Entangled states of neutrinos as qutrits (#8)

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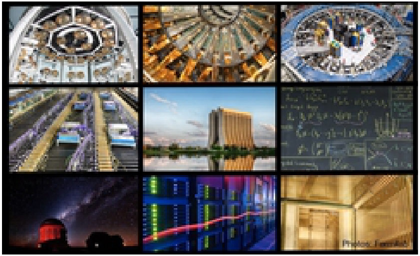


Overview

Introduction

- The neutrino flavor states $|\nu_\alpha\rangle$ ($\alpha = e, \mu, \tau$) are linear superposition of mass eigenstates $|\nu_j\rangle$ ($j = 1, 2, 3$): $|\nu_\alpha\rangle = \Sigma_j U^*_{\alpha j} |\nu_j\rangle$, where the asterisk denotes the complex conjugation of $U_{\alpha j}$ and $U_{\alpha j}$ are the elements of the lepton mixing matrix known as PMNS (Pontecorvo-Maki- Nakagawa-Sakita) matrix

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix} \tag{1}$$

  where, $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ and δ_{CP} is CP violation phase.
- The time evolution follows $|\nu_\alpha(t)\rangle = \Sigma_j e^{-iE_j t} U^*_{\alpha j} |\nu_j\rangle$, where E_j is the energy associated with the mass eigenstates $|\nu_j\rangle$. This is a superposition state. Therefore, we expect quantum entanglement [1,2].

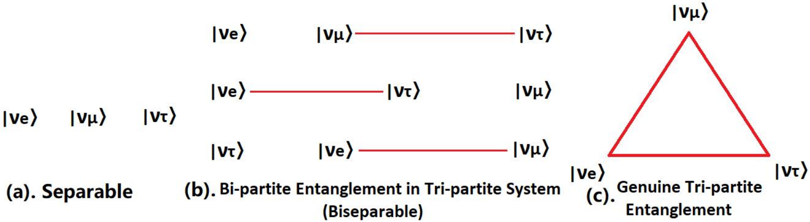
[1] M.Blasone et al. Phys.Rev.D 77 (2007) no.9, 096002

[2] M.Blasone et al. EPL **85** (2009), 50002

- Investigating quantum effects in neutrino oscillation can be a very interesting issue, since in this case, quantum correlations such as quantum coherence, which is a micro-scopic quantum effect, can be studied in large distances even up to several hundred kilometers away.
- We identify each flavor state $|\nu_\alpha\rangle$ $(\alpha = e, \mu, \tau)$ at t=0 as three qubit states:

$$
\begin{aligned}
|\nu_e\rangle &= |1\rangle_e \otimes |0\rangle_\mu \otimes |0\rangle_\tau \equiv |100\rangle_e \,, \\
|\nu_\mu\rangle &= |0\rangle_e \otimes |1\rangle_\mu \otimes |0\rangle_\tau \equiv |010\rangle_\mu \,, \\
|\nu_\tau\rangle &= |0\rangle_e \otimes |0\rangle_\mu \otimes |1\rangle_\tau \equiv |001\rangle_\tau \,.
\end{aligned}
$$

- These type of qubit states have these three types of entanglement:



Bi-partite entanglement in two-flavor neutrino oscillations

- The two neutrino state space $\mathcal{H}_\nu$ can be seen as the two-qubit Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ spanned by $\{|1\rangle_1 \otimes |0\rangle_2 , |0\rangle_1 \otimes |1\rangle_2\}$, by means of the unitary equivalence defined on the mass basis by $|\nu_1\rangle = |1\rangle_1 \otimes |0\rangle_2$ and $|\nu_2\rangle = |0\rangle_1 \otimes |1\rangle_2$. Then, observe that in this two-qubit representation is a bipartition of the space of quantum states available, relative to which entanglement can be considered. Correspondingly, a neutrino state which is entangled as a two qubit state is said to be mode entangled.
- In two-flavor ($\nu_\alpha \to \nu_\beta$) mixing, $|\nu_e(t)\rangle = \tilde{U}_{ee}(t)\,|10\rangle_e + \tilde{U}_{e\mu}(t)\,|01\rangle_\mu$, where $|10\rangle_e$ and $|01\rangle_\mu$ are two-qubit flavor mode states,

