New Exact Solution for a Beam in the Presence of Time-Dependent Isotropic Linear Focusing

IOTA Physics Meeting

February 8, 2021

Chad Mitchell
Lawrence Berkeley National Laboratory

BERKELEY LAB

U.S. DEPARTMENT OF ENERGY | Office of Science

ACCELERATOR TECHNOLOGY & APPLIED PHYSICS DIVISION ATAP

Introduction and Motivation

- The K-V beam (1959) remains the only widely-known[1] exact solution of the Vlasov-Poisson system describing an intense (4D) charged particle beam with nonzero emittance in the presence of time-dependent linear focusing.

- Unusual features of the K-V beam:
  - the distribution is not a true function, but a delta-function on the phase space
  - it is sensitive to growth of unstable collective normal modes
  - the procedure for constructing a K-V beam *cannot be generalized to 6D* (Sacherer, 1968)

- Building on the work of R. Kurth in stellar dynamics, we show how an exact solution can be constructed in either 4D (for an unbunched beam in the presence of 2D linear focusing) or in 6D (for a bunched beam in the presence of 3D linear focusing).

- The solution is limited to radially and spherically symmetric beams in linear (possibly time-dependent) isotropic focusing (*eg*, solenoid channels, ion traps).

- Why bother?
  - as a benchmark problem for codes computing 2D or 3D space charge
  - as a starting point for beam physics studies (vast literature on the isotropic K-V beam)

[1] However, see V. Danilov *et al*, Phys. Rev. ST Accel. Beams, 6, 094202 (2003)

The Stellar Dynamics Problem (Nonrelativistic)

Galaxy

10^5-10^{12} stars

Vlasov-Poisson system for a stellar distribution *f* :

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_x f - \nabla U \cdot \nabla_v f = 0$$

$$\nabla^2 U = 4\pi G \rho, \quad \rho = M \int f d\vec{v}$$

Collision time >> age of the universe (Vlasov description is valid)

confined by attractive self-forces with no external focusing

Kurth's exact solution 1978: constructed from invariants of motion

- distribution is rotationally symmetric
- linear gravitational self-forces
- allows for both steady-state and oscillating solutions

J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd ed, Princeton, 2008
R. Kurth, Quarterly Journal of Applied Mathematics, vol. 36, p. 325 (1978)

The Stellar Dynamics Problem (Nonrelativistic)

Galaxy

10^5-10^{12} stars

Vlasov-Poisson system for a stellar distribution *f* :

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_x f - \nabla U \cdot \nabla_v f = 0$$

$$\nabla^2 U = 4\pi G\rho, \quad \rho = M\int f d\vec{v}$$

Collision time >> age of the universe (Vlasov description is valid)

confined by attractive self-forces with no external focusing

Kurth's exact solution 1978: constructed from invariants of motion

- distribution is rotationally symmetric
- linear gravitational self-forces
- allows for both steady-state and oscillating solutions

Can be adapted to charged-particle beams.

J. Binney and S. Tremaine, *Galactic Dynamics*, 2nd ed, Princeton, 2008
R. Kurth, Quarterly Journal of Applied Mathematics, vol. 36, p. 325 (1978)

Application to Beam Physics: The 4D Kurth Distribution in Linear s-Dependent External Focusing (1)

Hamiltonian for an unbunched coasting beam in the presence of linear focusing:

$$H(x,p_x,y,p_y;s)=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}\Omega(s)^2(x^2+y^2)+\Phi(x,y;s)$$

Path length *s* is taken as the independent variable, and momenta are normalized by the design momentum.

Vlasov-Poisson system for the beam distribution function:

$$\frac{\partial f}{\partial s}+\{f,H\}=0,\qquad \nabla^2\Phi=-2\pi K_{pv}\int f\,dp_x dp_y \qquad (\star)$$

Here K_{pv} is the generalized beam perveance.

If $I_1,\ldots,I_n$ are invariant along the single-particle trajectories, then any distribution function of the form $f = G(I_1,\ldots,I_n)$, properly normalized, is a solution of ($\star$).

Application to Beam Physics: The 4D Kurth Distribution in Linear s-Dependent External Focusing (2)

We search for an axially-symmetric solution with linear space charge forces.

Let R be any solution of the envelope equation (ε > 0):

$$R''(s) + \Omega(s)^2 R(s) - \frac{K_{pv}}{R(s)} - \frac{\epsilon^2}{R(s)^3} = 0 \ .$$

Define normalized dimensionless variables by:

$$x_N = x/R(s), \qquad p_{xN} = \left[R(s)p_x - R'(s)x\right]/\epsilon,$$
$$y_N = y/R(s), \qquad p_{yN} = \left[R(s)p_y - R'(s)y\right]/\epsilon$$

Under the assumption of linear space charge forces, the following are invariants:

$$E = \frac{1}{2}\left(p_{xN}^2 + p_{yN}^2 + x_N^2 + y_N^2\right), \qquad L_z = x_N p_{yN} - y_N p_{xN}$$

energy (Hamiltonian) angular momentum

Application to Beam Physics: The 4D Kurth Distribution in Linear s-Dependent External Focusing (3)

This can be seen explicitly by verifying that along particle trajectories:

$$\frac{dE}{ds} = \frac{\partial E}{\partial s} + \{E, H\} = 0, \qquad \frac{dL_z}{ds} = \frac{\partial L_z}{\partial s} + \{L_z, H\} = 0 \; .$$

To obtain a K-V distribution, one would put:

$$f(x, p_x, y, p_y; s) = \frac{2}{\pi^3 \epsilon^2} \delta(1 - 2E) \; .$$

To obtain a Kurth distribution, put:

$$f(x, p_x, y, p_y; s) = \frac{1}{2\pi^2 \epsilon^2} \left(1 - 2E + L_z^2\right)_+^{-1/2}, \qquad |L_z| < 1$$

This notation means that *f* vanishes unless the argument () is positive and $|L_z|<1$.

Properties of the 4D Kurth Distribution

Integrating over the momentum variables shows that the spatial density is uniform inside a disk of radius *R*:

$$P(x,y;s) = \int\int f(x,p_x,y,p_y;s)dp_x dp_y = \frac{1}{\pi R(s)^2}\begin{cases} 1, & r \leq R(s) \\ 0, & r > R(s) \end{cases}$$

Similarly, the 2D projections P_x-P_y, X-P_x and Y-P_y are uniformly-filled ellipses.

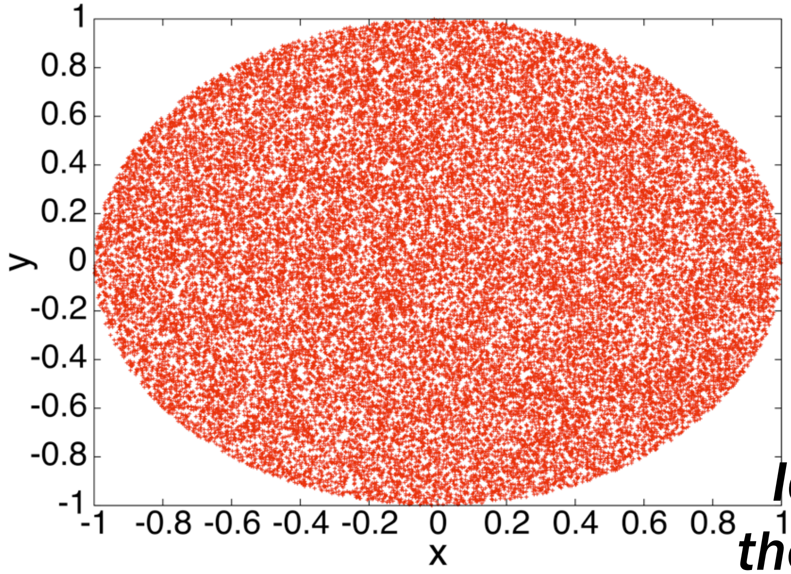
The rms beam sizes and emittances are given by:

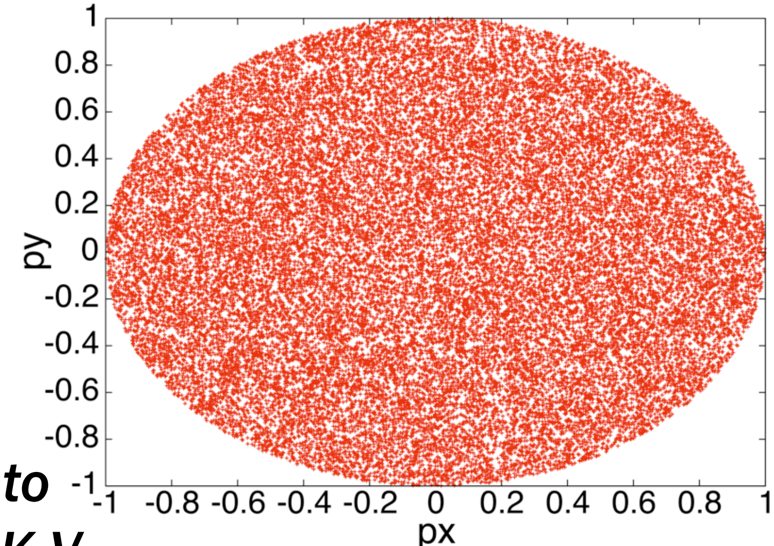
$$\sigma_x = \sigma_y = \frac{R}{2}, \qquad \epsilon_{x,rms} = \epsilon_{y,rms} = \frac{\epsilon}{4}$$

Note: These properties are the same as those of a K-V beam!

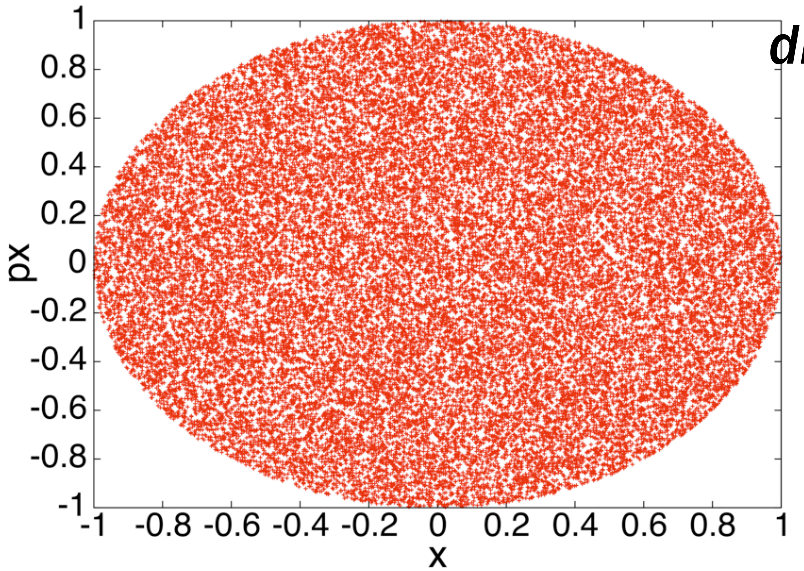
However, the details of the distribution function do make a difference.

