Minimization of Theoretical Minimum Emittance in Storage Rings

Wen Wei Ho
Mentor: Dr Chun-xi Wang

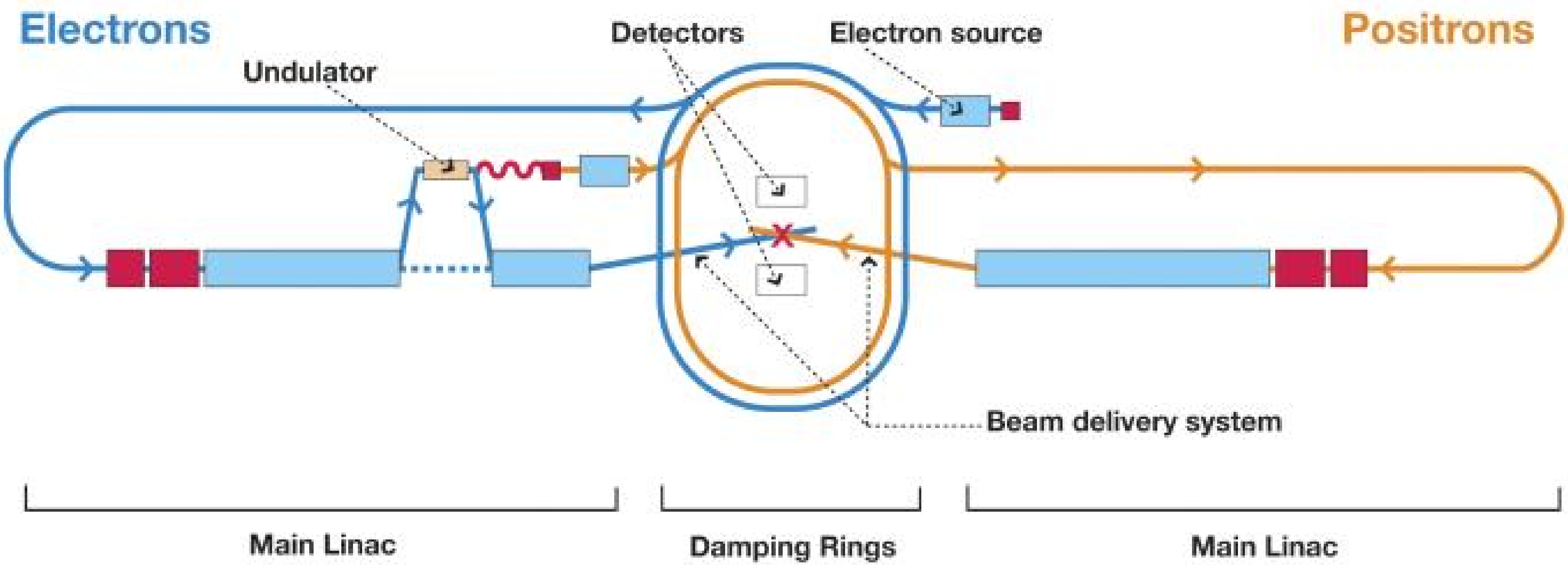
Misnomer... Minimization of the minimum?

- Emittance - area in phase space

  $\epsilon_x = C_q \gamma^2 \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{J_x \langle 1/\rho^2 \rangle}$ where $\mathcal{H} = \gamma \eta^2 + 2\alpha \eta \eta' + \beta \eta'^2$
- Emittance related to dispersion and Courant-Synder paramters
  - $\eta'' + \eta/\rho^2 = 1/\rho$
  - β, α, γ
- For a fixed bending profile
  - Can choose CS parameters β, α, and (hence) γ in such a way to minimize emittance
  - Theoretical minimum = TME
  - Achromatic minimum = ($\eta(0) = \eta'(0) = 0$) = AME
  - Effective minimum = EME
- For a fixed bending angle Θ
  - Can choose $\rho(s)$ to minimize the minimum emittance

Rationale

- Low emittance desirable
  - Greater brilliance of light from synchrotron radiation
  - Greater luminosity
  - Better emittance preservation in arcs
- Theoretical minimum
  - Theoretical interest
  - Practical value for machine design



