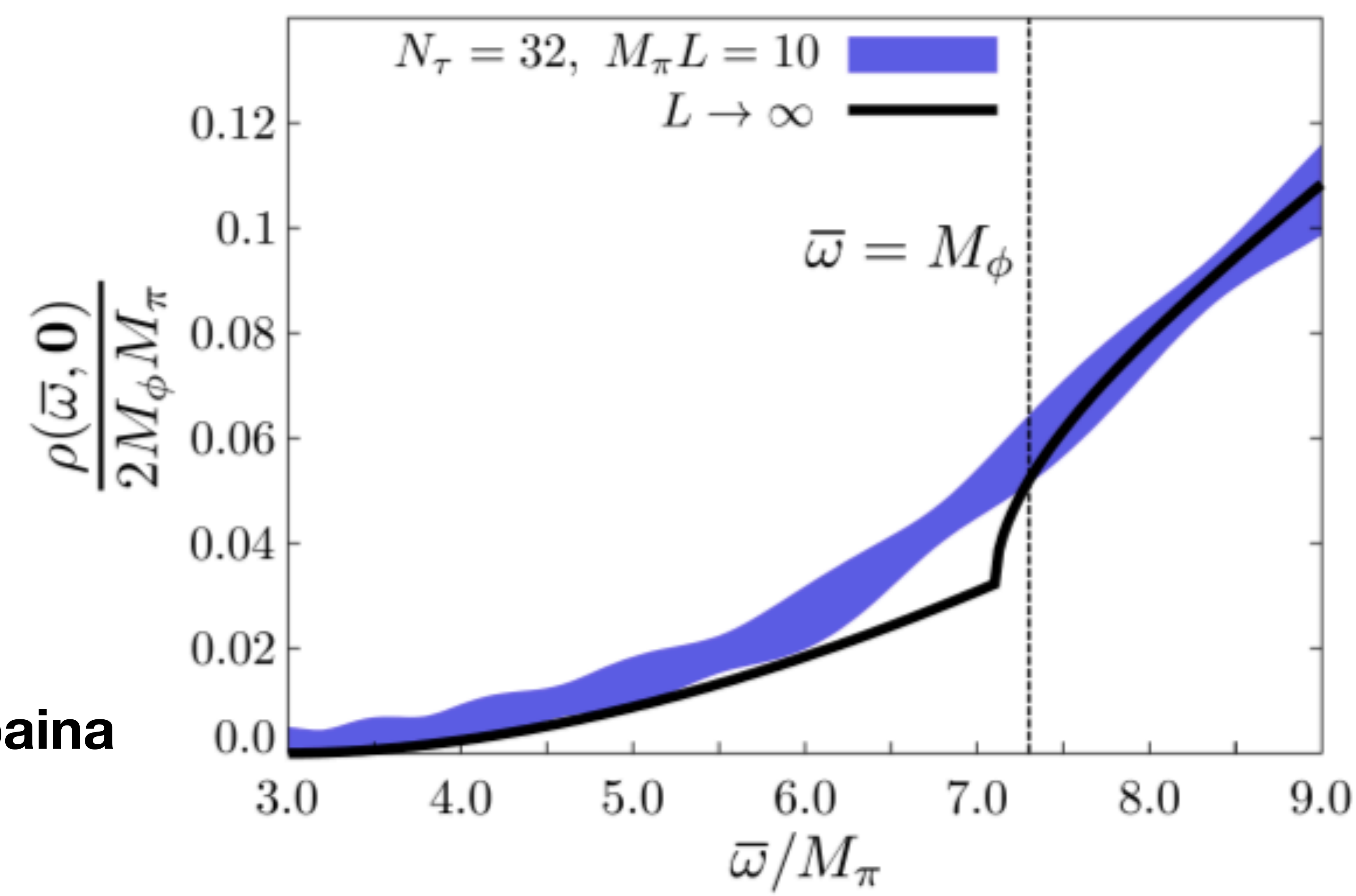



**Fig. 6 of M.T. Hansen, Meyer, Robaina arXiv:1704.08993**

# Backus-Gilbert reconstruction

- Ill-posed nature of unregulated problem $\Longleftrightarrow$ Results are very sensitive to small changes in data
- Phrase inversion problem as convex optimization
- Find kernel $\boldsymbol{\delta}_\Delta$ which minimizes the width functional
- Then work with smeared function $C_\Delta(\boldsymbol{\omega})$
- Upshots:
  - ☑ Smearing regulates and stabilizes the inversion
  - ☑ Smearing connects finite-volume to continuum

# Hansen, Lupo, & Tantalo

PRD 99 (2019) 9, 094508, arXiv:1903.06476

## On the extraction of spectral densities from lattice correlators

Martin Hansen,[1] Alessandro Lupo,[2] and Nazario Tantalo[3]

