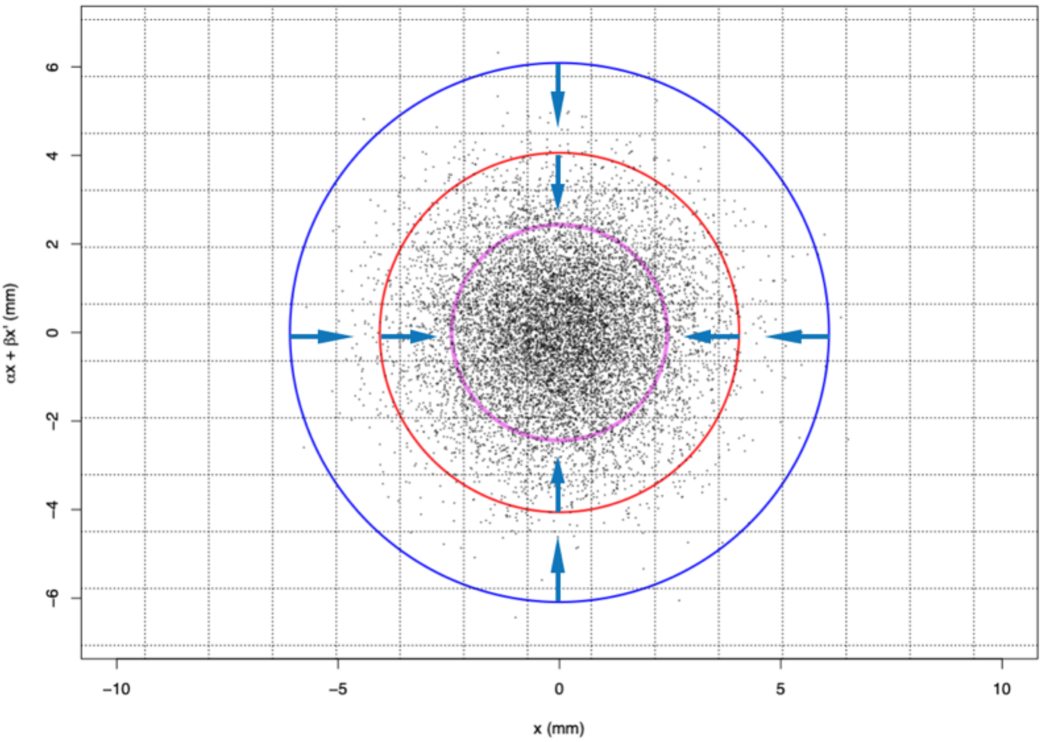
# 1 Computing Spill Quality Analytically

We know that as the system approaches the resonant tune, the area of the separatrix shrinks, and the particles become unstable and its position starts to increase non-linearly given enough time. The shape of the separatrix is triangular and the distribution of particles is very close to Gaussian.

But since we are interested only in the shape and form of the spill rate, we can assume the separatrix to be circular instead of triangular and try to analytically compute the rate of spill.




All the particles that are out of the circle are counted as 'extracted'. If we shrink the circle at just the right rate, we would extract equal number of particles with every time step. Let us see how to get the radius values in time that would give us the ideal constant spill rate.

## 1.1 Particle Density Function

If we want to find the total number of particles $N$ within a radius $r_0$ with the particle density function being $f(r)$, then.

$$N = \int_0^{r_0} \int_0^{2\pi} f(r) r \, \mathrm{d}r \, \mathrm{d}\phi$$

We see that if the particle density is a constant one ($n_0$, say), then,

$$\begin{aligned}
N &= \int_0^{r_0} \int_0^{2\pi} n_0 \; r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= 2\pi n_0 \int_0^{r_0} r \, \mathrm{d}r \\
&= 2\pi n_0 \left. \frac{r^2}{2} \right|_0^{r_0} \\
&= \cancel{2}\pi n_0 \frac{{r_0}^2}{\cancel{2}} \\
N &= n_0 \pi r_0^2
\end{aligned}$$

