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$CX = CNOT$



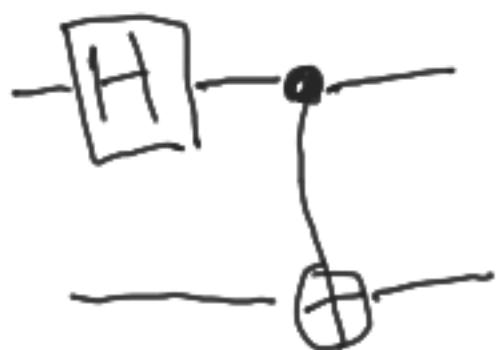
$$\begin{aligned}|00\rangle &\rightarrow |00\rangle \\|01\rangle &\rightarrow |01\rangle \\|10\rangle &\rightarrow |11\rangle \\|11\rangle &\rightarrow |10\rangle\end{aligned}$$

Ordered Basis
 $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad |c t\rangle \rightarrow |c, c+t \bmod 2\rangle$$

Afternoon lecture begins on slide 9

CX used to create entanglement



$$|00\rangle = |0\rangle \otimes |0\rangle$$

$$\xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle \\ = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$

$$\xrightarrow{\text{CX}} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

c7



$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |01\rangle$$

$$|10\rangle \rightarrow +|10\rangle$$

$$|11\rangle \rightarrow -|11\rangle$$



$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$



$$|00\rangle \rightarrow |00\rangle$$

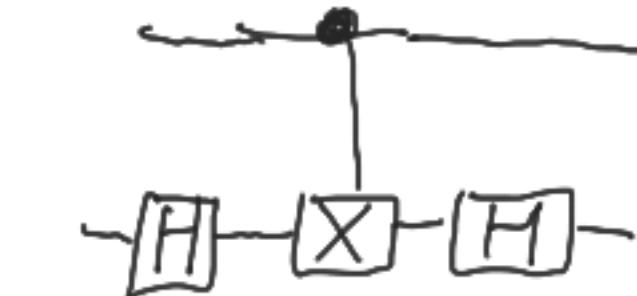
$$|01\rangle \rightarrow +|01\rangle$$

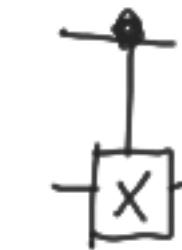
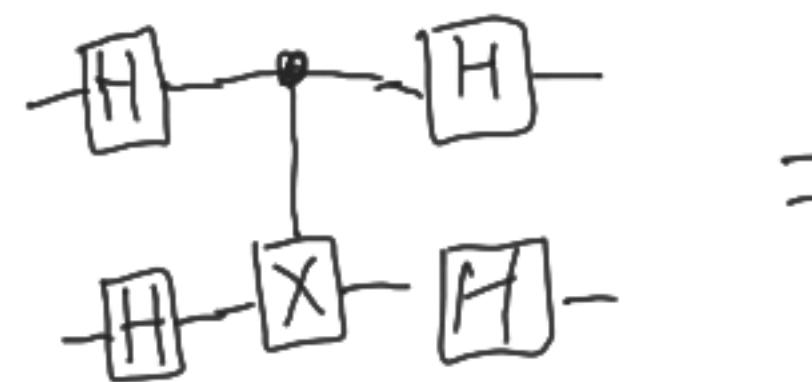
$$|10\rangle \rightarrow |10\rangle$$

$$|11\rangle \rightarrow -|11\rangle$$



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$00 \rightarrow 00$
 $01 \rightarrow 01$
 $10 \rightarrow |1\rangle \otimes X|0\rangle$
 $= |1\rangle \otimes |1\rangle$
 $= |11\rangle$