$$
\begin{aligned}
|\nu_e\rangle &= |1\rangle_e \otimes |0\rangle_\mu \equiv |10\rangle_e\,, \\
|\nu_\mu\rangle &= |0\rangle_e \otimes |1\rangle_\mu \equiv |01\rangle_\mu\,.
\end{aligned}
$$

- The appearance (P_a) and disappearance (P_d) probabilities are:

$$P_a = |\tilde{U}_{e\mu}(t)|^2 = \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\cos\left(\frac{\Delta m^2 t}{2E}\right),$$
$$\text{and} \quad P_d = |\tilde{U}_{ee}(t)|^2 = 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{\Delta m^2 t}{4E}\right). \qquad (2)$$

- The density matrix for $|\nu_e(t)\rangle$ can be expressed as

$$\rho^{e\mu}(t) = |\nu_e(t)\rangle\langle\nu_e(t)| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |\tilde{U}_{ee}(t)|^2 & \tilde{U}_{ee}(t)\tilde{U}^*_{e\mu}(t) & 0 \\ 0 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{ee}(t) & |\tilde{U}_{e\mu}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3)$$

- Positive Partial Transpose (PPT) criterion is a condition for determining entanglement in bi-partite system. It states that if the partial transposition $\rho^{T_e}_{pq,rs}(t) = \rho^{e\mu}_{rq,ps}(t)$ or $\rho^{T_\mu}_{pq,rs}(t) = \rho^{e\mu}_{ps,rq}(t)$ of a density matrix is a positive operator with all positive eigenvalues then the system is unentangled. If the system has even one negative eigenvalues then it is entangled [3] .
- Using PPT criterion for entanglement the eigenvalues of $\rho^{T_\mu}(t)$ are $\lambda_1 = P_d$, $\lambda_2 = P_a$, $\lambda_3 = \sqrt{P_d P_a}$, and $\lambda_4 = -\sqrt{P_d P_a}$ which shows the state $|\nu_e(t)\rangle$ is entangled since one of them is negative.
- Consequently, Negativity [4] $N(\rho^{e\mu}(t)) = ||\rho^{T_\mu}(t)|| - 1 = 2\sqrt{P_a P_d}$.

[3] P.Horodecki et al. Phys. Lett. A **232** (1997), 333

[4] Yong-Cheng. OU, et al.PhysRevA.75.062308 (2007)

Concurrence and Tangle

- Non-locality measures like concurrence and tangle are strong aspects of quantum correlations [5,6,7].
- Using the "Spin-flipped" density matrix,

$$\tilde{\rho}^{e\mu}(t) = (\sigma_y \otimes \sigma_y)\rho^{*\,e\mu}(t)(\sigma_y \otimes \sigma_y) \quad (4)$$

  where σ_x and σ_y are Pauli matrices, we calculate concurrence:

$$C(\rho^{e\mu}(t)) \equiv [max(\mu_1 - \mu_2 - \mu_3 - \mu_4, 0)] = 2\sqrt{P_a P_d} \quad (5)$$

  in which $\mu_1, ..., \mu_4$ are the eigenvalues of the matrix $\rho^{e\mu}(t)\tilde{\rho}^{e\mu}(t)$.
- Tangle:
  $\tau(\rho^{e\mu}(t)) \equiv [max(\mu_1 - \mu_2 - \mu_3 - \mu_4, 0)]^2 = 4P_a P_d$.
- Linear entropy [2]:
  $S(\rho^{e\mu}(t)) = 1 - [Tr_\mu(\rho^{e\mu}(t))]^2 = 4P_a P_d$.

