A GLOBAL PARTICULAR SOLUTION TO THE INITIAL-VALUE PROBLEM OF STELLAR DYNAMICS*

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Summary. An explicit global particular solution of the initial-value problem of stellar dynamics is presented.

1. After the local existence and uniqueness of a solution to the initial-value problem of stellar dynamics was demonstrated in 1952 [5], Batt proved, in 1963, the global existence and uniqueness for a modified, "mollified" initial-value problem, approximating the mass density function by a local average [2]. There was still the question whether or not there are global solutions to the original initial-value problem. Neunzert expressed doubts [10]. Recently, however, Batt has proven the global existence and uniqueness for a class of solutions which are distinguished by certain symmetries [3, 4]. Apparently no explicit example of a global solution has been known thus far. In this note, such an example is given. It exhibits the same symmetries as the solutions investigated by Batt, and generalizes a model of stellar systems discussed by v. d. Pahlen [11] and Scherrer [12] in which it was assumed that there is no velocity scattering. That model is related to Newtonian cosmology, which, however, is marred by a singularity on the time axis [5, 9]. The solution presented here yields a model of Newtonian cosmology without such a singularity (without an initial "big bang").

2. Let x and u be arbitrary points of the three-dimensional Euclidean space E^3 (x being the "position vector", and u being the "velocity vector"), and let t be the "time variable," defined on the real axis E^1. The initial-value problem of stellar dynamics then reads:

to determine, if possible, a non-negative real function f (the "frequency function"), defined on E^7, such that:

(i) there is a region $D \subseteq E^7$ in which f is positive (excepting, possibly, a set of measure 0) and has continuous partial derivatives with respect to all its variables. The intersection of D with every hyperplane t = constant in E^7 is nonempty. Outside the closure of D, f vanishes.

(ii) f satisfies, in D, Liouville's equation (the "Vlasov equation")

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} u - \frac{\partial f}{\partial u} \frac{\partial V}{\partial x} = 0$$

where

$$V(x \mid t) \equiv -G \cdot \int_{E^3} \frac{\rho(y \mid t)}{|x - y|} dy$$

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is the "gravitational potential" generated by the "mass density"

$$\rho(x \mid t) \equiv \int_{E^3} f(x, u \mid t)\, du.$$

G is the constant of gravitation, $\partial/\partial x$ and $\partial/\partial u$ denote gradients with respect to x and u; the products are scalar products.

(iii) $f(x, u \mid 0) = f_0(x, u)$ for all (x, u), where $f_0 : E^6 \to E^1$ is a given non-negative continuously differentiable function such that

$$0 < \int_{E^6} f_0(x, u)\, d(x, u) < \infty.$$

For any such function f and all values of t,

$$\int_{E^6} f(x, u \mid t)\, d(x, u) = \int_{E^6} f_0(x, u)\, d(x, u) = M,$$

the "total mass of the system."

3. Considering a system of spherical symmetry in the configuration space (the space of the position vectors x), we choose the units of length, time, and mass in such a way that $M = 1$, $G = 1$, and the initial radius of the system (i.e., its radius at the time $t = 0$) is unity. We introduce a function ϕ of the time variable t (which later will turn out to be the radius of the system):

Let ϕ be a real-valued function of the time variable t, defined in a neighbourhood of the time zero by

$$\phi^3\ddot{\phi} + \phi = 1,$$
$$\phi(0) = 1,$$
$$\dot{\phi}(0) = H = \text{const.}$$

($\dot{\phi}$ and $\ddot{\phi}$ denote the first- and second-order derivatives of ϕ.) Then the solution of this initial-value problem can be extended to the whole time axis, and $\phi(t)$ is positive for all t.

In particular:

(o) If $H = 0$, then $\phi(t) = 1$ for all t.

(i) If $0 < |H| < 1$, define the numbers v_0 and t_0 by

$$\cos v_0 = H, \qquad 0 < v_0 < \pi,$$
$$-(1 - H^2)^{3/2} t_0 = v_0 - H \sin v_0$$

and, on the whole real line, the function $t \to v(t)$ by

$$v - H \sin v = (1 - H^2)^{3/2}(t - t_0).$$

(Kepler's equation, elliptic case.) Then,

$$\phi = \frac{1 - H \cos v}{1 - H^2}.$$

(ii) If $|H| = 1$, define the function v by

$$v + \tfrac{1}{3}v^3 = 2(t + \tfrac{2}{3}).$$

Then,

$$\phi = \tfrac{1}{2}(1 + v^2).$$

(Parabolic case.)
(iii) If $|H| > 1$, let

$$\cosh v_0 = H,$$
$$-(H^2 - 1)^{3/2} t_0 = v_0 - H \sinh v_0 ,$$

and

$$v - H \sinh v = (H^2 - 1)^{3/2} (t - t_0).$$

Then,

$$\phi = \frac{H \cosh v - 1}{H^2 - 1}.$$

(Hyperbolic case.)
In the elliptic case, ϕ is a periodic function of t, the period being $2\pi(1 - H^2)^{-3/2}$. In the parabolic and hyperbolic cases, $\phi(t) \to \infty$ as $|t| \to \infty$.

