Quantum Gates: Lecture 3

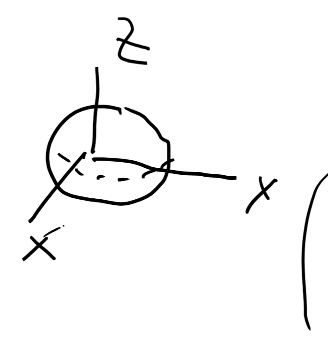
Gimbal Lock:

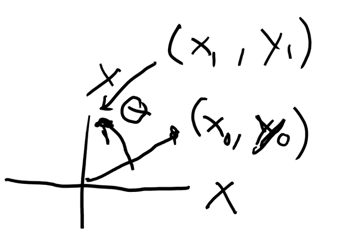
3D rotations: 3 choices (α, β, γ)

$$R = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \end{array} \begin{array}{c} \begin{matrix} x & x & z \end{matrix} \\ \begin{pmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{pmatrix} \end{array} \begin{array}{c} \begin{matrix} x & \cancel{y} & z \end{matrix} \\ \begin{pmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

if $\beta = \pi/2$: 1 choice $(\alpha+\gamma)$

$$R = \begin{pmatrix} 0 & 0 & 1 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & \sin(\alpha+\gamma) & 0 \end{pmatrix} \qquad \text{if } \begin{aligned} \alpha &\to \alpha+1 \\ \gamma &\to \gamma-1 \end{aligned}$$





$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

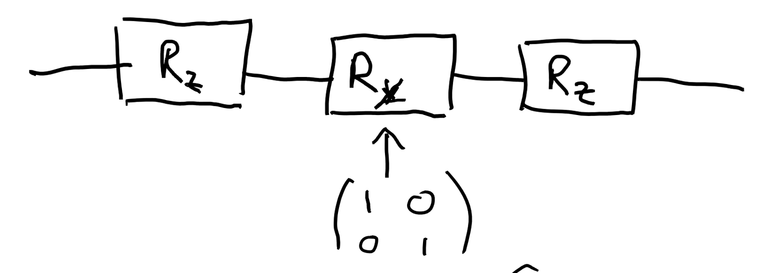
Quantum Gates:

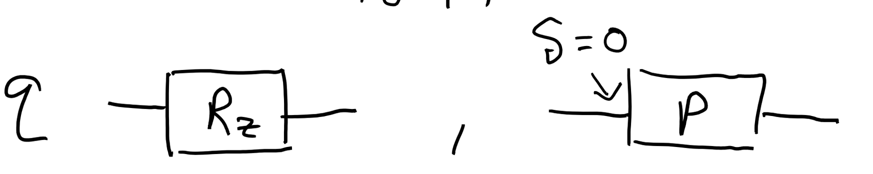
$$U(\underset{z}{\beta}, \underset{y}{\alpha}, \underset{z}{\gamma}, \delta) = \begin{pmatrix} \cos\alpha/2 & -e^{i\gamma}\sin\alpha/2 \\ e^{i\beta}\sin\alpha/2 & e^{i(\beta+\gamma)}\cos\alpha/2 \end{pmatrix} e^{i\delta}$$

$\alpha = 0, 2\pi$:

$$U = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm e^{i(\beta+\gamma)} \end{pmatrix} = P(\beta+\gamma)$$

Not bad. Because while we lose one choice of angle. We have a simpler quantum circuit.





Programming QC in Python:

QISKIT: IBM code (qiskit.org)

Answer 1:

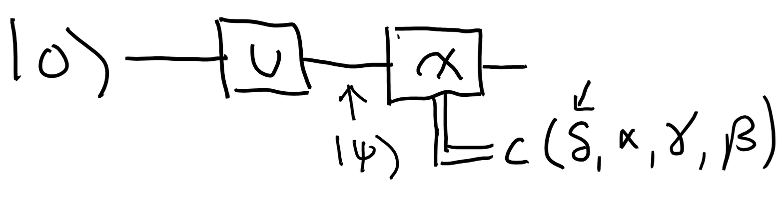
$$|\psi\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 \\ i(\sqrt{3}-1) \end{pmatrix}, \quad P(0) = \left(\frac{1}{2\sqrt{2}}(\sqrt{3}+1)\right)^2 = \frac{1}{2} + \frac{\sqrt{3}}{4} \approx 93.3\%$$

$$P(1) = 1 - P(0) = \frac{1}{2} - \frac{\sqrt{3}}{4} \approx 6.7\%$$

Gate Decomposition: (ZYZ)

$$U = e^{i\delta} \underbrace{\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}}_{R_z(\alpha)} \underbrace{\begin{pmatrix} \cos\gamma/2 & \sin\gamma/2 \\ -\sin\gamma/2 & \cos\gamma/2 \end{pmatrix}}_{R_y(\gamma)} \underbrace{\begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix}}_{R_z(\beta)}$$

Question 2: Does the probability of the output states depend on δ? Hint: $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$



Answer 2: $P(0) = |\langle 0|\psi\rangle|^2 = \langle 0|U^\dagger U|0\rangle$

$$U^\dagger = e^{-i\delta} R_z^\dagger(\beta) R_y^\dagger(\gamma) R_z^\dagger(\alpha)$$

$$U = e^{i\delta} R_z(\alpha) R_y(\gamma) R_z(\beta)$$

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = e^{i\gamma} R_z(\lambda) R_y(\theta) R_z(\phi)$$

$$x = \frac{1}{\sqrt{\det(U)}}$$

$$\gamma = -\arctan\left(\frac{Im(x)}{Re(x)}\right)$$

$$SU = xU = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\theta = 2\arctan\left(\frac{|b|}{|a|}\right)$$

$$\phi = \arctan\left(\frac{Im(d)}{Re(d)}\right) + \arctan\left(\frac{Im(b)}{Re(b)}\right)$$

$$\lambda = \arctan\left(\frac{Im(d)}{Re(d)}\right) - \arctan\left(\frac{Im(b)}{Re(b)}\right)$$

Question 3: What is the decomposition for Hadamard gate? $\arctan\left(\frac{x}{0}\right) = \pi/2$ $\quad \arctan(1) = \pi/4$ $\quad \arctan\left(\frac{-x}{0}\right) = -\pi/2$

Answer 3: $\gamma = \cancel{0}, \pi/2 \qquad R_z(\pi) R_y(\pi/2) R_z(0)$

$= 0 \qquad 0 \quad \pi/2 \quad 0$

$$\boxed{\gamma = \pi/2 \quad R_z(\pi) R_y(\pi/2) R_z(0)}$$

Transpilers & Compilers:

Transpiler: program in a language → optimization (change of gates) → program in same language

Compiler: program in one language (→ optimization) → program in another (usually lower-level C++ → assembly)

Transpilers needed to take gates that are user friendly → gates that are computer friendly in an automated way