[5] William K. Wooters, Phys.Rev.Lett.80.2245 (1998)

[6] A.K.Alok et al.Nucl. Phys. B **909** (2016)

[7] V.coffman et al.Phys. Rev. A 61, 052306 (2000)

- We see that entanglement measures between e and μ modes as:

$$\tau_{e\mu} = C^2_{e\mu} = N^2_{e\mu} = S_{e\mu} = 4P_aP_d. \tag{6}$$

- When $P_a = P_d = 0.5$, all measures of entanglement tend to 1 i.e, $N_{e\mu} = \tau_{e\mu} = C_{e\mu} = S_{e\mu} = 1$, which corresponds to maximally entangled state.
- This result shows that the two flavor neutrino oscillation is a bipartite entangled system of two qubit pure states and these quantum correlations have a direct experimental connection with physical quantities in neutrino oscillations.

Tri-Partite entanglement in three-flavor neutrino oscillations

- The density matrix of the time evolved electron neutrino flavor state is $\rho_{e\mu\tau}(t) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\tilde{U}_{ee}(t)|^2 & 0 & \tilde{U}_{ee}(t)\tilde{U}^*_{e\mu}(t) & \tilde{U}_{ee}(t)\tilde{U}^*_{e\tau}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{ee}(t) & 0 & |\tilde{U}_{e\mu}(t)|^2 & \tilde{U}_{e\mu}(t)\tilde{U}^*_{e\tau}(t) & 0 \\ 0 & 0 & 0 & \tilde{U}_{e\tau}(t)\tilde{U}^*_{ee}(t) & 0 & \tilde{U}_{e\tau}(t)\tilde{U}^*_{e\mu}(t) & |\tilde{U}_{e\tau}(t)|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Biseparable states are formed in a three particle system, by considering two out of three mode state as a single state. The relation between entanglement measures are:

$$N^2_{e(\mu\tau)} = C^2_{e(\mu\tau)} = \tau_{e(\mu\tau)} = S_{e(\mu\tau)} = 4P_aP_d \tag{7}$$

- For tri-partite entanglement, criterion is known as **Coffman-Kundu-Wooters (CKW) inequality**. It states that the sum of quantum correlations between e and μ, and between e and τ, is either less than or equal to the quantum correlations between e and $\mu\tau$ (treating it as a single object) [4,8]: $C^2_{e\mu} + C^2_{e\tau} \leq C^2_{e(\mu\tau)}$, $\tau_{e\mu} + \tau_{e\tau} \leq \tau_{e(\mu\tau)}$ and $N^2_{e\mu} + N^2_{e\tau} \leq N^2_{e(\mu\tau)}$.

Bi-separable entanglement measures	Results from $\rho^{e\mu\tau}(t)$
1. Concurrence equality	$C^2_{e\mu} + C^2_{e\tau} = C^2_{e(\mu\tau)}$
2. Tangle equality	$\tau_{e\mu} + \tau_{e\tau} = \tau_{e(\mu\tau)}$
3. Negativity inequality	$N^2_{e\mu} + N^2_{e\tau} < N^2_{e(\mu\tau)}$

[8] V.Coffman et al. Phys.Rev.A.61.052306,(2000)

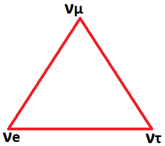
- There are two extra measure for genuine tri-partite entanglement quantified by **three-tangle** and **three-**π negativity known as residual entanglement.
- The residual entanglement three-π for electron neutrino flavor state $|\nu_e(t)\rangle$ is,

$$\begin{aligned}
\pi_{e\mu\tau} &= \frac{4}{3}[|\tilde{U}_{ee}(t)|^2\sqrt{|\tilde{U}_{ee}(t)|^4+4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\tau}(t)|^2} \\
&+ |\tilde{U}_{e\mu}(t)|^2\sqrt{|\tilde{U}_{e\mu}(t)|^4+4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\tau}(t)|^2} \\
&+ |\tilde{U}_{e\tau}(t)|^2\sqrt{|\tilde{U}_{e\tau}(t)|^4+4|\tilde{U}_{ee}(t)|^2|\tilde{U}_{e\mu}(t)|^2} \\
&- |\tilde{U}_{ee}(t)|^4-|\tilde{U}_{e\mu}(t)|^4-|\tilde{U}_{e\tau}(t)|^4].
\end{aligned} \tag{8}$$