Physics

- Balance between two effects
  - Radiation damping (synchrotron damping)
    - Serves to reduce phase space
  - Quantum excitation
    - Random fluctuations; causes emittance to be non-zero
- Ultimately, governed by Dispersion action:

$$\mathcal{H} = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2$$

$$\epsilon_x = C_q \gamma^2 \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{J_x \langle 1/\rho^2 \rangle}$$

- Usual treatment of subject in most literature

Physics

- New minimum-emittance theory in paper by C.-x. Wang, "Minimum emittance in storage rings with uniform or nonuniform dipoles", Phys. Rev. ST Accel. Beams 12,061001 (2009)
- Equations reduce to

$$\langle\langle \mathcal{H} \rangle\rangle \equiv \frac{\langle \mathcal{H}/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle} = \mathrm{Tr}(G_0 \sigma_0^+)$$ where

$$G_0 \equiv \check{\rho}\boldsymbol{\eta}_0\boldsymbol{\eta}_0^T + \boldsymbol{\eta}_0\langle\langle \hat{\boldsymbol{\xi}} \rangle\rangle^T + \langle\langle \hat{\boldsymbol{\xi}} \rangle\rangle\boldsymbol{\eta}_0^T + \langle\langle \hat{\boldsymbol{\xi}}\hat{\boldsymbol{\xi}}^T \rangle\rangle$$

$$\sigma^+ = \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix}$$ and

$$\langle\langle v \rangle\rangle \equiv \frac{\langle v/|\rho|^3 \rangle}{\langle 1/\rho^2 \rangle}$$

Physics

- One finds that:

$$2\sqrt{|A|}\begin{cases} 1 & \text{AME} \\ \sqrt{1-c} & \text{TME} \\ \sqrt{\frac{[1+(q+3)qc/2]\{1+[(1+\tau)q+3]qc/2\}}{1+qc}}, & \text{EME,} \end{cases}$$ where

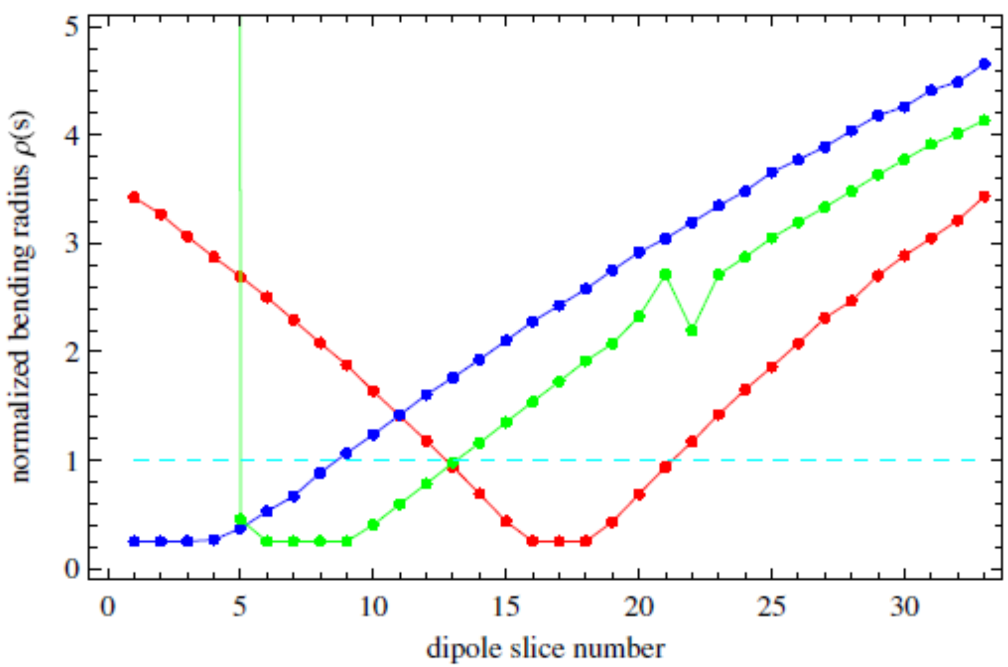
$A = \langle\langle \hat{\xi}\hat{\xi}^T \rangle\rangle$ and $B = \langle\langle \hat{\xi} \rangle\rangle \langle\langle \hat{\xi} \rangle\rangle^T / \check{\rho}$. and

$$c = -\frac{\mathrm{Tr}(JAJB)}{|A|}$$

- For any dipole, it is characterized by two numbers:
  - |A| and c
  - Minimal emittance conditions for a fixed profile is easy to calculate

