Note added after discussions.

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After further considerations, there are significant technical challenges with trying to use the Muon g-2 Storage Ring for an electric dipole moment (EDM) measurement:

- A deuteron EDM measurement at 10^{-26} $\mathrm{e{-}cm}$, equivalent to 10^{-27} $\mathrm{e{-}cm}$ for the neutron and 10^{-28} $\mathrm{e{-}cm}$ for the electron, is still an order of magnitude less sensitive than the current electron EDM limit. A hybrid-symmetric (HS) ring measurement would not suffer from this issue.
- The vertical E-field issue presents the main systematic error challenge that would need to be understood by developing a rather sophisticated Fabry-Perot E-field monitoring system, for the region between the E-field plates located at about $15\ \mathrm{m}$ apart. This system is expected to add to the cost and time that is needed to ensure the viability of the method. The storage is simultaneous and does not need to be tested ahead of time in the HS ring design.
- The geometrical phases are a major source of systematic errors, and the best chance of combating them is to have the horizontal spin precession cancelled locally and not just on average, see Yuri Orlov, EDM note $\#\ 26$, Dec. 5th , 2002 (attached). Using the Muon g-2 Storage Ring with 80% azimuthal coverage of the ring by E-field plates leads to an inevitable enhancement by a few orders of magnitude of the geometrical phase systematic error. This feature probably means that we are going to be limited by this geometry and we are not going to learn much more, i.e., stability of vertical E-field direction, etc. The HS ring method does not suffer from this issue by design.
- The deuterons need to be read out by an efficient polarimeter. At $0.5\ \mathrm{GeV}/c$ there is no known efficient polarimeter, and one would need to be developed for the needs of the experiment. This process will be time consuming and expensive to achieve. A HS ring can store magic momentum protons with an appropriate existing polarimeter design.
- The velocity of the deuteron beam means that the Coulomb scattering rate and the inter beam scattering (IBS) lifetime require very high vacuum, which will be quite expensive to achieve and maintain and will limit the stored beam intensity and EDM sensitivity. The proton velocity in the HS ring method is $\beta = 0.6$, and the vacuum requirements and IBS parameters have been studied and considered.
- The transition from the E-field region to no E-field region requires the modification of the pole-pieces in the no E-field region, shaving off about 10% from each pole piece. An appropriate modification of the pole-pieces is expected to be quite expensive. No such issue exists in the HS ring design.

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Muon EDM Note No. 26

Title: In the EDM Experiment, The Radial Electric Field Must Compensate the Vertical Magnetic Field Locally, not on the Average

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IN THE EDM EXPERIMENT, THE RADIAL ELECTRIC FIELD MUST COMPENSATE THE VERTICAL MAGNETIC FIELD LOCALLY, NOT ON THE AVERAGE

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In this EDM Note, I will show that the condition of zero g-2 precession

$$\omega_a = aB_V - \left(a - \frac{1}{\gamma^2 - 1}\right)\beta E_R = 0, \tag{1}$$

(the cancellation condition) must be satisfied locally, at every azimuth $\theta \equiv 2\pi s / L$, and not simply on the average over the whole ring. (L is the length of the ring.) That means that the canceling electric field E_R must be present at every point where the dipole magnetic field B_V is present. More precisely, the designed ω_a-value (1) must not vary as a function of the azimuth along the equilibrium orbit of muons. I will call such variations, when they are present, the oscillations of ω_a, because in the muon rest frame they are perceived as oscillations in time. In a lattice design with only the average cancellation, periodic rotations of the g-2 by the magnetic field B_V are periodically canceled by the inverse rotations of the g-2 by the electric field E_R, with some period *L/k*, or *T/k*. in time; *T* is the revolution period, and *k* is an integer depending on how we have designed the lattice. Such ω_a oscillations around its zero value are not dangerous by themselves. The problem is that there always exist different field perturbations, some of them having the same longitudinal period, *T/k* , as ω_a. These perturbations can also not be dangerous by themselves, but acting in resonance with the ω_a oscillations they can produce a significant g-2 rotation in the vertical plane. That, of course, is deadly dangerous for EDM. I have investigated the resonance between the oscillations of the longitudinal magnetic field, $B_L = B_L(s) = B_L(vt)$, and the ω_a oscillations. Below is a brief description of my results.