- When transition probabilities are $P_{\nu_{e\to e}}=0.39602$, $P_{\nu_{e\to\mu}}=0.435899$, and $P_{\nu_{e\to\tau}}=0.168081$, $\pi_{e\mu\tau}$ reaches the maximum value 0.436629.

Residual Entanglement	Tri-Partite results for ν_e disappearance
Three-tangle $\tau_{e\mu\tau} = C^2_{e(\mu\tau)} - C^2_{e\mu} + C^2_{e\tau}$	$\tau_{e\mu\tau} = 0$
Three-π $\pi_{e\mu\tau} = \frac{1}{3}(N^2_{e(\mu\tau)} + N^2_{\mu(e\tau)} + N^2_{\tau(e\mu)} - 2N^2_{e\mu} - 2N^2_{e\tau} - 2N^2_{\mu\tau})$	$\pi_{e\mu\tau} > 0$

- The residual entanglement three-tangle vanish, but the non-zero value of three-π result shows that the three flavor neutrino oscillations has a genuine form of tri-partite entanglement.
- Correlations exhibited by neutrino oscillations in tri-partite system are like the W-states which are legitimate physical resources for quantum information tasks [9].



Genuine Tri-partite Entanglement

[9] A.K.Jha et al., Modern Physics Letters A, Vol. 36, No. 09, 2150056 (2021), arXiv:2004.14853v2 [hep-ph]

Quantum computer circuit to simulate bipartite entanglement in the two flavor neutrino oscillation on the IBMQ platform

- A quantum computer is built from a quantum circuit containing quantum gates to carry quantum information [10] .
- We identified that the SU(2) rotation matrix $R(\theta)$ can be encoded in the IBM quantum computer by using the universal quantum gate $U3$:

$$U3(\Phi, \phi, \lambda) = \begin{pmatrix} cos\frac{\Phi}{2} & -sin\frac{\Phi}{2} e^{i\lambda} \\ sin\frac{\Phi}{2} e^{i\psi} & cos\frac{\Phi}{2} e^{i(\lambda+\psi)} \end{pmatrix} \tag{9}$$

- For the two-neutrino system, oscillation probabilities are depend on one of the parameters of U3 gate. Thus,

$$R(\theta) = U3(-2\theta, 0, 0) = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \equiv \begin{pmatrix} \tilde{U_{ee}} & \tilde{U_{e\mu}} \\ \tilde{U_{\mu e}} & \tilde{U_{\mu\mu}} \end{pmatrix} \tag{10}$$

[10] C.A. Argüelles et al.Phys.Rev.Research. 1 (2019) 033176

- The time-evolution operation can be identified as S-gate, where $\psi = \frac{\Delta m^2 t}{2E}$

$$S(\psi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\psi} \end{pmatrix} = U1(t). \tag{11}$$

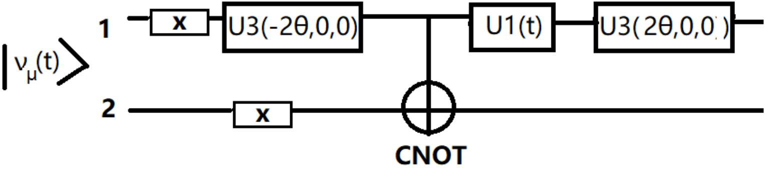


Figure: Using the Controlled-NOT (CNOT) gate, we construct quantum circuit for the time evolved muon flavor neutrino state in a linear superposition of flavor basis ($\nu_\mu \to \nu_e$), in the two qubit system: $|\nu_\mu(t)\rangle = \tilde{U}_{\mu e}(t)\,|10\rangle_e + \tilde{U}_{\mu\mu}(t)\,|01\rangle_\mu$. Here, 1 and 2 represent the input qubits first and second, respectively

