Optical Stochastic Cooling in IOTA

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IOTA review
Fermilab
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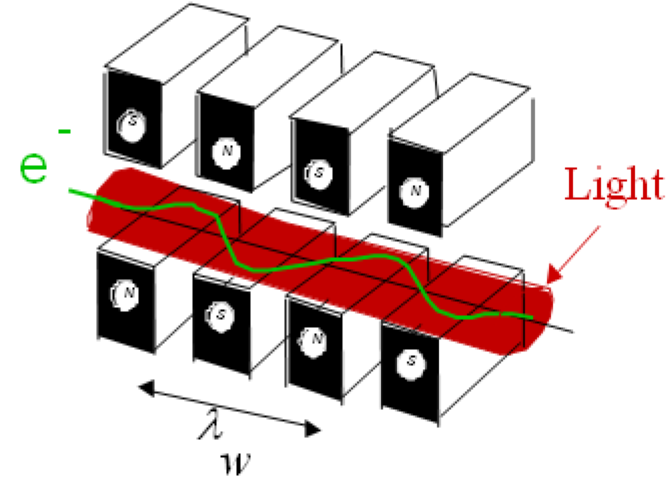
Objective

- Experimental test of Optical Stochastic cooling in IOTA

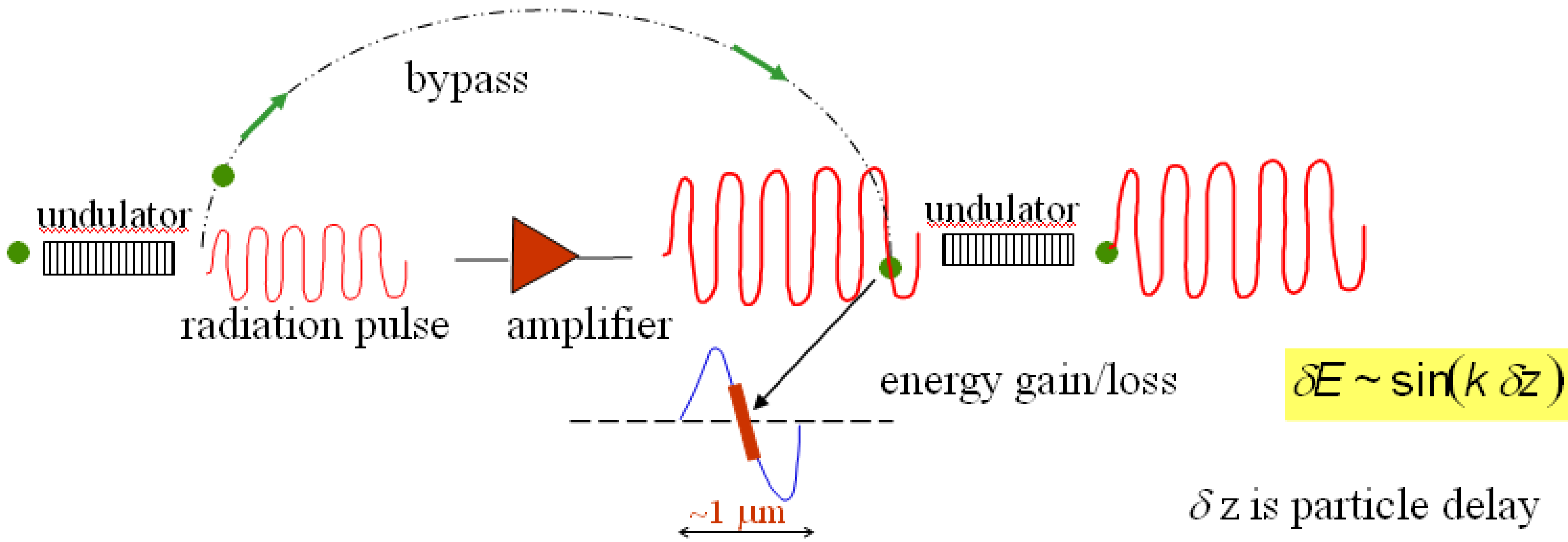
Outline

- Optical stochastic cooling principles
- Beam optics requirements
  - Damping rates
  - Cooling range
- Requirements to optical amplifier
- Future possible applications
- Conclusions

Optical Stochastic Cooling



- Suggested by Zolotorev, Zholents and Mikhailichenko (1994)
- Never tested experimentally
- OSC obeys the same principles as the microwave stochastic cooling, but exploits the superior bandwidth of optical amplifiers ~ 10^{14} Hz
- Undulator can be used as pickup & kicker
- Pick-up and Kicker should be installed at locations with nonzero dispersion to have both ⊥ and L cooling.