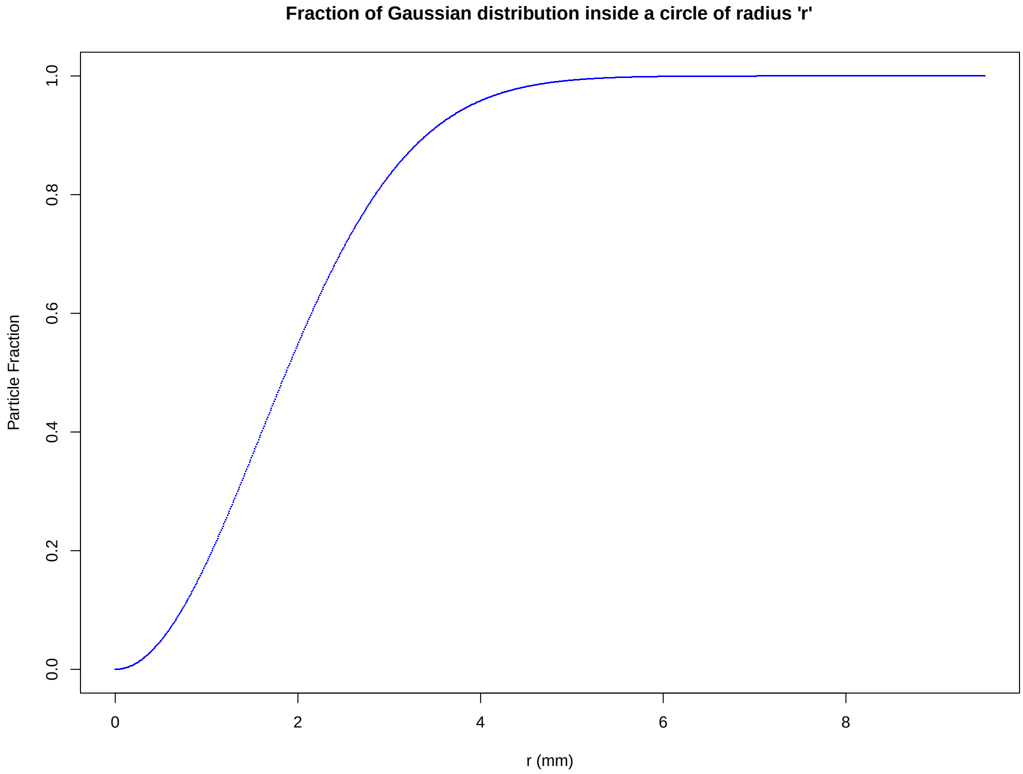
This is exactly what you would expect for a uniform density of particles. But if we instead had a Gaussian distribution in place of uniform distribution with a standard deviation $\sigma$, the total number of particles $N$ within a given radius $r_0$ would be,

$$\begin{aligned}
N &= \int_0^{r_0} \int_0^{2\pi} f(r) r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= \int_0^{r_0} \int_0^{2\pi} e^{-r^2/2\sigma^2} r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= 2\pi \int_0^{r_0} e^{-r^2/2\sigma^2} r \, \mathrm{d}r \\
N(r_0) &= 2\pi\sigma^2 (1 - e^{-r_0^2/2\sigma^2})
\end{aligned}$$

To normalize this, let us divide the integral by $2\pi\sigma^2$ so that when integrated from $r = 0 \to \infty$, we get the integral to be 1 (this way, we could talk of the fraction of particles). Thus the normalized density function would be,

$$f(r) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

Even though the Gaussian tail technically extends till infinity, let us have a cut-off radius of $r_{\mathrm{max}} = 6\sigma$ to account for *almost* all the particles. If we normalized the density function and plugged in a value of $\sigma_{\mathrm{beam}} = 1.58$ mm in $x$-coordinate, we get fraction of particles under a certain radius $r$, computed from $r = 0$ to $r = 6\sigma$.




We see that initially as $r$ increases from zero, the number of particles covered within that radius of circle $N$ increases. But as you keep increasing $r$, the number of particles covered within that circle does not increase as much because the particle density wanes off exponentially (as a Gaussian naturally does), and thus the fraction of particles contained within that radius starts to saturate asymptotically towards 1.

In our spill process, we need a uniform spill quality for over $N_{turn}$ turns. In other words, we need a fraction of $N_{\text{particles}}/N_{\text{turn}}$ to get extracted on every turn if we need a uniform extraction rate. This mathematically translates to the condition that the area under the density vs r curve in the above plot for $r_1$ to $r_2$ must be the same as the area under curve for $r_2$ to $r_3$.

$$\int_{r_1}^{r_2} f(r) r \, \mathrm{d}r = \int_{r_2}^{r_3} f(r) r \, \mathrm{d}r$$

For a Gaussian density, this means,

$$\int_{r_1}^{r_2} f(r) r \, \mathrm{d}r = [e^{-r_1^2/2\sigma^2} - e^{-r_2^2/2\sigma^2}]$$
$$\int_{r_2}^{r_3} f(r) r \, \mathrm{d}r = [e^{-r_2^2/2\sigma^2} - e^{-r_3^2/2\sigma^2}]$$

$$\implies [e^{-r_1^2/2\sigma^2} - e^{-r_2^2/2\sigma^2}] = [e^{-r_2^2/2\sigma^2} - e^{-r_3^2/2\sigma^2}]$$