4. Now the announced explicit solution to the initial-value problem can be given: The function $f: E^7 \to E^1$, defined by

$$f(x, u \mid t) = \frac{3}{2^3} \left[1 - \left(\frac{x}{\phi} \right)^2 - (\phi u - \dot{\phi} x)^2 + (x \times u)^2 \right]^{-1/2}$$

where the radicand is positive and $|x \times u| < 1$,

$= 0$ otherwise,

is a solution to the initial-value problem of stellar dynamics, f_0 being given by $f(. \mid 0)$. ("$\times$" denotes the vector product.) The corresponding mass density ρ and potential V are

$$\rho(x \mid t) = \left(\frac{4}{3} \pi \right)^{-1} [\phi(t)]^{-3} \quad \text{if} \quad |x| < \phi(t),$$
$$= 0 \quad \text{otherwise.}$$

and

$$V(x \mid t) = \tfrac{1}{2} x^2 [\phi(t)]^{-3} - \tfrac{3}{2} [\phi(t)]^{-1} \quad \text{for} \quad |x| < \phi(t).$$

The assertion is proven by straightforward verifications: the conditions (i) $-$ (iii) of Sec. 2 are satisfied.

(For the proof note that the condition $|x \times u| < 1$, which is invariant under the "phase flow", and the inequality $|x| < \phi(t)$ are equivalent. For, let

$$\xi \equiv x/\phi, \qquad \eta \equiv \phi u - \dot{\phi} x,$$

and define R^2 and T^2 by

$$R^2 + T^2 = \eta^2, \qquad \xi^2 T^2 = (\xi \times \eta)^2 = (x \times u)^2.$$

The radicand in the definition of f now reads:

$$1 - \xi^2 - (R^2 + T^2) + \xi^2 T^2.$$

It is positive if, and only if,

$$(1 - \xi^2)(1 - T^2) > R^2.$$

Therefore,

$$\xi^2 < 1 \Leftrightarrow T^2 < 1,$$

which implies that

$$(x \times u)^2 = \xi^2 T^2 < 1 \Leftrightarrow \left|\frac{x}{\phi}\right| = |\xi| < 1,$$

as has been asserted.)

The solution has been constructed by a combination of three results already known: (1) Schürer's space-time transformation [13], (2) its application to self-gravitating systems [7, 8], and (3) Ahmad's construction of a stationary self-gravitating system of uniform density [1].

5. If in the differential equation for ϕ given in Sec. 3 the right-hand side 1 is replaced by 0, the corresponding equation of v. d. Pahlen's and Scherrer's model [11, 12] and of Newtonian cosmology is obtained. The term ignored in these models corresponds to the velocity scattering, that is, in the hydrodynamical interpretation, to pressure [9]. It thus becomes understandable why, in these models, a collapse of the whole system into a single point, i.e. a singularity on the time axis, can (and does) occur: there is no pressure counteracting gravitation. Or, kinematically, the simultaneous arrival at the centre of all the "mass elements" is possible only because, at any given point x of the system, there is no local scattering of the initial velocities.

In a cosmological interpretation of the frequency function f, the real number H would be the (dimensionless) Hubble constant. Since its empirical value is of the order of unity, but is not known very precisely, the qualitative character of the function ϕ (i.e. of the radius of the "Universe" as a function of the time variable t) would (still) be empirically uncertain.

References

[1] A. Ahmad, *Uniform density equilibrium model of self-gravitating stellar systems*, Astrophys. Space Sci. **27**, 343 (1974)

[2] J. Batt, *Ein Existenzbeweis für die Vlasov-Gleichung der Stellardynamik bei gemittelter Dichte*, Arch. Rat. Mech. Anal **13**, 296–318 (1963)

[3] J. Batt, *Global symmetric solutions of the initial value problem of stellardynamics*, J. Differential Equations **25**, 342–364 (1977)

[4] J. Batt and E. Horst, *Global existence and conservation laws for the Vlasov equation*. Preprint Math. Inst. Univ. München, 1976

[5] H. Bondi, *Cosmology*, Cambridge University Press, 1961

[6] R. Kurth, *Das Anfangswertproblem der Stellardynamik*, Z. Astrophysik **30**, 213–229 (1952)

[7] R. Kurth, *Über Sternsystema zeitlich oder räumlich veränderlicher Dichte*, Z. Astrophysik **26**, 100–136 (1949)

[8] R. Kurth, *Zur Dynamik instationärer Sternsysteme*, Z. Astrophysik **26**, 168–175 (1949)

[9] R. Kurth, *Über den Zusammenhang zwischen dynamischer Kosmologie und Stellardynamik*, Z. Astrophysik **26**, 272–278 (1949)

[10] H. Neunzert, *Neuere qualitative und numerische Methoden in der Plasmaphysik,* Gastvorlesung im Rahmen des Paderborner Ferienkurses in Angewandter Mathematik. Manuskript 1975

[11] E. v. d. Pahlen, *Über die Entstehung sphärischer Sternsysteme,* Z. Astrophysik **24,** 68 (1947)

[12] W. Scherrer, *Über die Gravitation kontinuierlich ausgebreiter Massen,* Comm. Math. Helvetici **24,** 46–63 (1950)

[13] M. Schürer, *Beitrag zur Dynamik der Sternsysteme,* Astr. Nachrichten **273,** 230–242 (1943)