

Gimbal Lock + Loss of a degree of freedom with Euler Angles:

3D rotations:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\kappa & -\sin\kappa \\ 0 & \sin\kappa & \cos\kappa \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if $\beta = \pi/2$:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ \sin(\kappa+\gamma) & \cos(\kappa+\gamma) & 0 \\ -\cos(\kappa+\gamma) & \sin(\kappa+\gamma) & 0 \end{pmatrix}$$

Now changing κ or γ gives same result. Thus, we lost a degree of freedom.

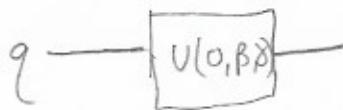
For quantum gates, recall

$$U(\kappa, \beta, \gamma) = \begin{pmatrix} \cos \frac{\kappa}{2} & -e^{i\gamma} \sin \frac{\kappa}{2} \\ e^{i\beta} \sin \frac{\kappa}{2} & e^{i(\beta+\gamma)} \cos \frac{\kappa}{2} \end{pmatrix}$$

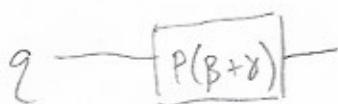
if we take $\kappa=0$:

$$U(0, \beta, \gamma) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\beta+\gamma)} \end{pmatrix} = P(\beta+\gamma)$$

And changing β or γ gives the same result. In this case, we lose one degree of freedom. However, this is something we can take advantage of. In the circuit diagram:



we can replace it with the simpler gate:



Programming Quantum Circuits in Python:
 QISKit: Python library developed by IBM
 (qiskit.org)

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Answer 1:

$$\begin{aligned} & \begin{pmatrix} \cos \frac{\pi}{6} & -i\sin \frac{\pi}{6} \\ -i\sin \frac{\pi}{6} & \cos \frac{\pi}{6} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{3} & -i \\ -i & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} & -i \\ -i & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} + 1 \\ i(\sqrt{3} - 1) \end{pmatrix} \end{aligned}$$

$$P(0) = \left(\frac{1}{2\sqrt{2}} (\sqrt{3} + 1) \right)^2 = \frac{1}{8} (4 + 2\sqrt{3}) = \frac{1}{2} + \frac{\sqrt{3}}{4} \approx 0.933 (93.3\%)$$

$$P(1) = \left(\frac{1}{2\sqrt{2}} (\sqrt{3} - 1) \right)^2 = \frac{1}{8} (4 - 2\sqrt{3}) = \frac{1}{2} - \frac{\sqrt{3}}{4} \approx 0.0670 (6.7\%)$$

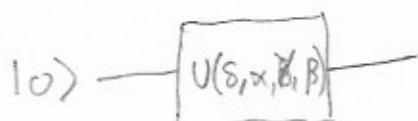
$$P(0) + P(1) = 1 \checkmark$$

Gate Decomposition: (ZYZ)

Can write:

$$U = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos \gamma/2 & \sin \gamma/2 \\ -\sin \gamma/2 & \cos \gamma/2 \end{pmatrix} \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}$$

Question 2: Given the following circuit, how does the probability of the state $|0\rangle + |1\rangle$ depend on δ ? Hint: $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$



Answer 2:

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$$\begin{aligned} U^+U &= \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} \begin{pmatrix} \cos \gamma/2 & -\sin \gamma/2 \\ \sin \gamma/2 & \cos \gamma/2 \end{pmatrix} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix} \\ &\times \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos \gamma/2 & -\sin \gamma/2 \\ -\sin \gamma/2 & \cos \gamma/2 \end{pmatrix} \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix} \\ &= R_z(-\beta) R_y(-\gamma) R_z(-\alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_z(\alpha) R_y(\gamma) R_z(\beta) \end{aligned}$$

\Rightarrow The probability is independent of S.
Overall phases do not change the results,
only relative phase differences do.

We can find $\gamma, \phi, \theta, \lambda$ for an arbitrary unitary matrix using the zyz decomposition:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = e^{i\gamma} R_z(\phi) R_y(\theta) R_z(\lambda)$$

Using the following algorithm:

$$X = \frac{1}{\sqrt{\det(U)}}$$

$$\gamma = -\arctan\left(\frac{\text{Im}(X)}{\text{Re}(X)}\right) \quad \# \text{ Remove global phase}$$

$$SU = XU = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\theta = 2\arctan\left(\frac{|b|}{|a|}\right)$$

$$\phi = \arctan\left(\frac{\text{Im}(d)}{\text{Re}(d)}\right) + \arctan\left(\frac{\text{Im}(b)}{\text{Re}(b)}\right)$$

$$\lambda = \arctan\left(\frac{\text{Im}(d)}{\text{Re}(d)}\right) - \arctan\left(\frac{\text{Im}(b)}{\text{Re}(b)}\right)$$

Question 3: What is the decomposition
of the Hadamard gate in the zyz-decomposition?

Answer 3: $\det\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \left(\frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right) = -1$

$$x = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i$$

$$\gamma = -\arctan\left(-\frac{1}{0}\right) = -\arctan(-\infty) = -(-\pi/2) = \pi/2$$

$$SU = xU = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\theta = 2\arctan\left(\frac{\sqrt{2}}{1/\sqrt{2}}\right) = 2\arctan(1) = 2\pi/4 = \pi/2$$

$$\phi = \arctan(\infty) + \arctan(-\infty) = \pi/2 + (-\pi/2) = 0$$

$$\lambda = \arctan(\infty) - \arctan(-\infty) = \pi/2 - (-\pi/2) = \pi$$

$$H = e^{i\pi/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\pi/4 & \sin\pi/4 \\ -\sin\pi/4 & \cos\pi/4 \end{pmatrix} \begin{pmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

Transpilers and Compilers:

Transpilers take a program in a given language as input, optimizes the code, and returns a program written in the same language.

Compilers take a program in a given language as input, optimizes the code, and returns a program written in a different language (typically a lower level language, ie. C++ \rightarrow assemble)

Quantum computer uses transpilers to take the circuit from one using arbitrary gates to one using only the allowed gates. Along the way, the transpiler attempts to remove as many gates as possible.

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Answer 4:

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$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$R_z(\pi/2) = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$S_x = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$\begin{aligned} & \left(\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \right) e^{i\pi/4} \\ &= \frac{1}{2} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} i\sqrt{2} & -i\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix} = \frac{(1+i)}{2} \begin{pmatrix} i & -i \\ -i & -i \end{pmatrix} e^{i\pi/4} \\ &= \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \checkmark \end{aligned}$$