We can thus solve for the *next* radius $r_3$ which would give you the same value of area under the curve between $r_2$ and $r_3$ as the value of area under the curve between $r_1$ and $r_2$. We get,

$$\implies \boxed{r_3 = \sqrt{-2\sigma^2 \log\left(2e^{-r_2^2/2\sigma^2} - e^{-r_1^2/2\sigma^2}\right)}} \tag{1.1}$$

And we can find all of the subsequent radii $r_4, r_5, ..., r_{\mathrm{N_{turn}}}$ iteratively! To kick-start this process from $r = 0$ to $r = r_1$ and find the first radius $r_1$, we need to know the exact fraction we want to extract. Our spill requirements are that a uniform fraction must be extracted every turn. If the total spill time is $T_{\text{spill}}$ and if our cyclotron time period is $T_{\text{cycl.}}$, then we get a total number of turns to be $T_{\text{spill}}/T_{\text{cycl.}} = N_{\text{turn}}$. We hence need a fraction of $1/N_{\text{turn}}$ particles extracted every turn. And that gives us our first $r_1$,

$$\int_0^{r_1} f(r) r \, \mathrm{d}r = 1/N_{\text{turn}} \tag{1.2}$$

$$= \frac{1}{2\pi\sigma^2} \int_0^{r_1} 2\pi e^{-r^2/2\sigma^2} r \, \mathrm{d}r \tag{1.3}$$

$$1/N_{\text{turn}} = \frac{1}{2\pi\sigma^2} 2\pi\sigma^2 (1 - e^{-r_1^2/2\sigma^2}) \tag{1.4}$$

$$e^{-r_1^2/2\sigma^2} = 1 - \frac{1}{N_{\text{turn}}} \tag{1.5}$$

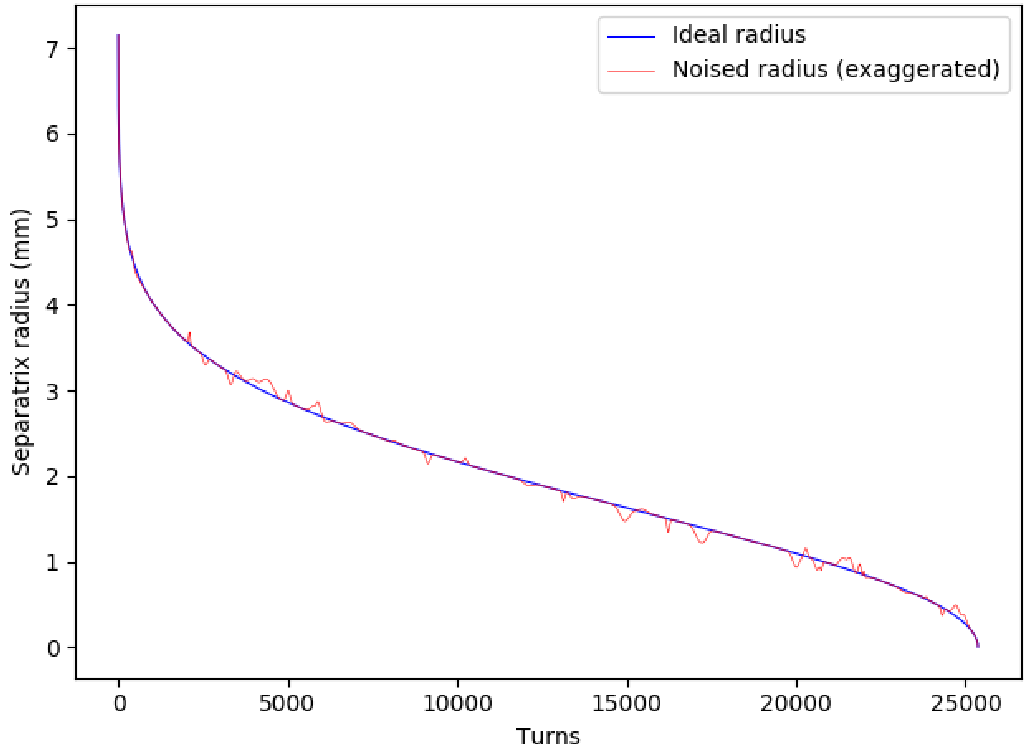
$$\boxed{r_1 = \sqrt{-2\sigma^2 \log\left(1 - \frac{1}{N_{\text{turn}}}\right)}} \tag{1.6}$$

With a beam rms size of $\sigma_{\text{beam}} = 1.58$ mm and an $N_{\text{turn}}$ of 25371, we get $r_1$ to be about 0.00354 mm. And we can thus iteratively compute the next '$N_{\text{turn}} - 1$' number of radii values that would give us uniform fraction of particles contained in the area enclosed between $r_n$ and $r_{n+1}$.