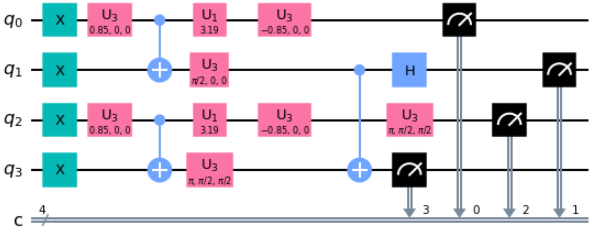


Figure: Implementation of concurrence circuit for $|\nu_\mu(t)\rangle$ on IBMQ processor [11].

- In order to measure concurrence, first we prepare two decouple copies of bi-partite neutrino state $|\nu_\alpha(t)\rangle \otimes |\nu_\alpha(t)\rangle$ in the two flavor system (where $\alpha = e, \mu$), and apply a "spin-flipped" operation $\sigma_y \otimes \sigma_y$ on one of the two copies to prepare an arbitrary global state of neutrino in the four qubit system followed by CNOT and Hadamard gate (H) .
- We find that the concurrence value of the time evolved flavor neutrino oscillation can be extracted from the global state [12].

[11] I. Esteban, et al.JHEP 09 (2020)

[12] G.Romero et al.Phys.Rev.A.75.032303 (2007)

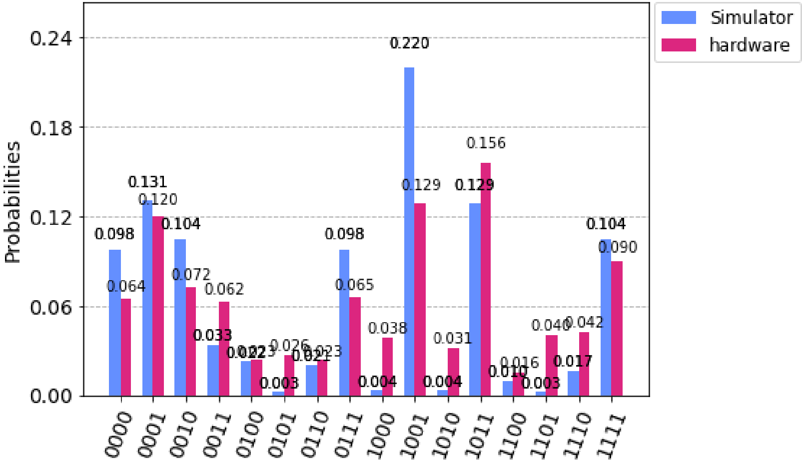


Figure: The concurrence varies with time at the IBMQ computer for an initial muon neutrino flavor state. The concurrence information encoded in the coefficients of four qubit global state basis are shown through Histogram (probabilities in percentage) plot.

Poincare$^{'}$ sphere representation of two-flavor neutrino oscillations

- The density matrix correspond to a pure state $|\psi\rangle$ in two dimensional Hilbert space $\mathcal{H}^2$ is given by the projection $\rho = |\psi\rangle\langle\psi|$. Its expansion in terms of Pauli matrices σ_j leads to the Poincare$'$ sphere construction [13]:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \hat{n}.\vec{\sigma}) \tag{12}$$

where $\rho^\dagger = \rho^2 = \rho \geq 0,\ Tr\rho = 1 \implies \hat{n}^* = \hat{n},\ \hat{n}.\hat{n} = 1 \iff \hat{n} \in S^2$ is the unit vector on the sphere .

- Thus, there is a one to one correspondence between pure qubit states and points on the unit sphere S^2 embedded in R^3. This is known as the Poincare$'$ sphere construction (of which the Bloch sphere is a special case).