Work done so far

- C.-x. Wang, Y. Wang, and Y. Peng, "Optimal dipole-field profiles for emittance reduction in storage rings", Phys. Rev. ST Accel. Beams 14, 034001 (2011)
- Evolutionary program used to find optimal bending profile ρ (s) (proportional to 1/B(s)) for TME, AME and EME



Work done so far

- Profiles are relatively simple
  - Amenable to piece-wise analytic approximations
  - Can obtain (approximate) analytic expressions for A and c
  - Can obtain (approximate) numerical values of emittance reduction factor (1/ Normalized TME , 1/Normalized AME or 1/Normalized EME) against $\kappa = B_{max}/B_{ref}$

Work done (Addendum to theory)

- Spotted an area overlooked in the theory
- Different values of A and c for different reference points; theory valid only for A and c calculated at beginning, but some reference points allow for easier calculations
- Together with mentor, developed the connection formulas to relate A and c from any point to the beginning

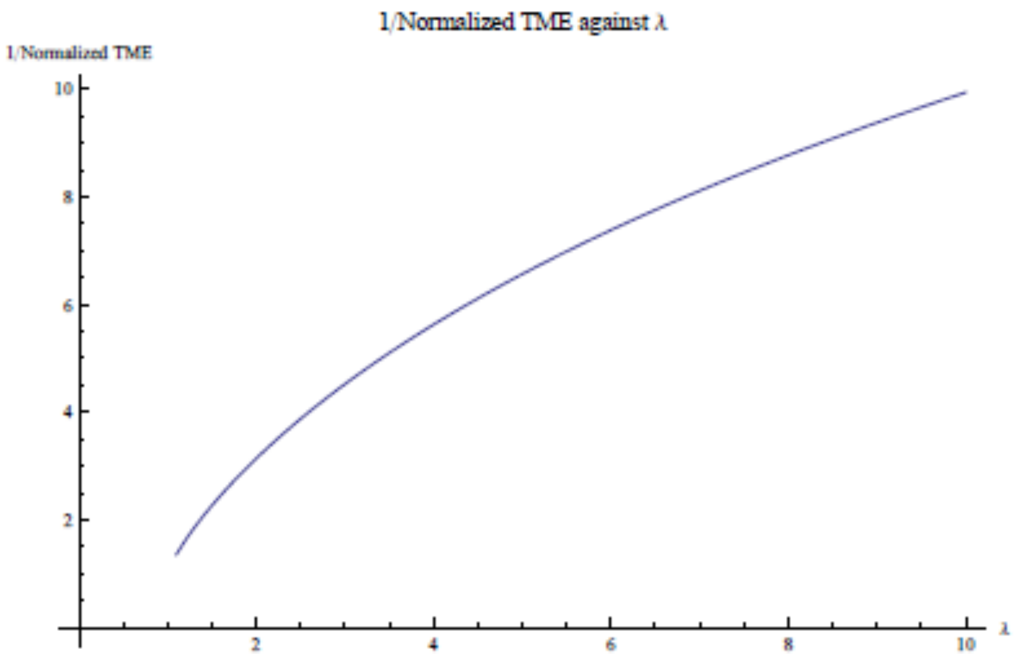
$$A_0 = M(s_0|s_1)^{-1}(A_1 - B_1 + \vec{\zeta}\vec{\zeta}^T)(M(s_0|s_1)^{-1})^T$$