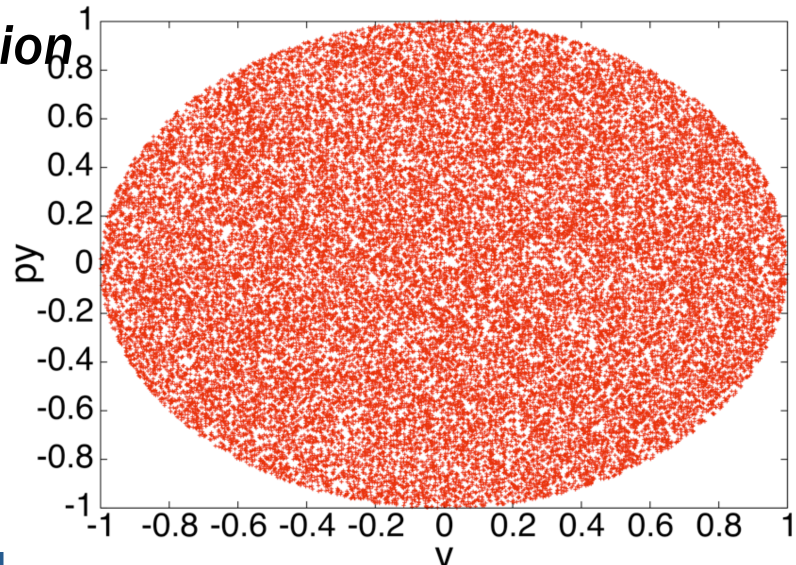
Sampled Particles from a 4D Kurth Distribution (shown in scaled dimensionless coordinates)



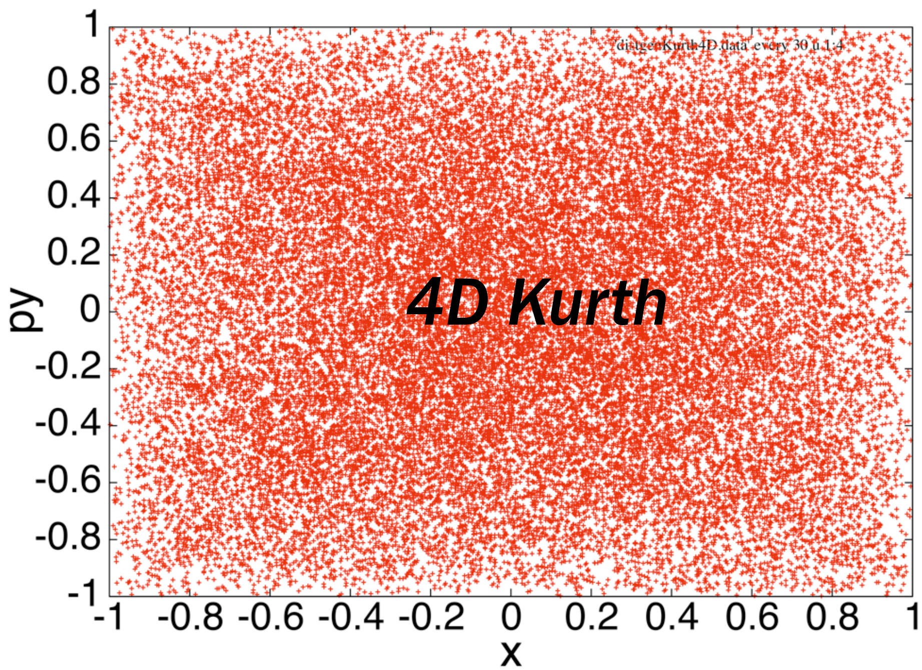


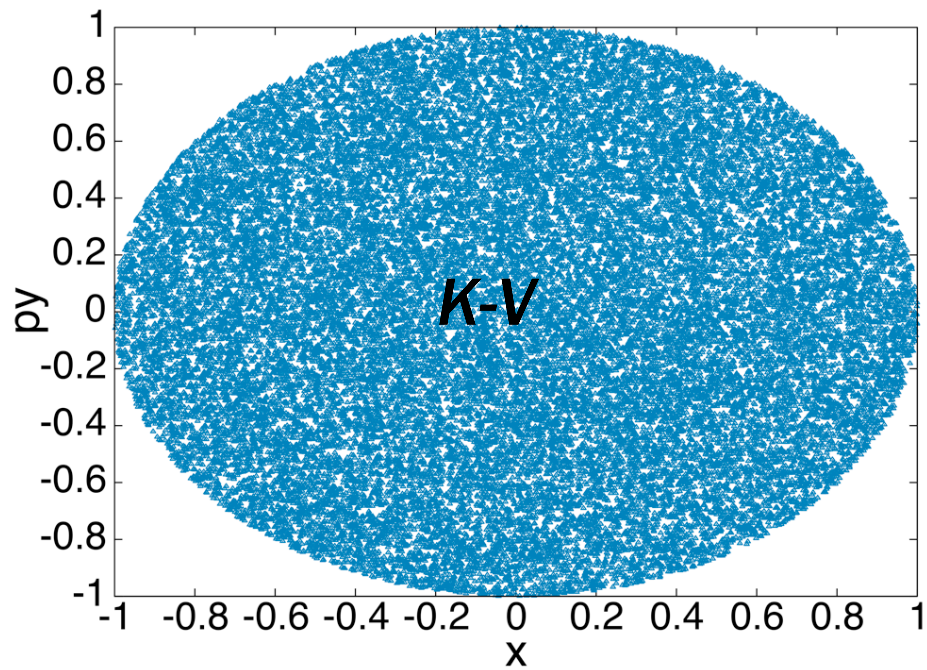
Identical to those of a K-V distribution





Not All 2D Projections Are Identical to Those of a K-V





X, P_y are statistically independent;

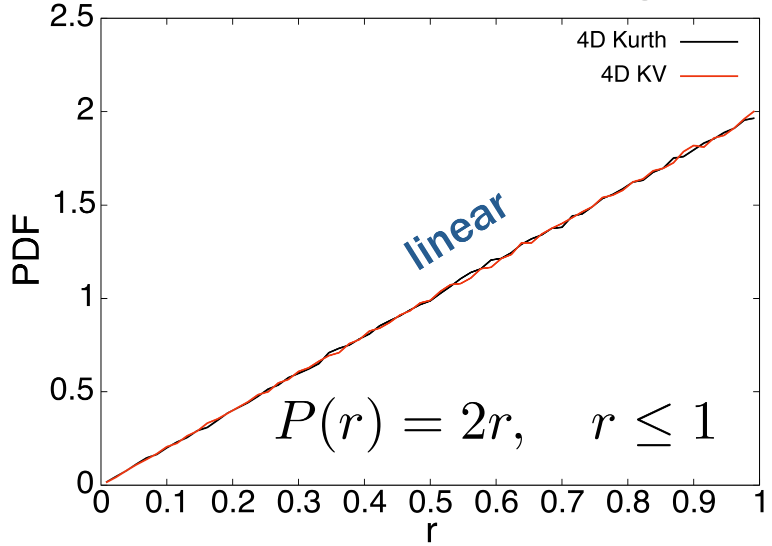
$$P(x, p_y) = P(x)P(p_y)$$

X, P_y projection is a uniformly-filled ellipse

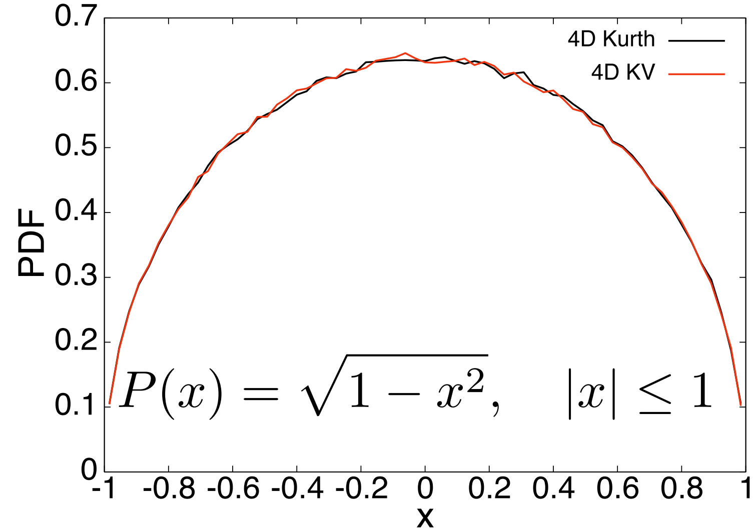
Identical behavior for Y, P_x projection.

Some 1-D Histograms for the 4D Kurth Distribution

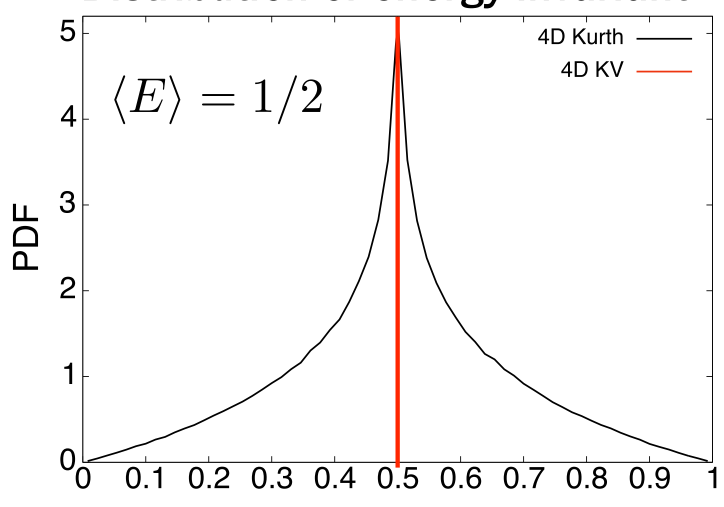
Distribution of coordinate magnitude



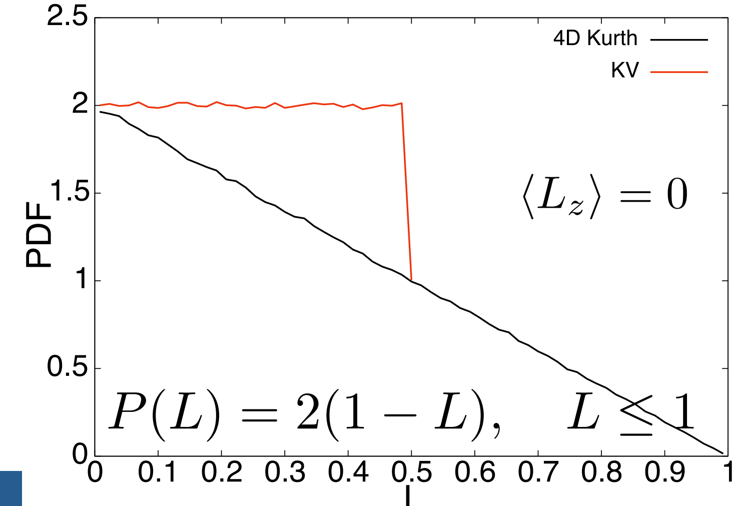
Distribution of coordinate component



Distribution of energy invariant



Distribution of angular momentum



Application to Beam Physics: The 6D Kurth Distribution in Linear Time-Dependent External Focusing (1)

Hamiltonian for a bunched beam in the presence of isotropic linear focusing:

$$H(\vec{r},\vec{p};\tau) = \frac{1}{2}|\vec{p}|^2 + \frac{1}{2}\Omega(\tau)^2|\vec{r}|^2 + \Phi(\vec{r};\tau)$$

Time $\tau = ct$ is taken as the independent variable, and momenta are normalized by the value mc. We assume that we are working in the bunch rest frame.