## 1.2 Introducing Noise

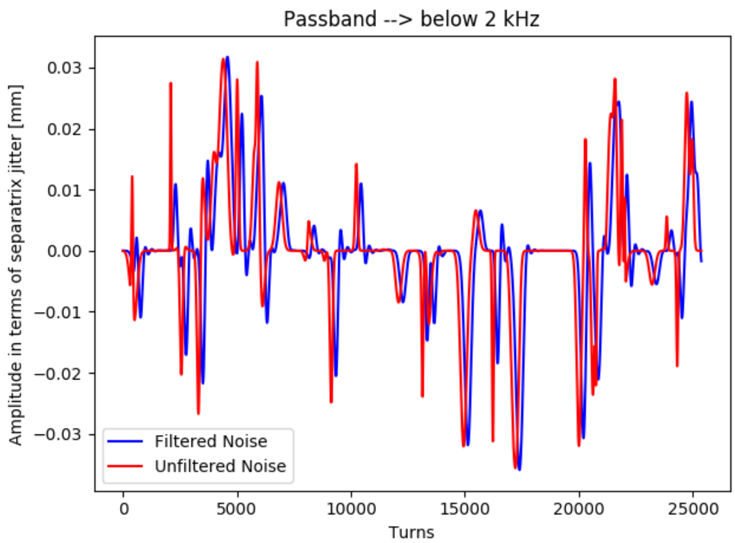
The value of the radius of separatrix is controlled directly by the excitation current given to the quadrupole. Since the excitation current is drawn from the power supply and there is noise from the power supply in the form of random spikes and ripples, the excitation current given to the quadrupole would be marred and superimposed with the noise content.

The random noise and spikes arising from the power supply can thus be included to the ideal spill by adding the noise to the radius value for each turn $(r_1 + \delta_1, r_2 + \delta_2, ..., r_{\mathrm{N_{turns}}} + \delta_{\mathrm{N_{turns}}})$.

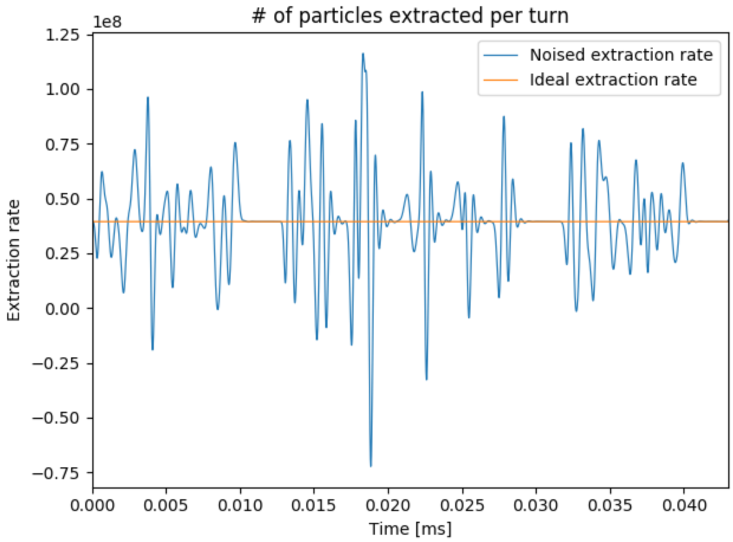



The fast varying noise in the power supply (and thus consequentially the fluctuations in magnetic field in the quadrupole) can have many range of frequencies in it. However, not all of the resulting fluctuation in the magnetic field penetrates through the steel beam pipe and affects the beam. The steel beam pipe shields any frequency larger than around 2000 hertz.

Hence in our analytical model, we need to pass the randomnly generated noise profile through a low-pass filter. We have some empirical measurement of how much magnetic field the beam pipe shields, so we can put in the order of the filter to simulate the real world as close as possible.



After adding the noise to the ideal radius values, the extraction rate won't be a flat line but will vary with respect to the added noise profile.