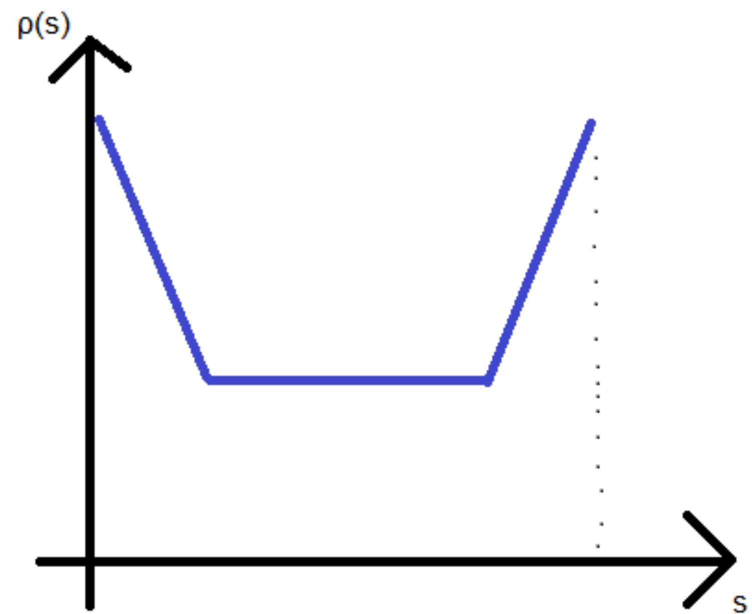
$c = 1 - |A_1 - B_1|/|A_0|$ where

$$\zeta \equiv \frac{\langle\!\langle \hat{\xi}_1 \rangle\!\rangle - \breve{\rho}\,\hat{\xi}_1(s_0)}{\sqrt{\breve{\rho}}}$$

Work done (TME)

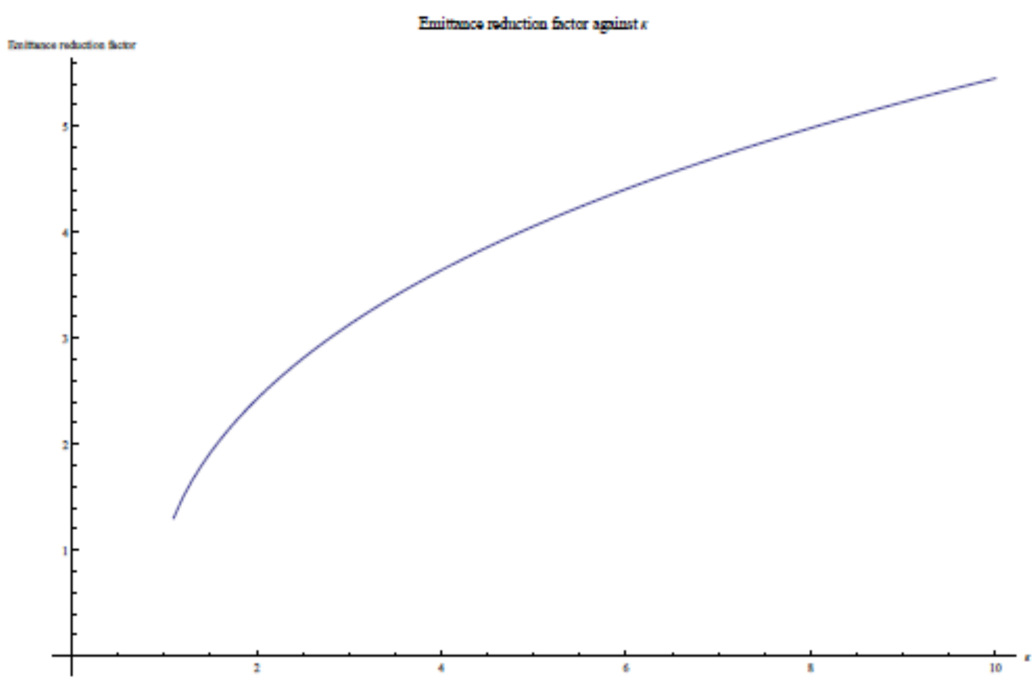
- Approximate analytic profile - linear ramp down, flat section, linear ramp up, symmetric
- Emittance improvement factor: $-1.214 + 2.826\kappa - 0.39712\kappa^2 + 0.034693\kappa^3 - 0.0012124\kappa^4$
- Matches 3 points obtained by program