[13] Arvind et.al, J.Phys.A30:2417-2431,1997

- We parameterize a one-qubit state $|\nu_e(t)\rangle$ with θ and ϕ as:

$$|\nu_e(\theta,\phi)\rangle = e^{-iE_1 t/h}(cos\theta\,|0\rangle + sin\theta e^{-i\phi}\,|1\rangle) \quad (13)$$

where $E_1 = (p^2 + m_1^2)^{1/2}$ and $E_2 = (p^2 + m_2^2)^{1/2}$ and in the ultra-relativistic limit $\phi = \frac{(E_2 - E_1)t}{\hbar} \equiv \frac{\Delta m^2 t}{2E}$.

- We see that $|\nu_e(\theta,\phi)\rangle$ is an eigenstate of an operator $\hat{O}$ with eignvalue $+1$

$$\hat{O}\,|\nu_e(\theta,\phi)\rangle = |\nu_e(\theta,\phi)\rangle\,, \quad (14)$$

where

$$\hat{O} = \hat{n}(\theta,\phi).\vec{\sigma} = \begin{pmatrix} cos2\theta & sin2\theta e^{i\phi} \\ sin2\theta e^{-i\phi} & -cos2\theta \end{pmatrix}. \quad (15)$$

Here, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\hat{n}(\theta,\phi)$ is a real unit vector, $\hat{n}(\theta,\phi) = sin2\theta cos\phi\hat{e}_1 + sin2\theta sin\phi\hat{e}_2 + cos2\theta\hat{e}_3$ called the Poincare$'$ unit vector.

- Thus a state $|\nu_e(\theta,\phi)\rangle \in \mathcal{H}^2$ is expressed in terms of a unit vector $\hat{n}(\theta,\phi)$ on the surface of the Poincare$'$ sphere. This correspondence is one-to-one if the ranges of θ and ϕ are restricted to $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.
- The density matrix is given by

$$\rho^e = \begin{pmatrix} cos^2\theta & e^{i\phi} sin\theta cos\theta \\ e^{-i\phi} sin\theta cos\theta & sin^2\theta \end{pmatrix} = \frac{1}{2}(I + \hat{n}.\vec{\sigma}), \qquad (16)$$

  which is the same as Eq.[12].
- The eigenvalues of ρ^e are 1 and 0, therefore ρ^e is a rank 1 density matrix. This maps the neutrino state $|\nu_e(t)\rangle$ to the the surface of the unit sphere in the three dimensional vector space.
- A similar mapping can be done for the time evolved muon flavor neutrino state $|\nu_\mu(t)\rangle$.
- When $\theta \rightarrow \frac{\theta}{2}$ then the Poincare$'$ sphere becomes the Bloch sphere used in quantum optics.

Poincare$'$ sphere representation of three-flavor neutrino oscillations

- A qutrit [14] is a unit of quantum information that is realized by three mutually orthogonal states: $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
- The three flavour states of a neutrino system can be written in the qutrit basis by identifying the mass eigenstates with the qutrit basis states of the three dimension Hilbert space $\mathcal{H}^3$ as

$$|1\rangle = |\nu_1\rangle \quad ; |2\rangle = |\nu_2\rangle \quad ; |3\rangle = |\nu_3\rangle \tag{17}$$

- Using the PMNS matrix (Eq.(1)), we take $\theta_{12} = \phi$, $\theta_{13} = \theta$ and $\theta_{23} = \eta$ and assuming $\delta_{CP} = 0$. In the ultra-relativistic limit $L \approx t$ ($c = 1$), we define $\xi_1 = (E_3 - E_1)t/\hbar = \Delta m^2_{31} t/2E$, and $\xi_2 = (E_2 - E_1)t/\hbar = \Delta m^2_{21} t/2E$ (where $\hbar = 1$).