Vlasov-Poisson system for the beam distribution function:

$$\frac{\partial f}{\partial \tau} + \{f, H\} = 0, \quad \nabla^2\Phi = -4\pi r_c N \int f dp_x dp_y dp_z \qquad (\star)$$

Here r_c is the classical particle radius and N is the bunch population.

Construction goes through just as in the 4D case, except that L_z is replaced by the total angular momentum $L = |\vec{L}|$.

Application to Beam Physics: The 6D Kurth Distribution in Linear Time-Dependent External Focusing (2)

We search for a radially symmetric solution with linear space charge forces.

Let R be any solution of the envelope equation (ε > 0):

$$R''(\tau) + \Omega(\tau)^2 R(\tau) - \frac{r_c N}{R(\tau)^2} - \frac{\epsilon^2}{R(\tau)^3} = 0 \, .$$

Define normalized dimensionless variables by:

$$\vec{r}_N = \vec{r}/R(\tau), \qquad \vec{p}_N = \left[R(\tau)\vec{p} - R'(\tau)\vec{r}\right]/\epsilon$$

Under the assumption of linear space charge forces, the following are invariants:

$$E = \frac{1}{2}\left(|\vec{p}_N|^2 + |\vec{r}_N|^2\right), \qquad L = |\vec{r}_N \times \vec{p}_N|$$

energy (Hamiltonian) angular momentum

Properties of the 6D Kurth distribution

The 6D Kurth distribution is given by:

$$f(\vec{r},\vec{p};\tau) = \frac{3}{4\pi^3\epsilon^3}\left(1-2E+L^2\right)_+^{-1/2}, \quad L<1$$

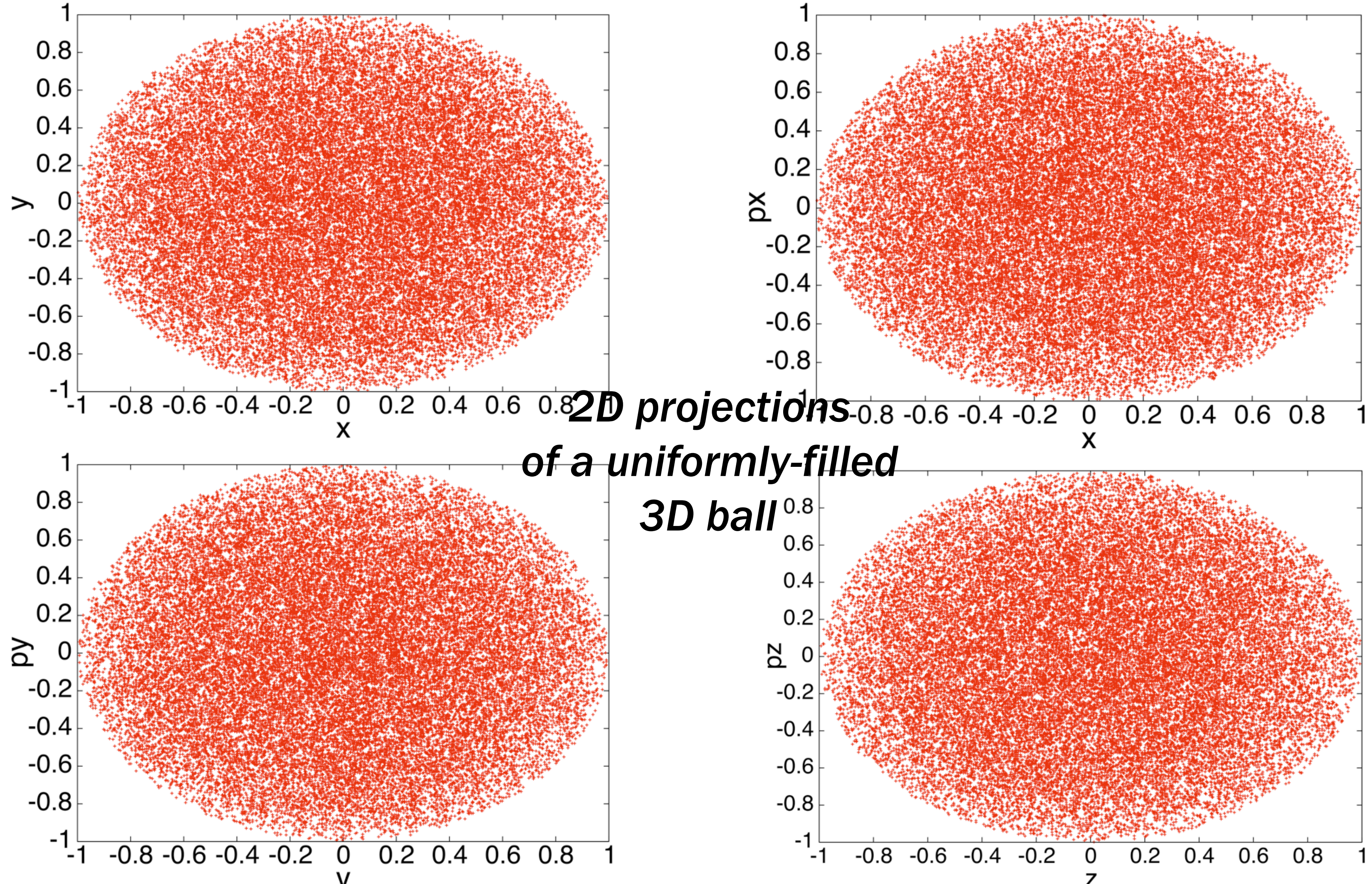
Integrating over the momentum variables shows that the spatial density is uniform inside a 3D ball of radius *R*. The space charge forces are again linear.

The functional form is a natural generalization of the 4D case.

The rms beam sizes and emittances are given by:

$$\sigma_x = \sigma_y = \sigma_z = \frac{R}{\sqrt{5}}, \qquad \epsilon_{x,rms} = \epsilon_{y,rms} = \epsilon_{z,rms} = \frac{\epsilon}{5}$$

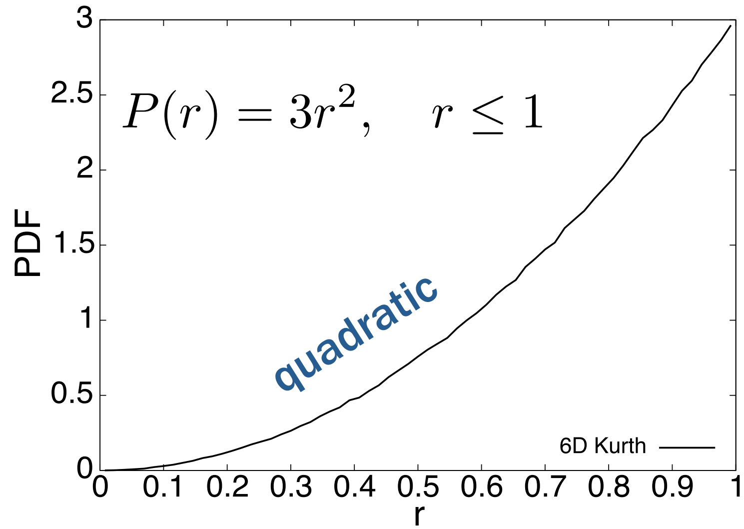
Sampled Particles from a 6D Kurth Distribution (shown in scaled dimensionless coordinates)



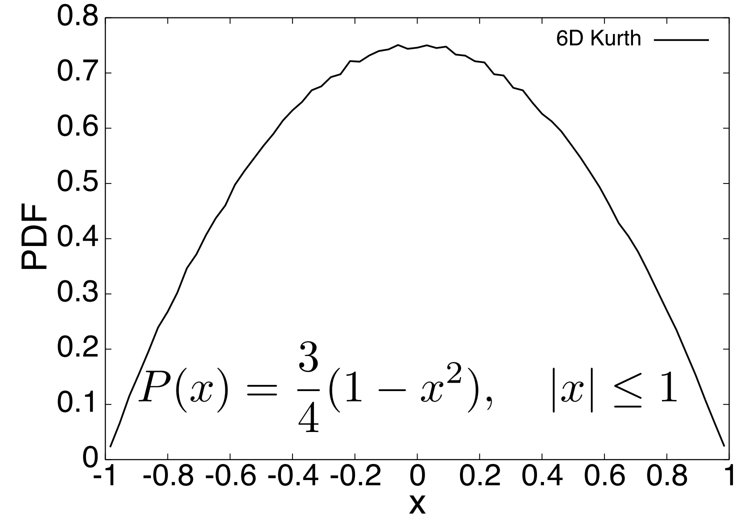
2D projections of a uniformly-filled 3D ball