Let the lattice have k identical substructures, and inside every substructure (having the length L/k, or, in the time variable, T/k), the designed fields change as follows:

$$B(s)=\begin{cases} B_V, & 0<s<l \\ 0, & l<s<L/k-l \end{cases} \tag{2}$$

$$E(s)=\begin{cases} 0, & 0<s<l \\ E_R, & l<s<L/k-l \end{cases} \tag{3}$$

with the cancellation condition

$$aB_V l=\left(a-\frac{1}{\gamma^2-1}\right)\beta E_R(L/k-l). \tag{4}$$

Then ω_a oscillates with the frequency $2\pi k/L$ in space, that means, with the frequency f=$2\pi k/T$ in time, and with the amplitude $\delta\omega_a$,

$$\delta\omega_a=\frac{2}{\pi}\frac{L/k}{(L/k-l)}a\frac{eB_V}{mc}\sin\frac{\pi kl}{L}. \tag{5}$$

Note that by virtue of the condition (4), $\delta\omega_a\neq 0$ when $l\to 0$ or $l\to L/k$. The case without oscillations corresponds to $k=0$.

Let there also exist a longitudinal magnetic field perturbation, $B_L=B_L(s)$, with the same longitudinal mode, $2\pi k/L$, as ω_a. As a function of time, $t=s/v$, $L=vT$,

$$\omega_a(t)=\delta\omega_a\cos(2\pi t/T), \tag{6}$$

$$B_L(t)=b\cos(2\pi kt/T+\phi). \tag{7}$$

The longitudinal magnetic field does not perturb the beam dynamics of the muons moving along the equilibrium orbit. Let us consider only these ideal muons, assuming also the absence of any perturbations but (6)—which can be a result of a lattice design—and (7), the undesigned perturbation. The BMT equations written for the 4-dimensional lab-frame spin in the cylindrical coordinates are very simple in the presence of only (6) and (7). The successive transformation of the lab-frame spin projections into the rest frame is also elementary for the ideal muons:

$$s'_R=s_R;\quad s'_V=s_V;\quad s'_L=\gamma(s_L-\beta s_0), \tag{8}$$

where s_0 is the 4th component of the spin. Taking into account the orthogonality of the 4-spin and 4-momentum, we have $s_0=\beta s_L$, so

$$s'_L = s_L / \gamma \,. \tag{9}$$

s denotes the lab-, and s' the rest-frame spin. All this leads to the following spin equations:

$$ds'_L / dt = s'_R \delta\omega_a \cos 2\pi kt / T, \tag{10}$$

$$ds'_R / dt = -s'_L \delta\omega_a \cos 2\pi kt / T + s'_V \omega_L \cos(2\pi kt / T + \phi), \tag{11}$$

$$ds'_V / dt = -s'_R \omega_L \cos(2\pi kt / T + \phi). \tag{12}$$

Here

$$\omega_L \equiv eb(1+a) / mc\gamma \,, \tag{13}$$

while $2\pi / T \equiv \omega_C$, the muon revolution frequency, so $2\pi kt / T \equiv k\omega_C t$.