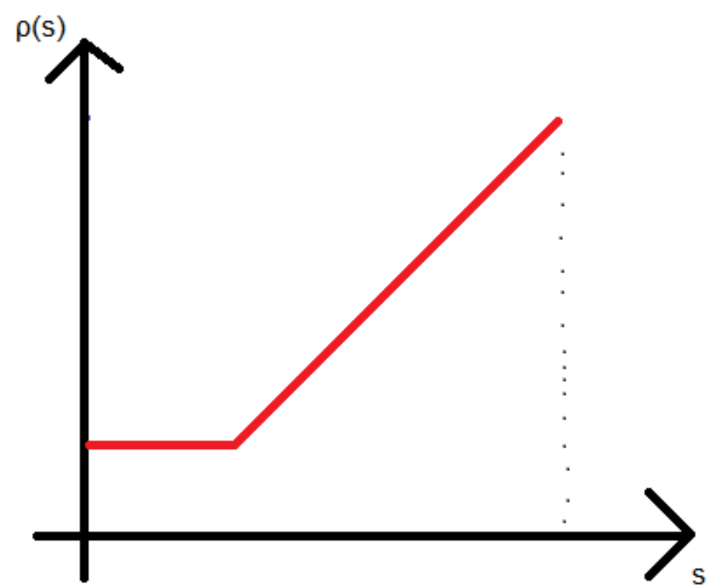




Work done (AME)

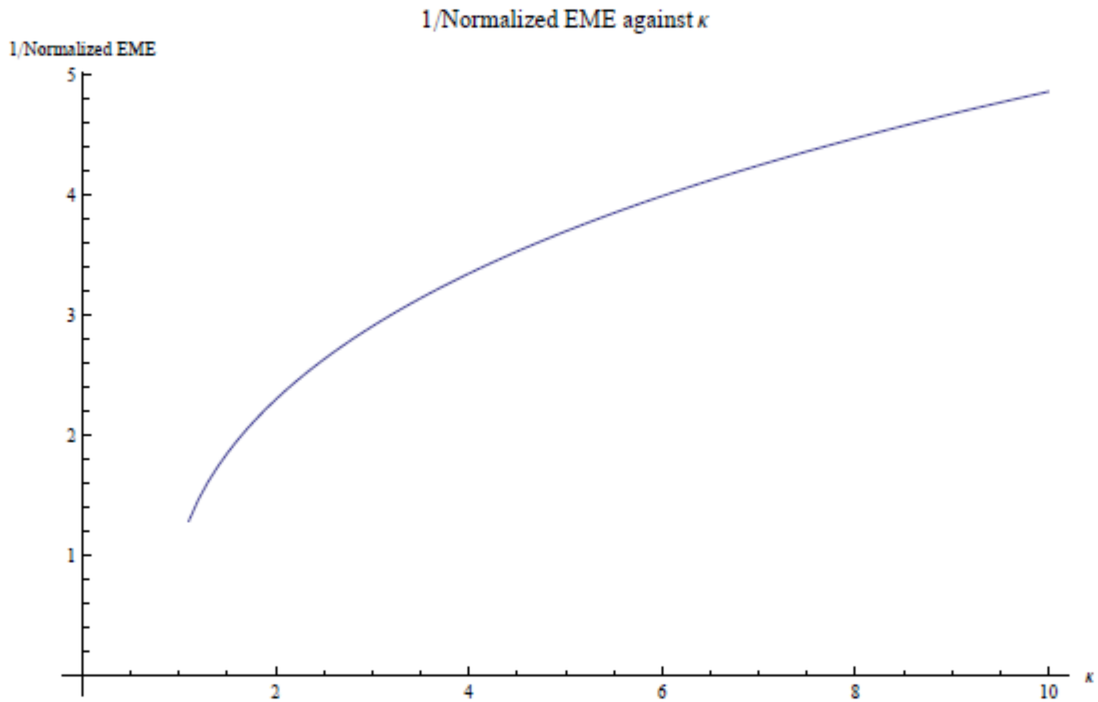
- Approximate analytic profile - linear ramp down, flat section, linear ramp up, asymmetric.
  - Optimal profile found to be have no linear ramp down
- Emittance improvement factor: $-0.2992 + 1.894\kappa - 0.3353\kappa^2 + 0.031983\kappa^3 - 0.001168\kappa^4$