Some 1-D Histograms for the 6D Kurth Distribution

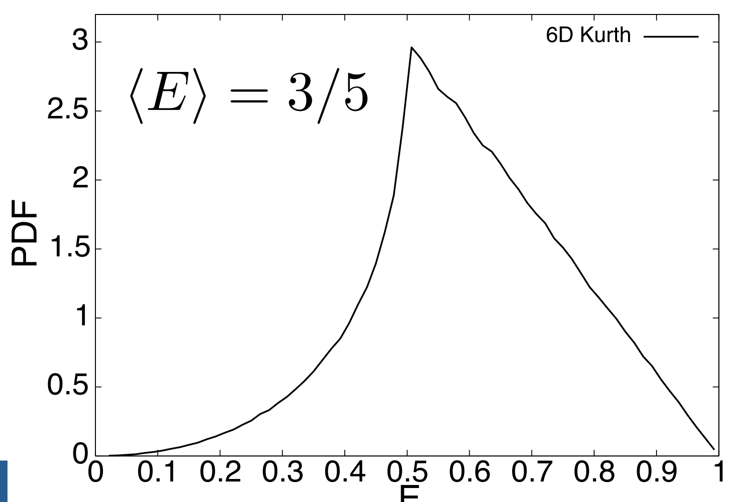
Distribution of coordinate magnitude



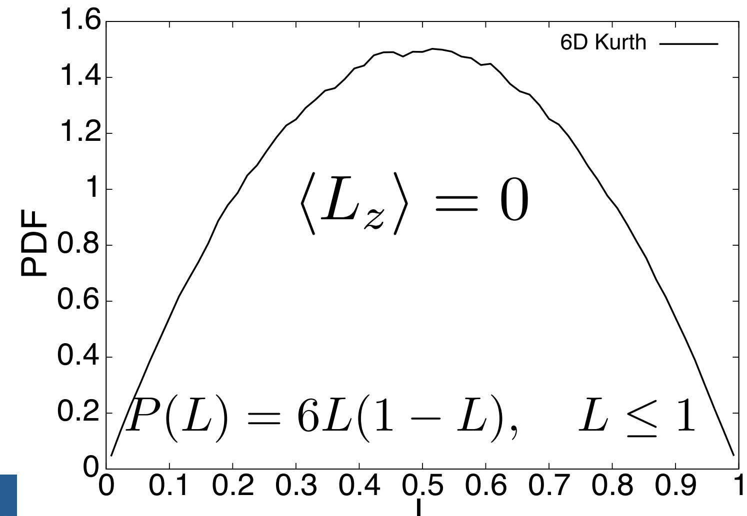
Distribution of coordinate component



Distribution of energy invariant



Distribution of angular momentum



- Test: Breathing 4D beam in a linear focusing channel

4D Kurth Beam Test, Constant Focusing: Problem Setup

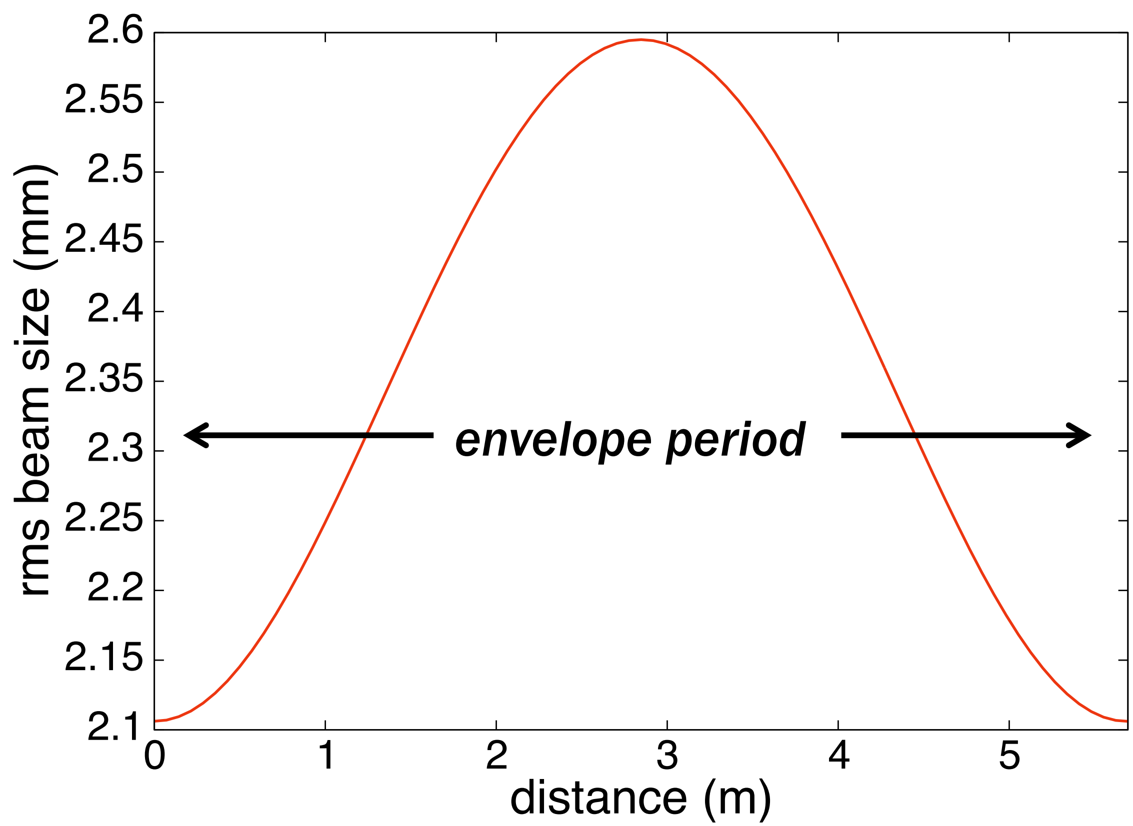
100K particles, 1D Gauss' Law solver - similar results obtained using 2D solvers

200 MeV proton beam
Focusing: B=2.7 T
ε = 10.14 mm-mrad
20 A current

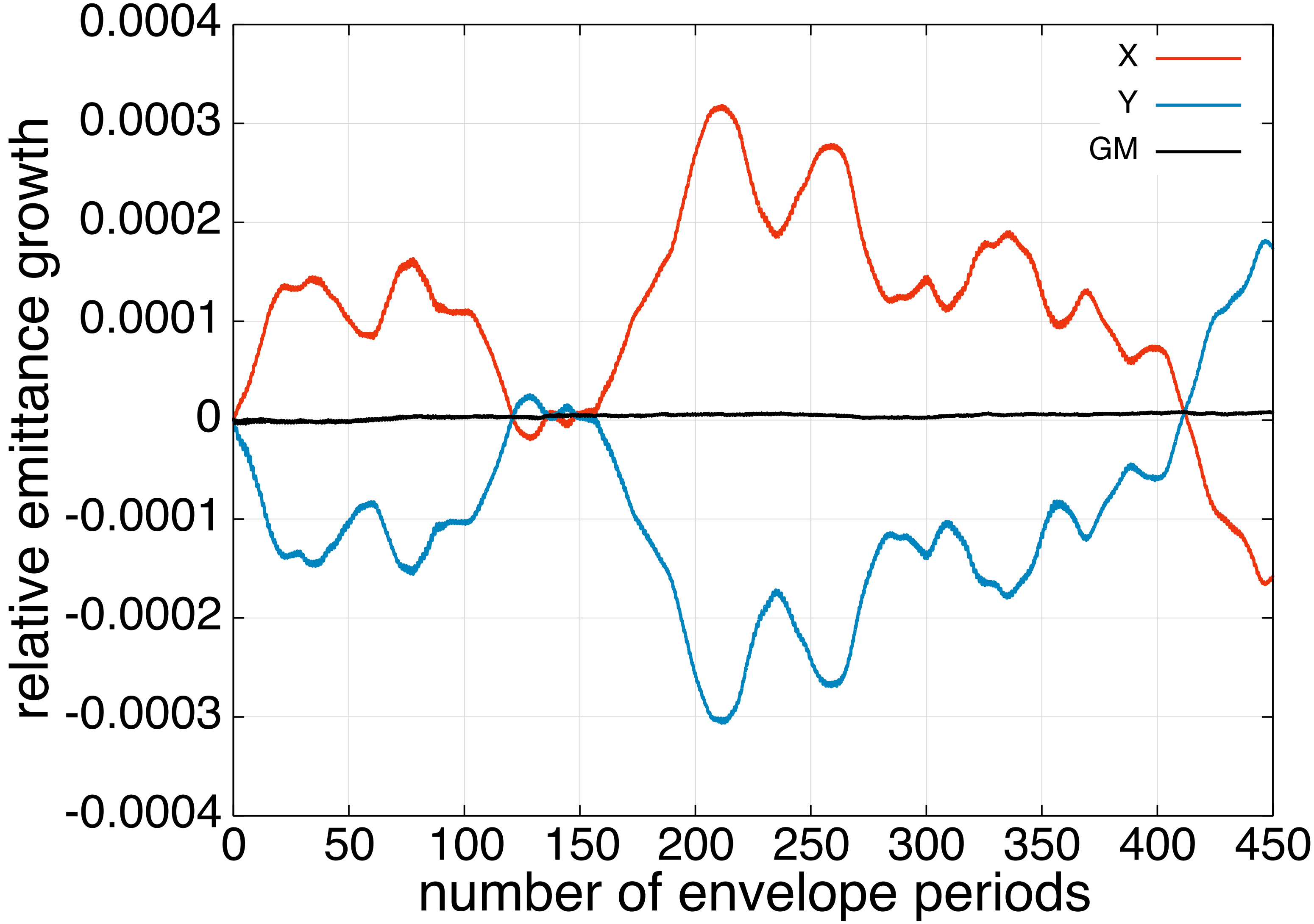
SC tune depression:

$$\omega/\omega_0 = 0.74$$

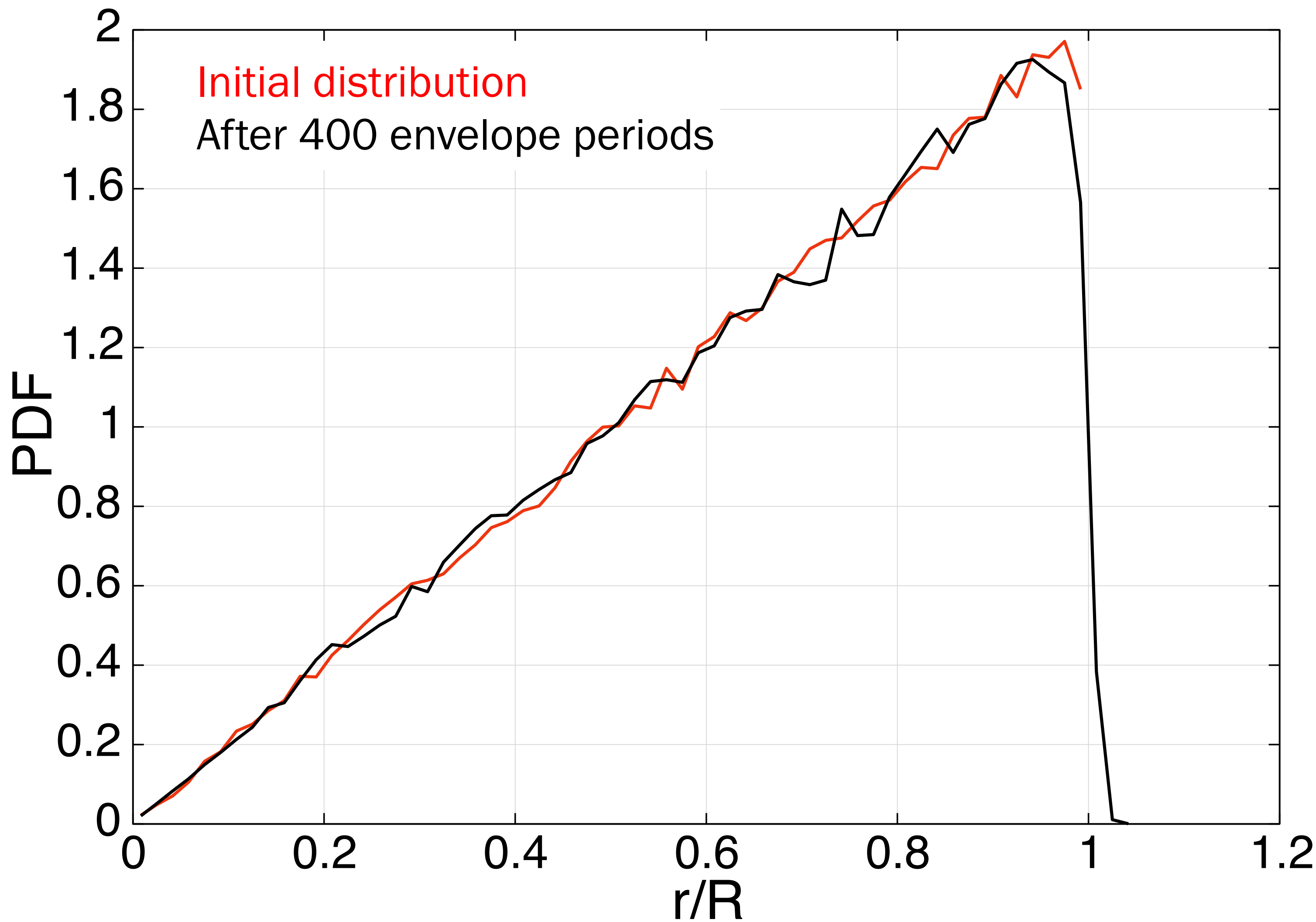
Initial beam size is mismatched by 10%.