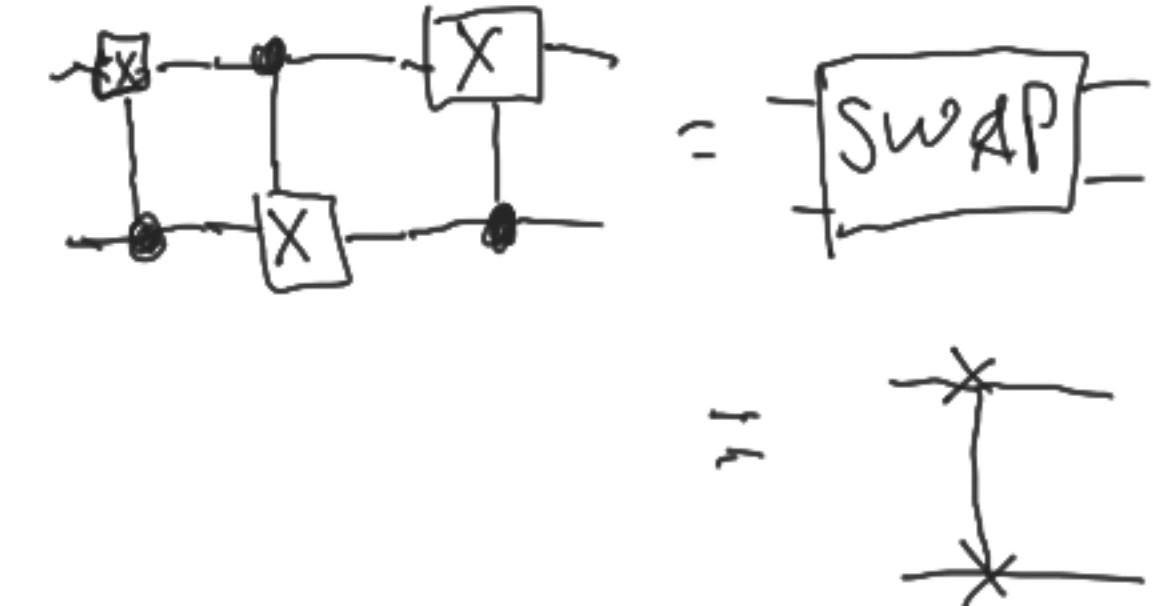
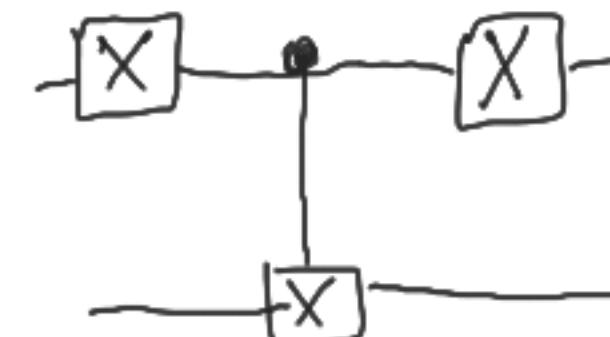
what if we want to swap them?

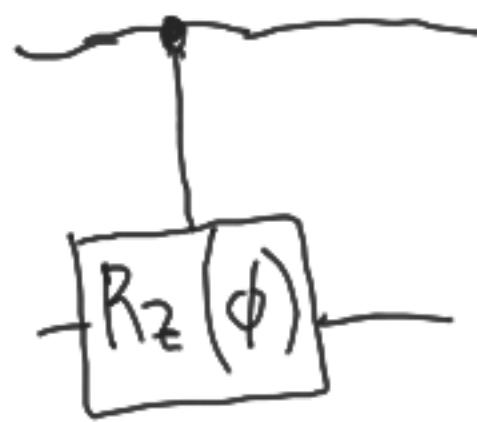


$00 \rightarrow 00$
 $01 \rightarrow 10$
 $10 \rightarrow 01$
 $11 \rightarrow 11$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$00 \rightarrow 01$
 $01 \rightarrow 00$
 $10 \rightarrow 10$
 $11 \rightarrow 11$





$$R_z(\phi) = e^{\frac{-i\phi}{2}Z} \quad \checkmark$$

Typical gate in quantum simulation

$$e^{-i\theta Z, Z_2}$$

$$Z_1 Z_2 = Z \otimes Z$$

$$(z|0\rangle) \otimes (z|0\rangle)$$

↷

$$Z \otimes Z |00\rangle = (+)(+)|00\rangle \otimes |00\rangle = |00\rangle$$

$$0+0 \bmod 2 = 0$$

$$0+1 \bmod 2 = 1$$

$$\dots = 1$$

$$|01\rangle = (+)(-1)|0\rangle \otimes |1\rangle = -|01\rangle$$

$$|10\rangle = (-1)(+1)|1\rangle \otimes |0\rangle = -|10\rangle$$

$$1+1 \bmod 2 = 0$$

$$0 \bmod 2 = 0$$

$$1 \bmod 2 = 1$$

$$2 \bmod 2 = 0$$

$$3 \bmod 2 = 1$$

11:00 + 5 hrs → 16:00

$$|11\rangle = (-1)(-1)|1\rangle \otimes |1\rangle = |11\rangle$$

$$e^{-i\theta Z \otimes Z} :$$

$$|00\rangle \rightarrow e^{-i\theta(+1)(+1)} |00\rangle = e^{-i\theta} |00\rangle$$

$$|01\rangle \rightarrow e^{-i\theta(+1)(-1)} |01\rangle = e^{+i\theta} |01\rangle$$

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

$$|10\rangle \rightarrow e^{-i\theta(-1)(+1)} |10\rangle = e^{+i\theta} |10\rangle$$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

$$|11\rangle \rightarrow e^{-i\theta(-1)(-1)} |11\rangle = e^{-i\theta} |11\rangle$$

Action is dictated by sum modulo 2:

$$\text{if } 0 \rightarrow \text{give } e^{-i\theta}$$

$$\text{if } 1 \rightarrow \text{give } e^{+i\theta}$$

$$e^{-i\theta Z \otimes Z}$$

Apply $e^{-i\theta}$ for sum = 0

$e^{+i\theta}$ for sum = 1



$$|c, t\rangle \rightarrow |c, t+c \text{ mod } 2\rangle$$

$$e^{-iz}$$

$$|c\rangle \dots |c\rangle$$

$$|t\rangle - \boxed{x} - |t + c \text{ mod } 2\rangle \dots$$



$$\boxed{-i\theta Z \otimes Z}$$

$$e$$

$$H \rightarrow \boxed{e^{-i\theta Z}} \rightarrow H$$

$$= \boxed{-i\theta X}$$

$$e$$

$$HZH = X$$

$$HXH = Z$$

$$e^{-i\theta Z \otimes Z}$$

generalizes to 3 or n qubits

$$e^{-i\theta Z \otimes Z \otimes Z}$$

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Turned a 3-qubit operator into
a sequence of CX's and
a single-qubit rotation.