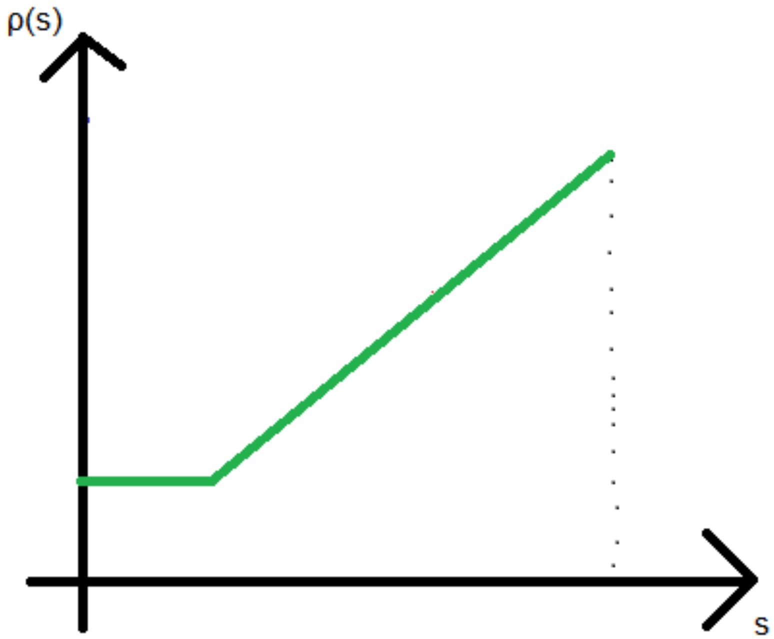




Work done (EME)

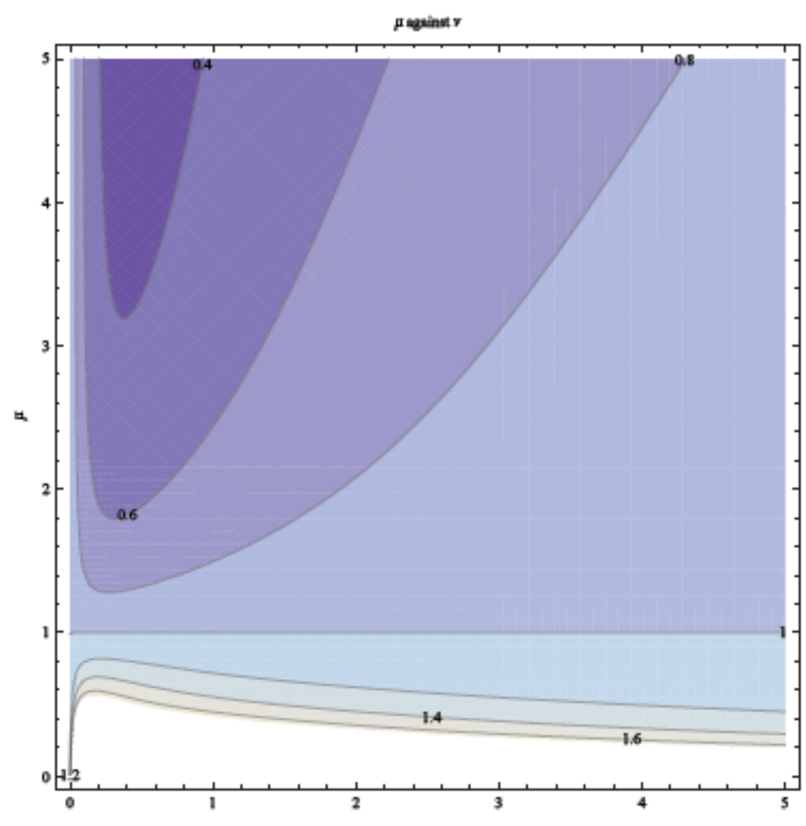
- Approximate analytic profile - drift section, flat section, linear ramp up
  - Optimal profile found to be have no drift section (!)
- Emittance improvement factor: $-0.1460 + 1.718\kappa - 0.3128\kappa^2 + 0.03013\kappa^3 - 0.001106\kappa^4$