Mismatched 4D Kurth Beam in a Constant Focusing Channel: Emittance Evolution

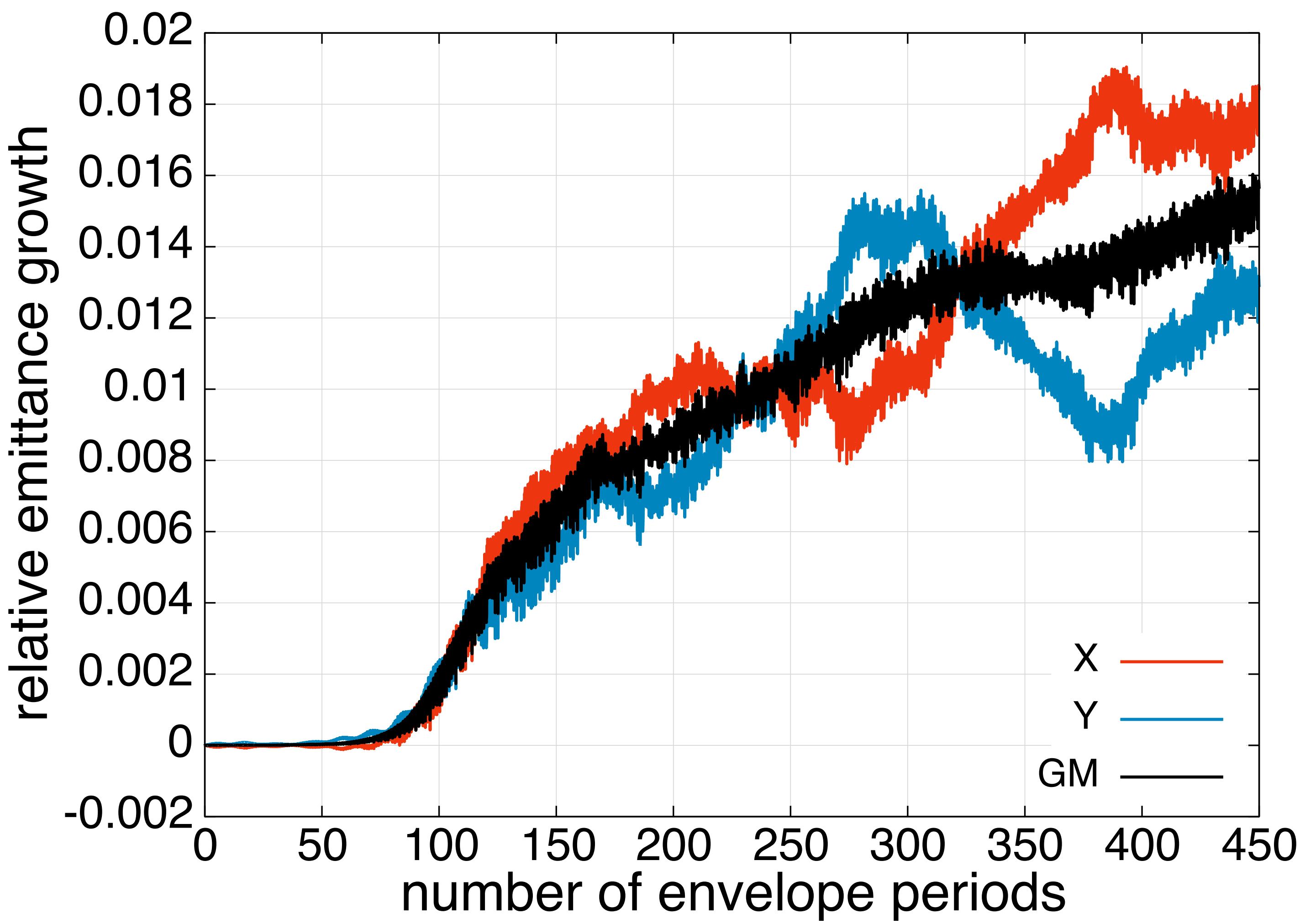


Mismatched 4D Kurth Beam in a Constant Focusing Channel: Radial Profile



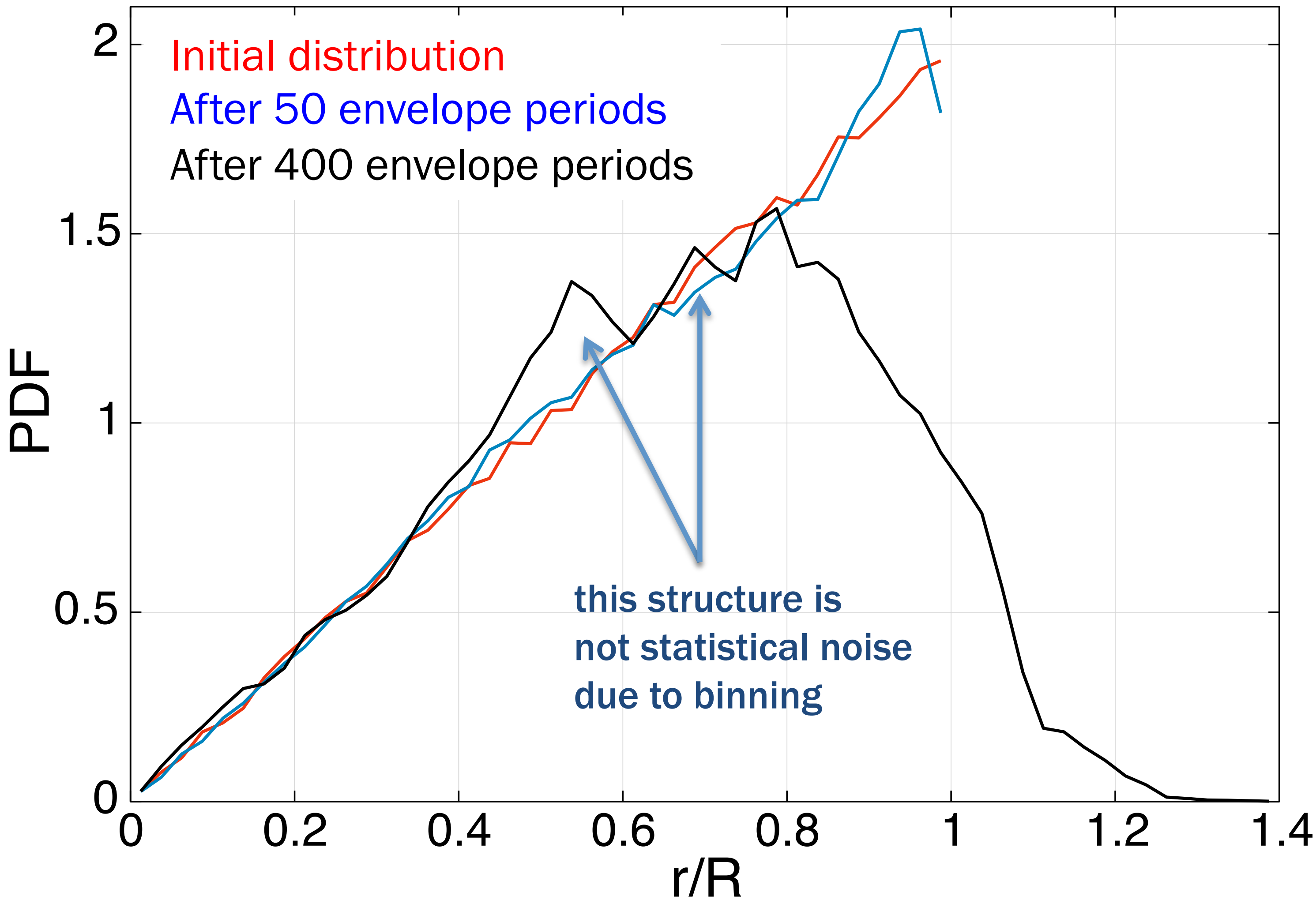
uniform spatial distribution is well-preserved

Mismatched K-V Beam in a Constant Focusing Channel for the Same Parameters: Emittance Evolution



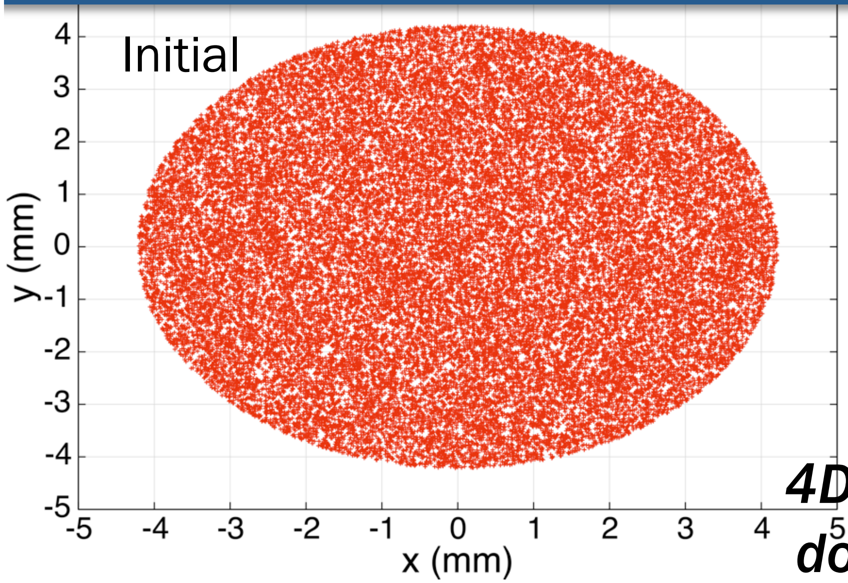
Differences in the details of the initial distribution have resulted in a dramatic difference in stability behavior