[1] *INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133, Rome, Italy.*

[2] *University of Rome Tor Vergata, Via della Ricerca Scientifica 1, I-00133, Rome, Italy.*

[3] *University of Rome Tor Vergata and INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133, Rome, Italy.*

Hadronic spectral densities are important quantities whose non–perturbative knowledge allows for calculating phenomenologically relevant observables, such as inclusive hadronic cross–sections and non–leptonic decay–rates. The extraction of spectral densities from lattice correlators is a notoriously difficult problem because lattice simulations are performed in Euclidean time and lattice data are unavoidably affected by statistical and systematic uncertainties. In this paper we present a new method for extracting hadronic spectral densities from lattice correlators. The method allows for choosing a smearing function at the beginning of the procedure and it provides results for the spectral densities smeared with this function together with reliable estimates of the associated uncertainties. The same smearing function can be used in the analysis of correlators obtained on different volumes, such that the infinite volume limit can be studied in a consistent way. While the method is described by using the language of lattice simulations, in reality it is completely general and can profitably be used to cope with inverse problems arising in different fields of research.

# Hansen, Lupo, & Tantalo

- Observation: It would be nice to be able to choose the smearing kernel.
- Idea: choose a different functional from BG

Minimize: $$\int d\omega \left[\hat{\delta}_\Delta(\omega,\omega') - \delta_\Delta(\omega,\omega')\right]^2$$

- As before, regulate with data covariance matrix

# Hansen, Lupo, & Tantalo

- Blue curve: target smearing function
- Red curve: HLT $\delta_\Delta(\omega)$
- Yellow curve: difference