Consider first the case $\phi = 0$. In this case, equations (10)-(12) have a simple analytical solution from which we can see that the perturbation $B_L(t)$ with the same frequency and phase as $\omega_a(t)$, see (6), (7) is not dangerous. To get the solution we can simply introduce a new variable u, $du = dt \cdot \cos 2\pi kt / T$, after which we have equations with constant coefficients. I will show here only the solution for $s_V(t)$, our main concern. If the initial time $t_0 = 0$, and the initial spin projections $s'_V(0) = s'_{V0}$, $s'_R(0) = s'_{R0}$, then for $\phi = 0$

$$s'_V(t) = s'_{V0} - s'_{R0} \frac{\omega_L}{k\omega_C} \left[1 - \frac{\delta\omega_a^2 + \omega_L^2}{6(k\omega_C)^2} \sin^2 k\omega_C t \right] \sin k\omega_C t \,. \tag{14}$$

In (14), there are only small fast oscillations of the vertical spin around its initial value.

The situation is radically different when $\phi \neq 0$. The out-of-phase mode of B_L, $b \sin\phi \cdot \sin k\omega_C t$ in this case, since we define the ω_a-oscillations as $\omega_a = \delta\omega_a \cos k\omega_C t$, rotates spin in the vertical plane. Let us consider a case with only this mode. The equations now are:

$$ds'_L / dt = s'_R \delta\omega_a \cos k\omega_C t \tag{15}$$

$$ds'_R / dt = -s'_L \delta\omega_a \cos k\omega_C t - s'_V \omega_L \sin\phi \sin k\omega_C t \tag{16}$$

$$ds'_V / dt = s'_R \omega_L \sin\phi \sin k\omega_C t \tag{17}$$

There are no variables now that can help to simplify the equations. The solution of (15)-(17) can be found, however, as an infinite Fourier series. The preliminary analysis had shown that this series is a sum of terms with a low frequency connecting only the longitudinal, s'_L , and the vertical, s'_V, projections, and terms with high frequencies of the orders $k\omega_C$, $2k\omega_C$, etc. Thus, assume

$$s'_L(t) = s'_{L,slow}(t) + s'_{L,fast}(t)\,; \quad s'_V(t) = s'_{V,slow}(t) + s'_{V,fast}(t)\,. \tag{18}$$

In the first approximation, we can neglect the time derivatives of the slow functions. This immediately gives us the first approximation for s'_R, which is a fast function:

$$s'_R(t) = -\frac{\delta\omega_a}{k\omega_C} s'_{L,slow} \sin k\omega_C t + \frac{\omega_L}{k\omega_C} s'_{V,slow} \sin\phi \cos k\omega_C t\,. \tag{19}$$

Substituting this into equations (15) and (17), and keeping there only slow functions (in this first approximation), we get equations connecting the vertical spin projection with the longitudinal one.

$$ds'_{L,slow}/dt = \frac{\omega_L \delta\omega_a}{2k\omega_C} \sin\phi \cdot s'_{V,slow} \tag{20}$$

$$ds'_{V,slow}/dt = -\frac{\omega_L \delta\omega_a}{2k\omega_C} \sin\phi \cdot s'_{L,slow}. \tag{21}$$

Therefore, the muon spin in this case is rotated in the vertical plane with the angular frequency

$$\Omega_V = \left|\frac{\omega_L \delta\omega_a \sin\phi}{2k\omega_C}\right|. \tag{22}$$

It is amazing that the spin rotates around the radial, not the longitudinal axis—as a result of the combination of the "oscillating rotations" around the longitudinal axis (perturbation ω_L), and the "oscillating rotations" around the vertical axis (perturbation $\delta\omega_a$).

The next approximations are not important. The technique of separating slow and fast oscillations that I have used here is a well-known one; it was apparently first developed by Lyapunov (in the middle of the 19th century) and then by Bogolyubov and others (in the middle of the 20th century).

Let us estimate the permitted $\delta\omega_a$, assuming (for a strong focusing ring) $\omega_L / \omega_C > 10^{-5}$, k=8, $\sin\phi = 1$, and the permitted vertical spin angle, developing during some $10^{-4}\,\mathrm{sec}$, $\theta_t < 10^{-7}$. Under these assumptions, the permitted $\delta\omega_a$ must be less than 7 sec^{-1}. Obviously, we will need to carefully minimize the local violations of condition (1).