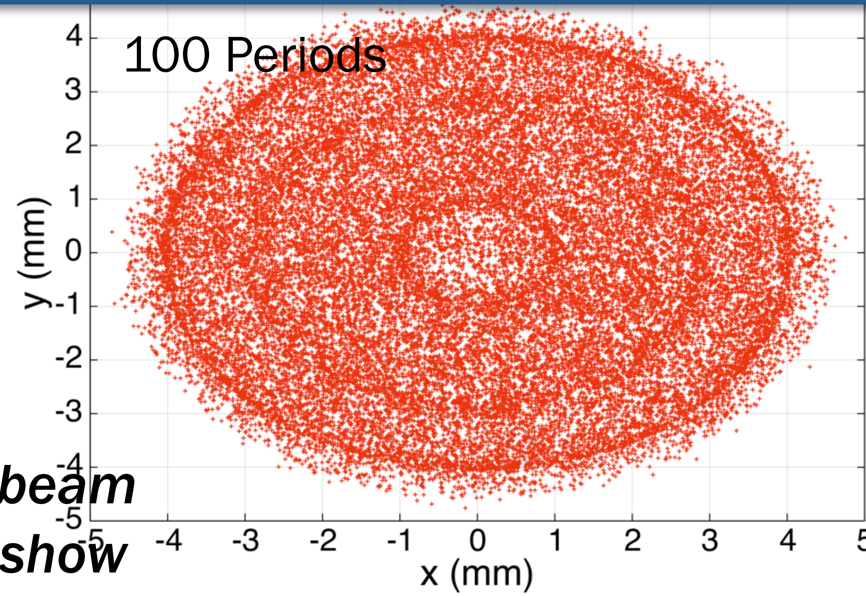
Mismatched K-V Beam in a Constant Focusing Channel for the Same Parameters: Radial Profile



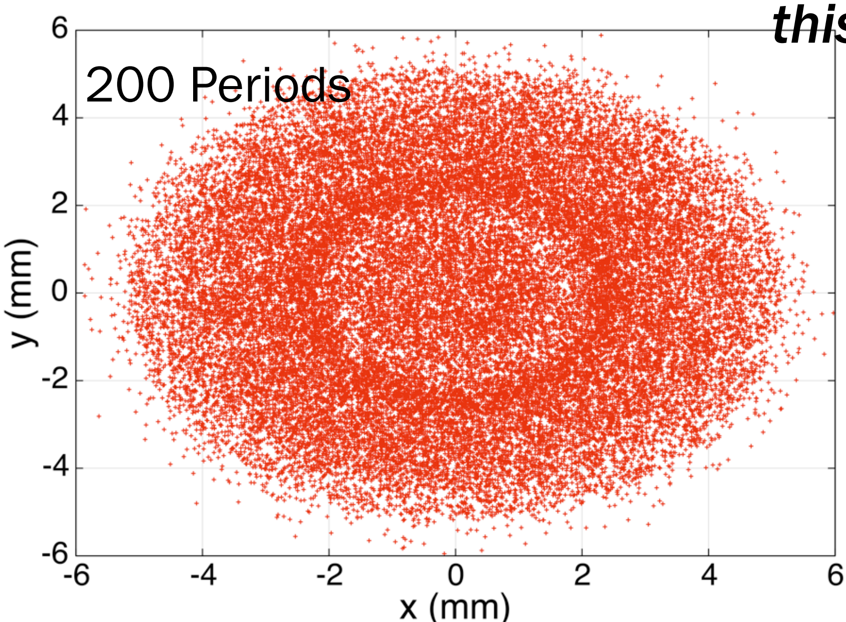
Note significant profile evolution beyond period 50

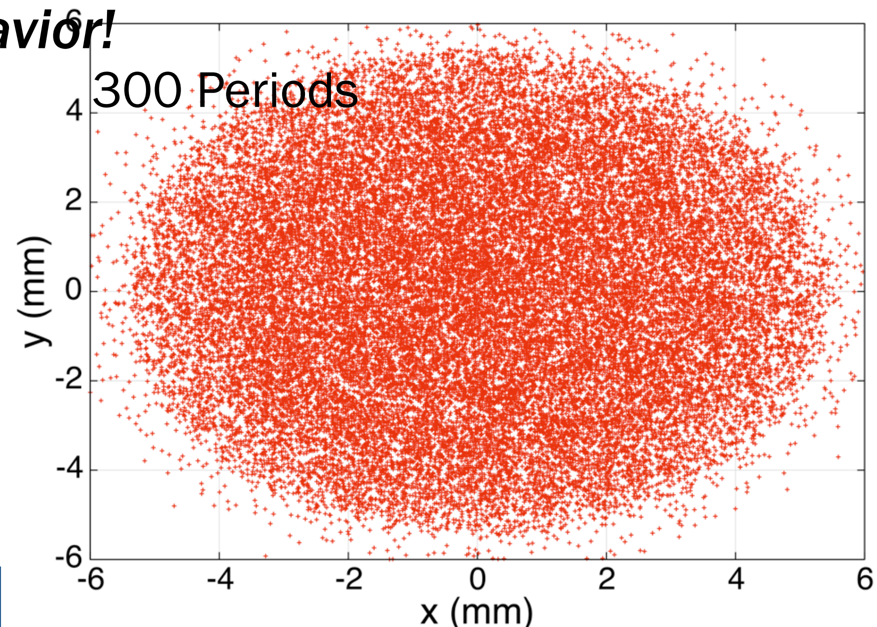
Mismatched K-V Beam in a Constant Focusing Channel for the Same Parameters: Evolution of Instability





4D Kurth beam does not show this behavior!





- Linear channel with periodic s-dependent focusing

4D Kurth Beam Test, Periodic Focusing: Problem Setup

100K particles, 1D Gauss' Law solver

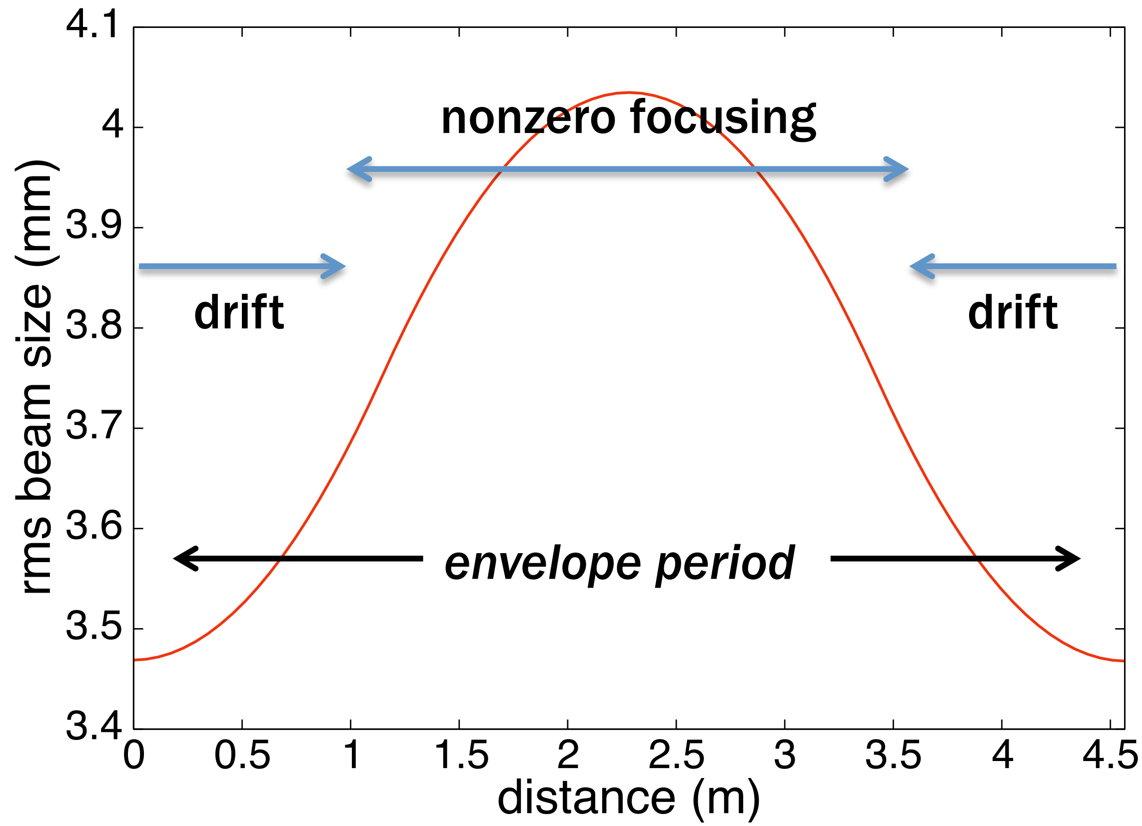
200 MeV proton beam
Focusing: B=0 or 1.91 T
ε = 10.14 mm-mrad
20 A current

SC tune depression:

$$\sigma/\sigma_0 = 0.56$$

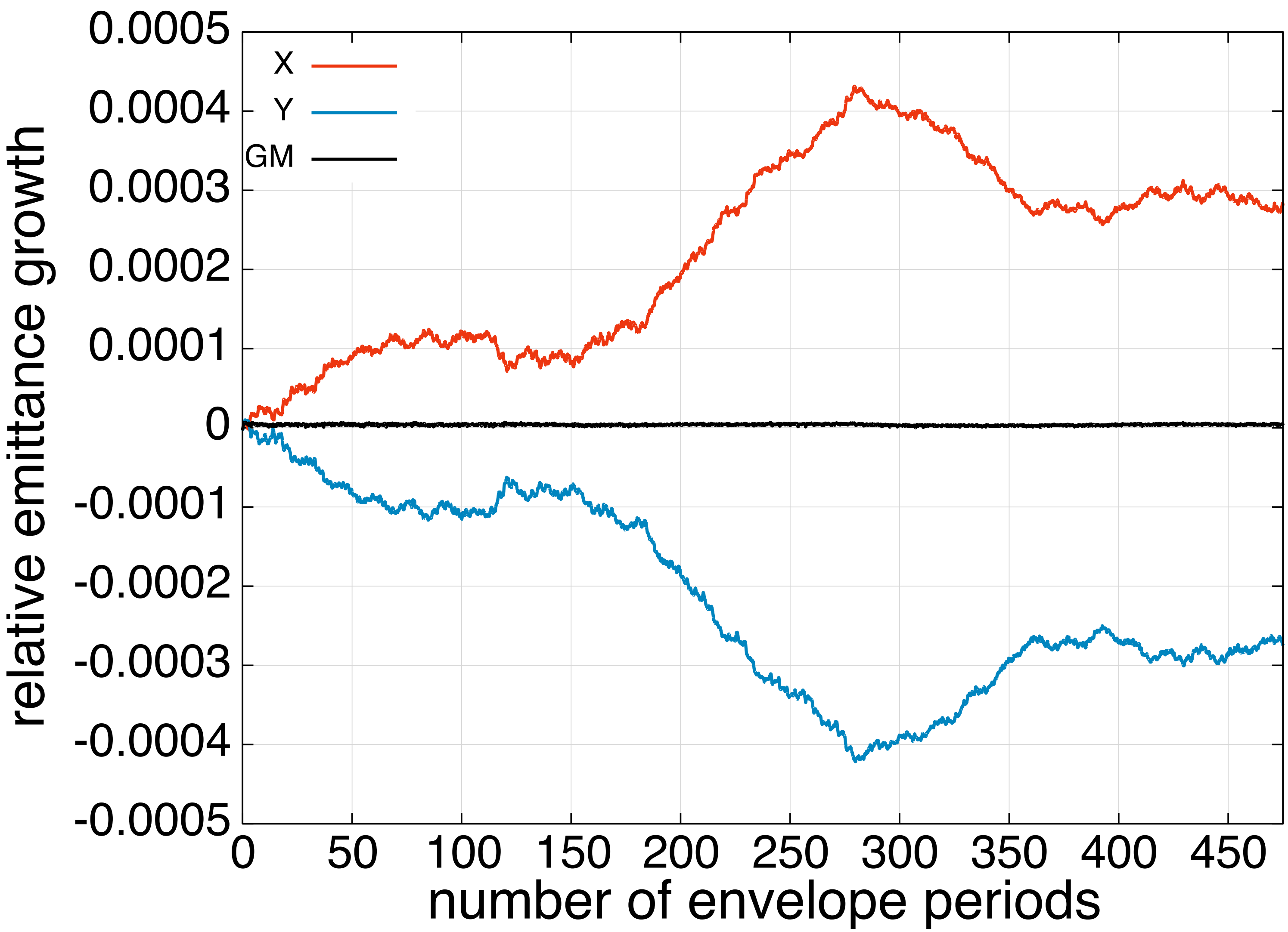
Beam is rms matched to the periodic channel.