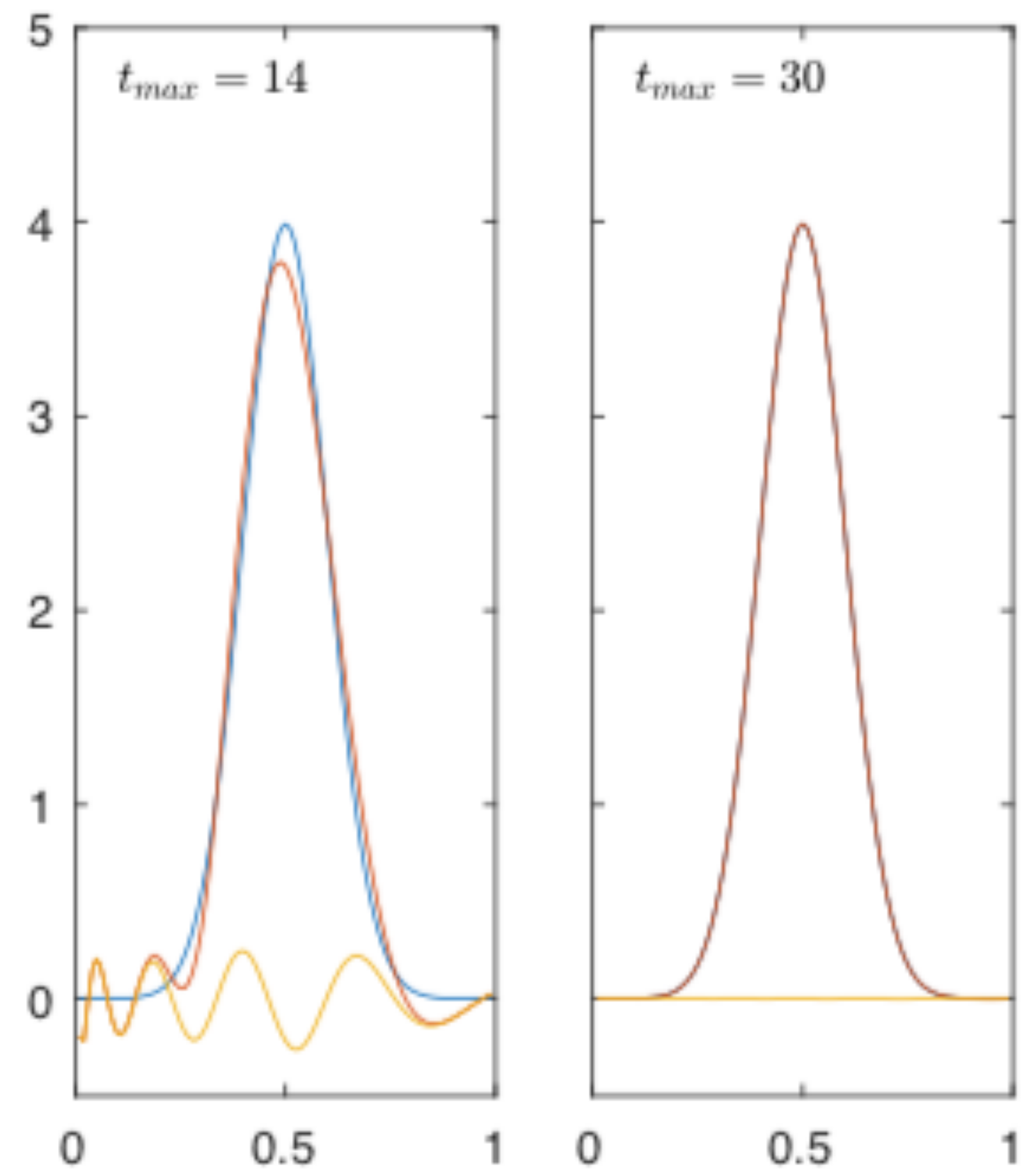



**Fig. 3 of Hansen, Lupo, Tantalo**
**arXiv:1903.06476**

# Hansen, Lupo, & Tantalo

- **Black line:** Exact <u>smeared</u> result for toy problem
- **Backus-Gilbert smearing**
- **HLT smearing**

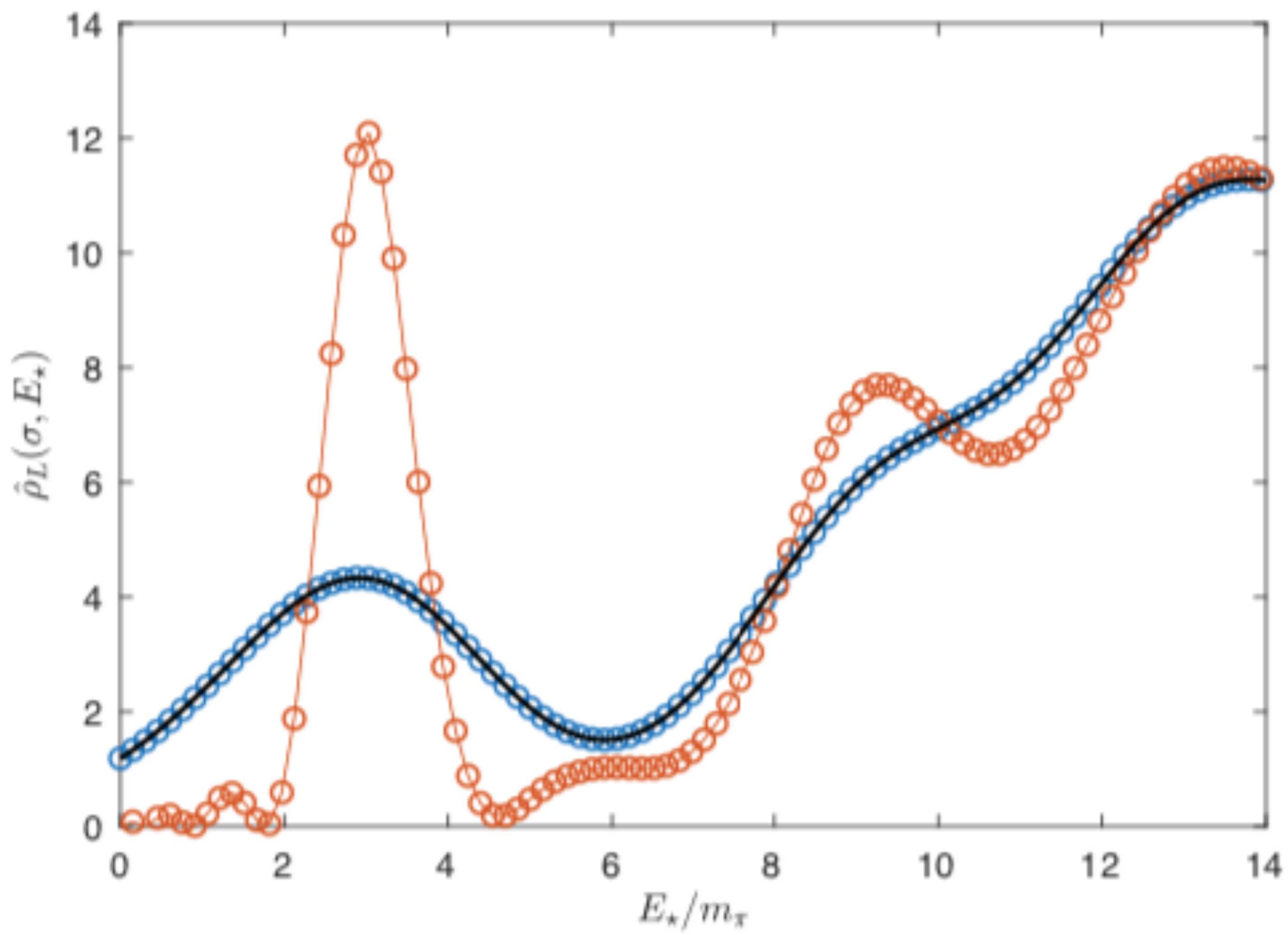



**Fig. 6 of Hansen, Lupo, Tantalo arXiv:1903.06476**

# Orthogonal polynomials

→ **See next talk from P. Gambino** for details

PHYSICAL REVIEW LETTERS **125,** 032001 (2020)