## 1.3 Relation to tune distance

In a regular sextupole drive third-integer resonance extraction, the length of the side of triangular separatrix is approximately given by,

$$x_0 = \frac{16\pi\delta}{A}$$

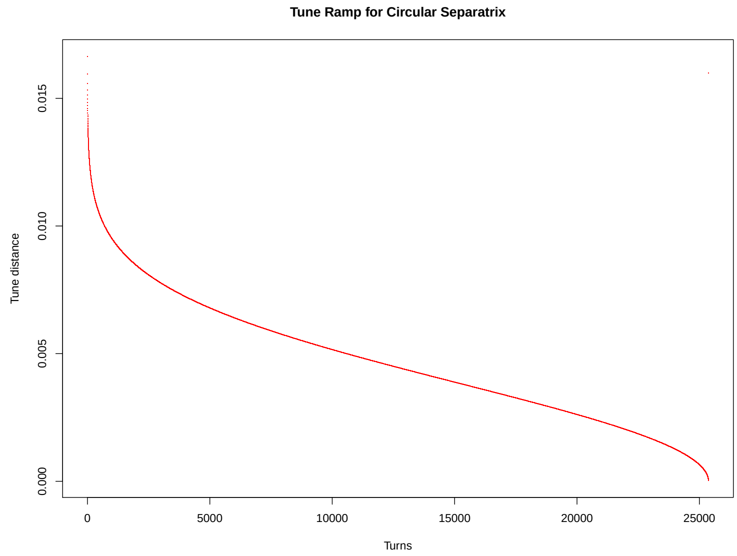
where $\delta$ is the tune distance of machine from the resonant tune, given by $\delta = \nu_{\mathrm{machine}} - \nu_{\mathrm{resonance}}$, and the parameter $A$ is proportional to the sextupole strength, given by,

$$A = \frac{\beta_0}{B\rho} \oint \left(\frac{\beta}{\beta_0}\right)\left(\frac{B''}{2}\right) \cos\left(3\nu_0\phi\right) \mathrm{d}s$$

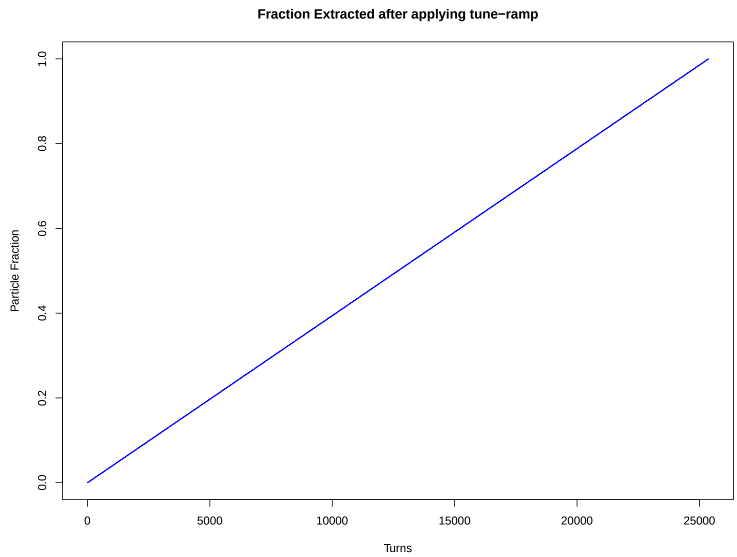
Since we have approximated the sepratrix to be circular, let us continue with that scheme and see how the radius evolution relates to tune evolution. We can take the radius of our circle to be about half of the side of the triangle, $r = x_0/2 = 8\pi\delta/A$. So if the sextupole strength is constant, the tune distance is directly proportional to the radius (barring numerical factors).

$$\delta(r) \approx \frac{A}{8\pi} r$$

If we have a sextupole strength of 500 $\mathrm{T/m^2}$ and a magnetic rigidity of 29.15 T-m, and assume that the Courant-Snyder $\beta$ at the observation point to be 12 m and at sextupole to be 9 m, and assuming a sextupole length of 0.5 m, we get $A$ to be 0.0596. Since we already have computed the $r$-s on the RHS, we can get the respective tune values, and thus, the tune-ramp function!




And the corresponding density function (in congruence, the number of extracted particles) when computed with the respective $r$-s is now a linear function, as expected:




The above scheme employed is under the assumption that the separatrix is circular and the beam distribution is Gaussian. In real life, the separatrix is triangular, the beam distribution in the stable region may not remain Gaussian throughout the extraction, the beam may not always be perfectly centered, the particles near the stationary points are going to move slower than the ones away from it, and so on. The above scheme also assumes the particle is 'immediately' extracted once it goes out of the separatrix, which is also not the case as the particle takes a finite transit time (and more than one jump) to get past the septum, and this transit time may also vary depending on where the particle radially is in the phase-space.

That said, we could expect the tune ramp curve for the real world triangular separatrix to be similar in quality as the one we derived above.