Matched beam size is 34% larger than zero-current match.



$$\sigma = 47.6^\circ, \quad \sigma_0 = 84.4^\circ$$

Matched 4D Kurth Beam in a Periodic Focusing Channel: Emittance Evolution

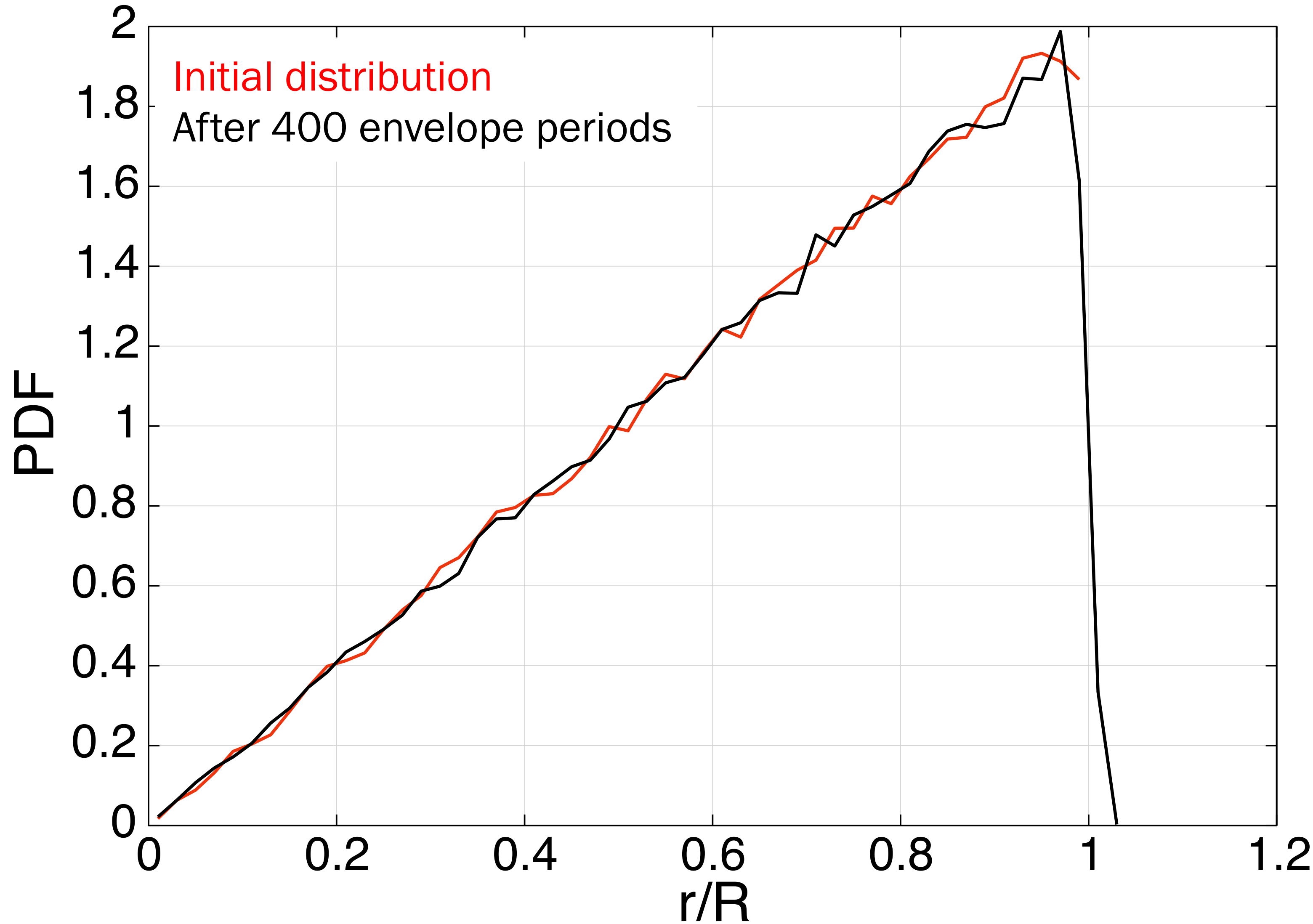


Emittance is well-preserved.

The Kurth beam appears stable for this problem.

A K-V beam also appears stable for the same problem.

Matched 4D Kurth Beam in a Periodic Focusing Channel: Radial Profile



Distribution is well-preserved.

4D Kurth Beam Test, Periodic Focusing: Problem Setup

100K particles, 1D Gauss' Law solver

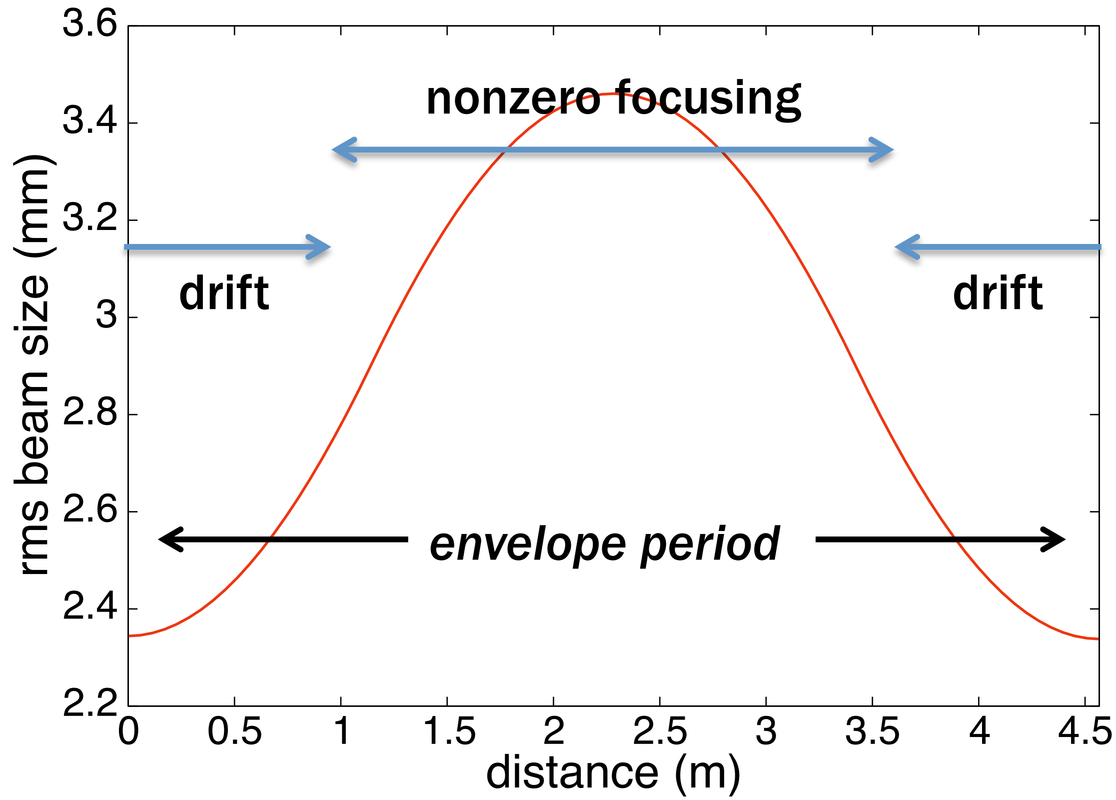
200 MeV proton beam
Focusing: B=0 or 2.7 T
ε = 10.14 mm-mrad
20 A current

SC tune depression:

$$\sigma/\sigma_0 = 0.67$$

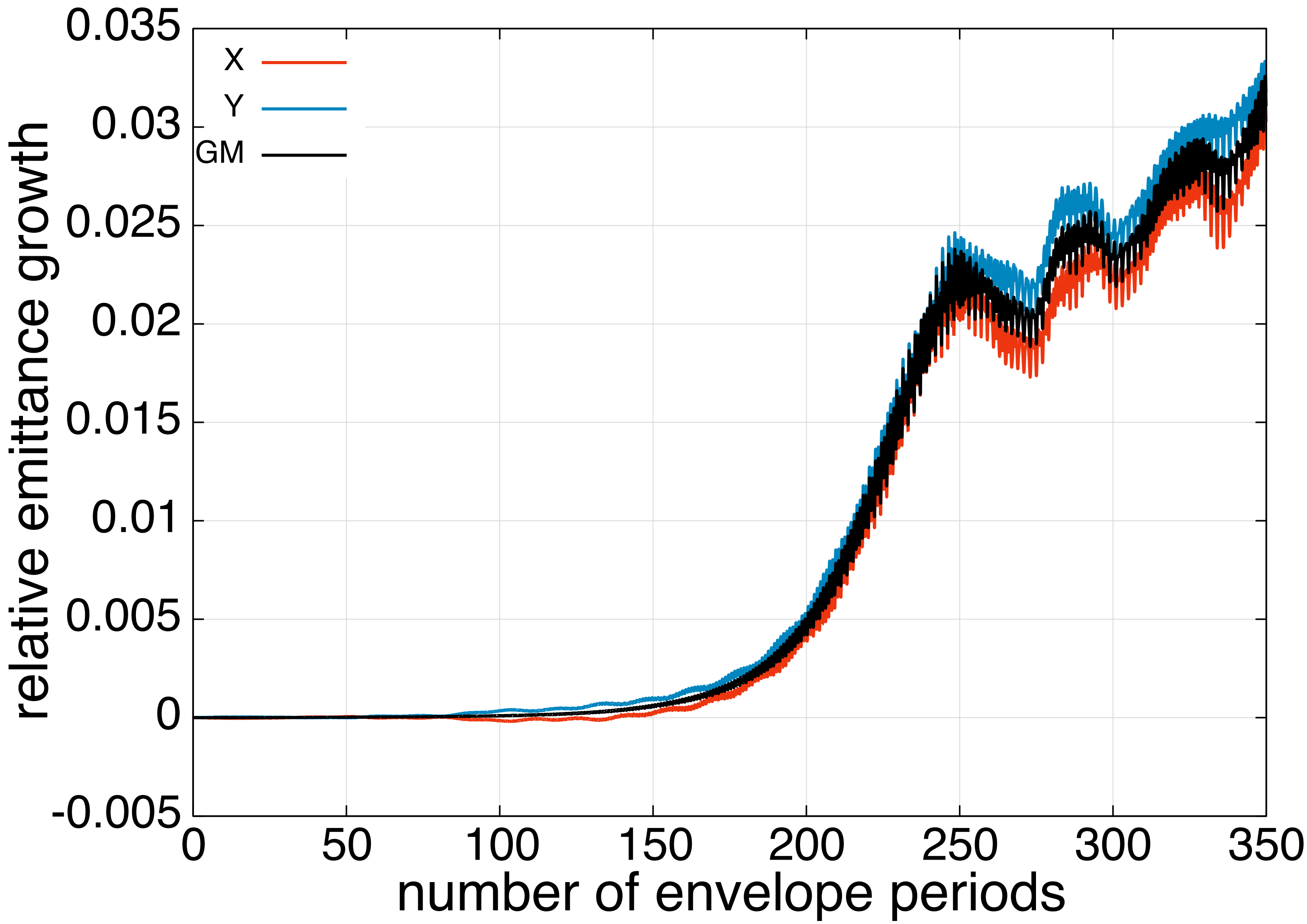
Beam is rms matched to the periodic channel.

Matched beam size is 28% larger than zero-current match.



$$\sigma = 83.8^\circ, \quad \sigma_0 = 125^\circ$$

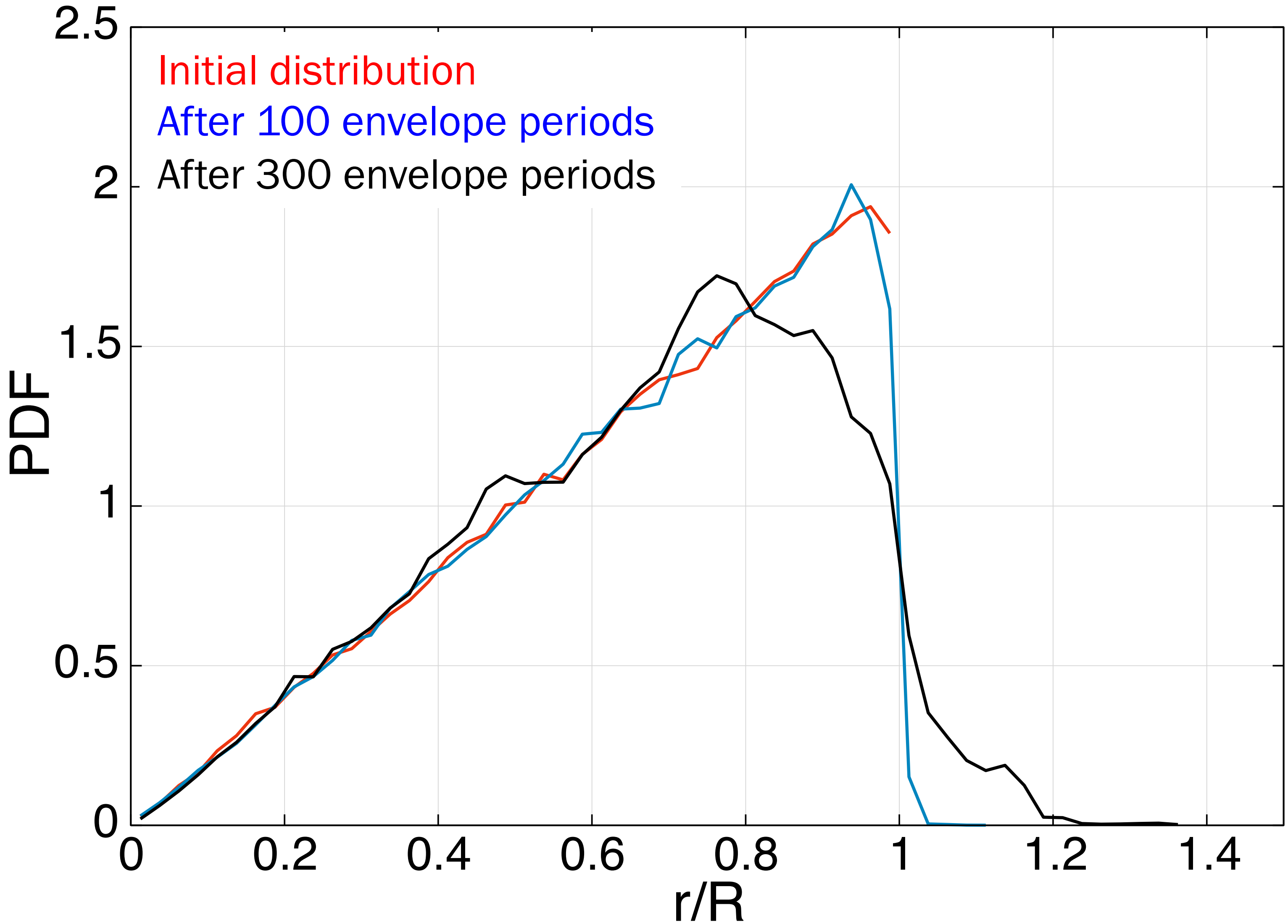
Matched 4D Kurth Beam in a Periodic Focusing Channel: Emittance Evolution



The Kurth beam is not always stable.

Tests in progress to see how this case compares to K-V.

Matched 4D Kurth Beam in a Periodic Focusing Channel: Radial Profile

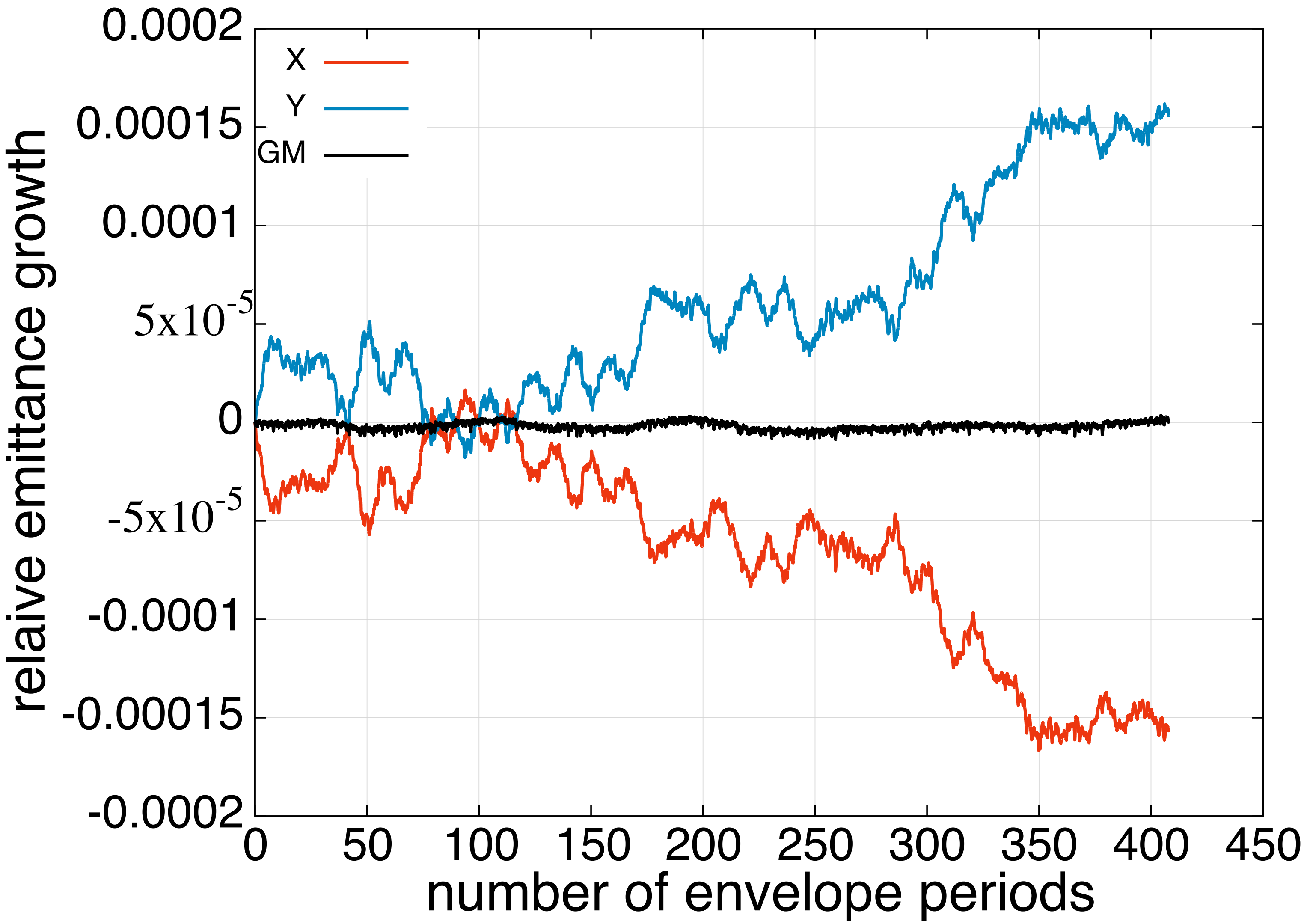


Distribution is well-preserved for the first 100 envelope periods

Conclusions

- The Kurth distribution represents an exact solution of the Vlasov-Poisson equations for an intense beam in the presence of isotropic time-dependent linear focusing.
- Unlike the K-V distribution, it can be constructed in 4D and 6D, and it is defined by a true function on the phase space. The treatment breaks down for non-symmetric beams in non-symmetric focusing channels.
- In studies of two 4D benchmarks at high space charge intensity, the Kurth distribution behaves as expected. The distribution (and emittance) are well-preserved on moderate time scales.
- The 4D Kurth beam shows different behavior with regard to space charge instabilities compared to the K-V beam, and this deserves detailed exploration.
- Benchmarks will next be performed for a 6D Kurth beam with 3D space charge.

Matched K-V Beam in a Periodic Focusing Channel: Emittance Evolution



Emittance is well-preserved.

The K-V beam appears stable for this problem.