**Inclusive Semileptonic Decays from Lattice QCD**

Paolo Gambino
*Dipartimento di Fisica, Università di Torino and INFN, Torino Via P. Giuria 1, I-10125 Torino, Italy*

Shoji Hashimoto
*Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan*
*and School of High Energy Accelerator Science, The Graduate University for Advanced Studies (SOKENDAI), Tsukuba 305-0801, Japan*

(Received 28 May 2020; accepted 23 June 2020; published 14 July 2020)

We develop a method to compute the inclusive semileptonic decay rate of hadrons fully nonperturbatively using lattice QCD simulations. The sum over all possible final states is achieved by a calculation of the forward-scattering matrix elements on the lattice, and the phase-space integral is evaluated using their dependence on the time separation between two inserted currents. We perform a pilot lattice computation for the $\bar{B}_s \to X_c \ell \bar{\nu}$ decay with an unphysical bottom quark mass and compare the results with the corresponding OPE calculation. The method to treat the inclusive processes on the lattice can be applied to other processes, such as the lepton-nucleon inelastic scattering.

DOI: 10.1103/PhysRevLett.125.032001

# Orthogonal polynomials

- Recall: want $\rho(\omega) \propto \langle\psi| \delta(\hat{H} - \omega) |\psi\rangle$

- Have: $$C(\tau) = \langle\psi| e^{-H\tau} |\psi\rangle$$
$$= \langle\psi| z^{\tau} |\psi\rangle$$
$z = e^{-H}$ = transfer matrix

- Actually want smeared version:

$$\rho_\Delta(\omega) = \int d\omega' \delta_\Delta(\omega, \omega')\rho(\omega')$$
$$= \langle\psi| \delta_\Delta(\omega, \hat{H}) |\psi\rangle$$

# Orthogonal polynomials

- Smearing kernel is arbitrary / we can pick
- Choose, say, gaussian of width Δ
- Try expanding in orthogonal polynomials

$$\delta_\Delta(\omega, \hat{H}) = \frac{c_0(\omega)}{2} + \sum_j c_j(\omega) T_j(\hat{z})$$

Easy "Jackson" problem to compute coefficients $c_j(\boldsymbol{\omega})$ for any desired kernel

Matrix elements $\langle T_j(z) \rangle$ are just linear combos of $\langle z^t \rangle$, which is what we have!

# Orthogonal polynomials

- Computing ⟨$T_N(z)$⟩ for order N requires C(t) on N different time slices C(1), C(2), … C(N)

where the last term $\langle T_j^*(\hat{z})\rangle$ may be constructed from the correlator $\bar{C}(t)$ by replacing the power of the transfer matrix $\hat{z}^t$ appearing in $T_j^*(\hat{z})$ by $\bar{C}(t) = C(t+2t_0)/C(2t_0)$. Therefore, the first few terms are obtained as

$$\begin{aligned}
\langle T_0^*(\hat{z})\rangle &= 1, \\
\langle T_1^*(\hat{z})\rangle &= 2\bar{C}(1) - 1, \\
\langle T_2^*(\hat{z})\rangle &= 8\bar{C}(2) - 8\bar{C}(1) + 1, \\
\langle T_3^*(\hat{z})\rangle &= 32\bar{C}(3) - 48\bar{C}(2) + 18\bar{C}(1) - 1, \qquad (23) \\
&\vdots
\end{aligned}$$

# Orthogonal polynomials

- Computing $\langle T_N(z) \rangle$ for order N requires C(t) on N different time slices C(1), C(2), … C(N)
- Kernel choice is completely arbitrary
  - Could use Gaussian or dipole
  - Could also use “leptonic tensor” for process of interest

# Summary

- Understanding the discrepancy between inclusive and exclusive determinations of CKM elements is a long-standing problem
- Frontier calculations using numerical lattice QCD hold the promise of determining matrix elements for inclusive decays of B-mesons non-perturbatively
- The techniques developed will have broad applications throughout hadronic physics
- Reaching physical kinematics from Euclidean space requires solving a delicate inverse problem
- Look for important progress over the next 5-10 years.