[14] Carlton M. Caves and Gerard J. Milburn, doi:10.1016/S0030-4018(99)00693-8, arXiv:quant-ph/9910001v2

- The normalized time evolved electron neutrino flavor state $|\nu_e(t)\rangle$ in qutrit basis, parametrized by three different mixing angle θ, ϕ, η and with two arbitrary phases ξ_1 and ξ_2 can be written as

$$\begin{aligned} |\nu_e(\theta,\phi,\eta,\xi_1,\xi_2)\rangle &= e^{i\xi_1} cos\theta cos\phi \,|1\rangle \\ &+e^{i\xi_2}(-cos\eta sin\phi - cos\phi sin\theta sin\eta)\,|2\rangle \\ &+(sin\eta sin\phi - sin\theta cos\phi cos\eta)\,|3\rangle \end{aligned} \tag{18}$$

- The density matrix of a state $|\nu_e(\theta,\phi,\eta,\xi_1,\xi_2)\rangle$ is

$$\rho^e = |\nu_e(\theta,\phi,\xi_1,\xi_2)\rangle\langle\nu_e(\theta,\phi,\xi_1,\xi_2)| = \frac{1}{3}(1+\sqrt{3}\hat{n}.\vec{\lambda}) \tag{19}$$

$$= \begin{pmatrix} [cos^2\theta cos^2\phi] & \begin{matrix}[cos\phi cos\theta e-^{i(\xi_2-\xi_1)} \\ (-sin\phi cos\eta - cos\phi sin\phi sin\eta)]\end{matrix} & \begin{matrix}[cos\phi cos\theta e^{i\xi_1} \\ (sin\phi sin\eta - cos\phi cos\theta \cos\eta)]\end{matrix} \\ \begin{matrix}[cos\phi cos\theta e^{i(\xi_2-\xi_1)} \\ (-sin\phi cos\eta - cos\phi sin\theta sin\eta)]\end{matrix} & (-sin\phi cos\eta - cos\phi sin\theta sin\eta)^2 & \begin{matrix}[(-sin\phi cos\eta - cos\phi sin\theta sin\eta)e^{i\xi_2} \\ (sin\phi sin\eta - cos\phi cos\theta cos\eta)]\end{matrix} \\ \begin{matrix}[cos\phi cos\theta e^{-i\xi_1} \\ (sin\phi sin\eta - cos\phi cos\theta cos\eta)]\end{matrix} & \begin{matrix}[(sin\phi sin\eta - cos\phi cos\theta cos\eta)e^{-i\xi_2} \\ (-sin\phi cos\eta - cos\phi sin\theta sin\eta)]\end{matrix} & (sin\phi sin\eta - cos\phi cos\theta cos\eta)^2 \end{pmatrix}$$

where n_j are the components of unit vector $\hat{n}$ and λ_j are the Gell-Mann matrices (j=1,...,8).

The components of $\hat{n}$ are defined as:

$$n_j = \frac{\sqrt{3}}{2} tr(\rho^e \lambda_j) = \frac{\sqrt{3}}{2} \langle \nu_e(\theta, \phi, \eta, \xi_1, \xi_2)| \lambda_j |\nu_e(\theta, \phi, \eta, \xi_1, \xi_1)\rangle \tag{20}$$

where,

$$\begin{aligned}
n_1 &= \sqrt{3} cos\theta cos\phi(-sin\phi cos\theta - sin\phi sin\theta sin\eta) cos(\xi_2 - \xi_1) \\
n_2 &= \sqrt{3} cos\theta cos\phi(-sin\phi cos\eta - cos\phi sin\theta sin\eta) sin(\xi_2 - \xi_1) \\
n_3 &= \frac{\sqrt{3}}{2}[cos^2\phi cos^2\theta - (-sin\phi cos\eta - cos\phi sin\theta sin\eta)^2] \\
n_4 &= \sqrt{3} cos\theta cos\phi(sin\phi sin\eta - cos\phi cos\theta cos\eta) cos\xi_1 \\
n_5 &= -\sqrt{3} cos\phi cos\theta(sin\phi sin\eta - cos\phi cos\theta cos\eta) sin\xi_1 \\
n_6 &= \sqrt{3}(-sin\phi cos\eta - cos\phi sin\theta sin\eta)(sin\phi sin\eta - cos\phi cos\theta cos\eta) cos\xi_2 \\
n_7 &= -\sqrt{3}(sin\phi cos\eta - cos\phi sin\theta sin\eta)(sin\phi sin\eta - cos\phi cos\theta cos\eta) sin\xi_2 \\
n_8 &= \frac{1}{2}[cos^2\theta cos^2\phi + (-sin\phi cos\eta - cos\phi sin\theta sin\eta)^2 \\
&\quad -2(sin\phi sin\eta - cos\phi cos\theta cos\eta)^2]
\end{aligned}$$