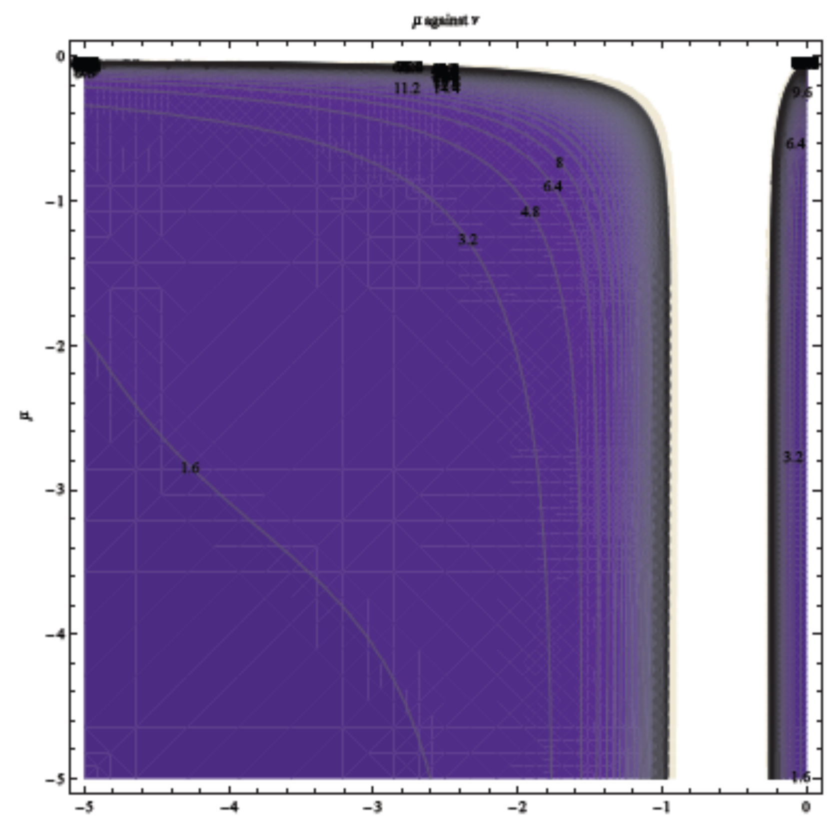


Work done (Sandwich dipole with reverse field)

- Applied theory to 'wiggler' - where field can switch polarity
- Obtained the following results (normalized TME)



(a) Contour plot of the normalized TME in the first quadrant.



(b) Contour plot of the normalized TME in the third quadrant.

Variational approach

- Attempt to apply functional analysis to <<H>>
- Form the "fundamental function"

$$f = \lambda_1 \left(\frac{\gamma(s)\eta(s)^2 + 2\alpha(s)\eta(s)\eta'(s) + \beta(s)\eta'(s)^2}{|\rho(s)|^3} \right)$$
$$+ \lambda_2 \left(\frac{1}{\rho(s)^2} \right)$$
$$+ \lambda_3 \left(\eta''(s) + \frac{1}{\rho(s)^2}\eta(s) - \frac{1}{\rho(s)} \right)$$
$$+ \lambda_4 \left(\beta(s)\beta''(s) - \beta'(s)^2 + \frac{4\beta(s)^2}{\rho(s)^2} - 4 \right)$$

- Apply Euler-Lagrange equations that include 2nd order terms:
    - $$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial y''} = 0$$

Variational approach

- Theoretically plausible
- Practically, equations intractable
- Offers little to no value in solving optimization problem

Conclusion

- Added addendum to minimum-emittance theory
- Obtained analytic expressions for A and c for optimal TME, AME and EME and the corresponding reduction factor
- Obtained analytic expressions for wiggler
- Derived transfer matrix for linear ramp
- Considered variational approach to optimization problem
- Finalizing results for publication in Phys. Rev. ST-AB

The End

Acknowledgments: Dr Chun-xi Wang
Question & Answer