Universal Gate Sets

Afternoon Lecture

A set of gates is universal for quantum computation if any unitary U acting on the qubits can be approximated to arbitrary accuracy using those gates.

Standard set:



Alternative set:

$\left\{ \text{CNOT} + \text{all single-qubit unitaries} \right\}$

G
 $\{a, b, c\} \subset G$
 Consider some $\epsilon > 0$.
 There is a V , for any unitary U , s.t.
 $\|U - V\| \leq \epsilon$ spectral norm

Q: How to get
 $\neg \boxed{X}$?

$$\neg \boxed{S} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\neg \boxed{S} - \boxed{S} = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix}^T = \begin{pmatrix} 1 & -1 \end{pmatrix} = \boxed{Z}$$

$$\neg \boxed{T} - \boxed{S} - \boxed{S} - \boxed{H} = \neg \boxed{X}$$

Including T and S in standard set
Has to do with marking

Note: $\neg \boxed{T} - \boxed{T} = \neg \boxed{S}$ fault tolerant operations.

Argument for universality - Sketch → (of CNOTs & arbitrary single-qubit rotations)

[st. Two-level unitaries are universal.]

Then: CNOTs & single-qubit rotations
are universal for two level unitaries

$$U = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & j \end{pmatrix}$$

want: $U = \text{product of two level unitaries}$

unitary which mixes at most two basis vectors

Ex: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ Mixes $|10\rangle$ and $|11\rangle$

Idea: Find matrices U_3, U_2, U_1 such that

$$U_3 U_2 U_1 U = I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Rightarrow U = U_1^+ U_2^+ U_3^+$$

Reference:
Nielsen &
Chuang
§4.5

U_1

$$b=0 \rightarrow U_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$b \neq 0 \rightarrow$$

$$\Rightarrow U_1 U = \begin{pmatrix} a' & d' & g' \\ 0 & e' & h' \\ c' & f' & j' \end{pmatrix}$$

$$U_1 = \frac{1}{\sqrt{|a|^2 + |b|^2}} \begin{pmatrix} a^* & b^* & 0 \\ b & -a & 0 \\ 0 & 0 & \sqrt{|a|^2 + |b|^2} \end{pmatrix}$$

Next: U_2

$$\begin{pmatrix} a' & d' & g' \\ 0 & e' & h' \\ c' & f' & j' \end{pmatrix}$$

If $c' = 0$, $U_2 = \begin{pmatrix} a'^* & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$c' \neq 0$, $U_2 = \frac{1}{\sqrt{|a'|^2 + |c'|^2}} \begin{pmatrix} a'^* & 0 & c'^* \\ 0 & \sqrt{|a'|^2 + |c'|^2} & 0 \\ c' & 0 & -a' \end{pmatrix}$

$$U_2 U_1 U = \begin{pmatrix} 1 & d'' & g'' \\ 0 & e & .'' \\ 0 & f'' & j'' \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e'' & h'' \\ 0 & f'' & j'' \end{pmatrix}$$

$d'' = 0$
 $g'' = 0$ due to
unitarity

Finally U_3

$$U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & f^{i\phi} \\ 0 & h^{i\phi} & j^{i\psi} \end{pmatrix}$$

$$U = U_1 U_2^* U_3^*$$

Find:

$$U_3 U_2 U_1 U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

Turned 3×3 unitary into product of two-level unitaries

\Rightarrow Two level unitaries are universal for $U(3)$

Tri-qubit: 4 dimensional

Proof: sketch

$$U = \begin{pmatrix} a & e \\ b & f \\ c & g \\ d & h \end{pmatrix}$$

Get zeros in the b, c, d slots,
with two level unitaries,
then recycle the 3×3 algorithm

→ $U(4)$ can be covered by two level unitaries

Generalizes to any dimension.

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

two level unitaries

$$\overline{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$U_d = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} e^{-i\theta} & & & \\ & e^{i\theta} & & \\ & & e^{-i\theta} & \\ & & & e^{i\theta} \end{pmatrix}$$

Acts nontrivially
on more than
two kets
 \Rightarrow Not a
two level
unitary

If working in d -dimensional vectors space

\rightarrow Need

$$\frac{d(d-1)}{2}$$

two-level unitaries
for this method.

Reminder: Show that CNOTs & single-qubit rotations can effect any two-level unitary on qubits.

out of time —————