Concurrence for two qutrits flavor neutrino state

- The density matrix of the two qutrits time evolved neutrino flavor state can be expanded uniquely as

$$\rho^{\alpha\alpha'} = \rho^{\alpha} \otimes \rho^{\alpha'} = (\frac{1}{3}(I + \sqrt{3}\hat{n}.\vec{\lambda}^{\alpha}) \otimes (\frac{1}{3}(I + \sqrt{3}\hat{n}'.\vec{\lambda}^{\alpha'})$$

$$= \frac{1}{9}(I \otimes I + \sqrt{3}\vec{\lambda}^{\alpha}.\hat{n} \otimes I + \sqrt{3}I \otimes \vec{\lambda}^{\alpha'}.\hat{n}' + \frac{3}{2}\sum_{i,j=1}^{8} \beta_{ij}\lambda_i^{\alpha} \otimes \lambda_j^{\alpha'}) \quad (21)$$

- The (real) expansion coefficients are given by

$$n_i = \frac{\sqrt{3}}{2} tr(\rho^{\alpha\alpha'}\lambda_i \otimes I)$$

$$n'_j = \frac{\sqrt{3}}{2} tr(\rho^{\alpha\alpha'} I \otimes \lambda_j)$$

$$\beta_{ij} = \frac{3}{2} tr(\rho^{\alpha\alpha'}\lambda_i \otimes \lambda_j) \quad (22)$$

- The entanglement measures concurrence for the two qutrits mixed state density matrix is defined as

$$C_3(\rho^{\alpha\alpha'}) = max\{0, 2\mu_1 - \sum_{i=1}^{9} \mu_i\} \tag{23}$$

where μ_i (with i=1,2,...,9) are the square roots of the eigenvalues of the non-Hermitian matrix $\rho^{\alpha\alpha'}\tilde{\rho}^{\alpha\alpha'}$ in decreasing order, and the spin-flipped density matrix is

$$\tilde{\rho}^{\alpha\alpha'} = (O_3 \otimes O_3)\rho^{*\alpha\alpha'}(O_3 \otimes O_3) \tag{24}$$

with $\rho^{*\alpha\alpha'}$ being the complex conjugate of $\rho^{\alpha\alpha'}$ and O_3 is the transformation for qutrits

$$O_3 = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}. \tag{25}$$

Summary

- We mapped the neutrino flavor state to two and three mode states which are like qubits. We find that two and three flavor neutrino oscillations exhibits bipartite and tripartite entanglement, respectively.
- In the tripartite system the three neutrino state shows the remarkable property of having a specific form of three-way entanglement similar to the W-state, used in quantum information theory.
- We proposed the implications of the implementation of bipartite entanglement in the two neutrino system on the IBM quantum platform.
- Poincare$'$ sphere constructed for two and three flavor neutrino oscillations in qubit and qutrit basis, respectively.
- We defined two qutrit entanglement measure concurrence $(C_3(\rho^{\alpha\alpha'}))$ for the time evolved two qutrit flavor neutrino state.
- Our further investigation of these quantum studies to model neutrinos on quantum computer is in progress.
- Thus, neutrinos can be considered as potential quantum information resources.

Thank You