MIT-Bates Proposal for Tevatron (2007)

OSC Demo With Electrons

- OSC considered for p, ions at several facilities, still unproven
  - Counteracts heating due to IBS, beam-beam (cooling of tails)
  - OSC rates, luminosity gain strongly depend on achievable parameters
- Technical requirements for cooling of heavy particles are severe ($$$)
  - Optics of particles in bypass controlled to fraction of λ
  - Very strong wiggler fields
  - Amplifier saturates far below optimal gain
  - Diagnostics predictive of OSC required (cooling time of order hours)
- Demonstration of OSC with e^- would point way to cooling at very high E, N
  - OSC of electrons much faster (~1 sec)
  - Modest technical requirements (wiggler, amplifier, chicane)
  - Develop techniques to achieve OSC, study physics for scaling to high E, N
  - Proposed at several facilities but not carried out

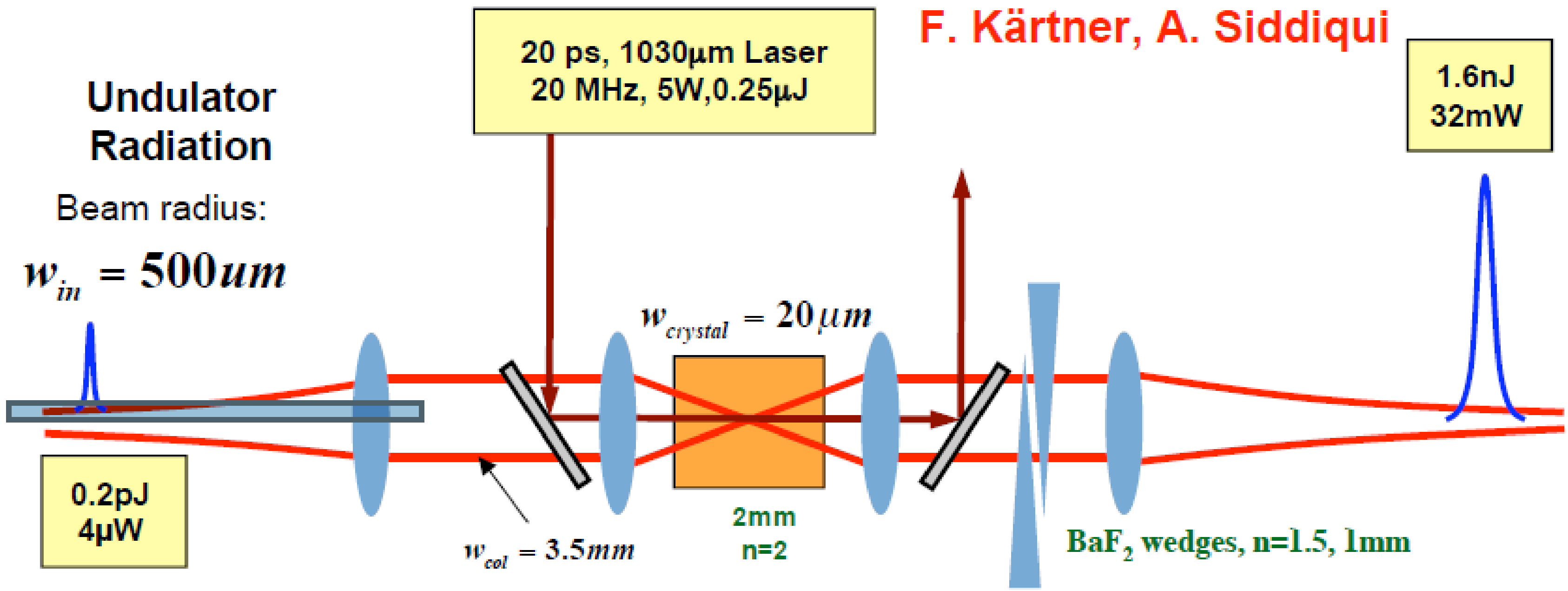
W. Franklin PAC 2007 5

- Some deficiencies of the proposal will be discussed later

Main parameters of the Bates proposal (W. Franklin, PAC-2007)

Beam energy	300 MeV
SR transverse damping time	4.83 s
Machine circumference	190.2 m
Number of particles per bunch	10^8
Number on bunches	12
Rms horizontal emittance (SR equilibrium)	$98*10^{-7}$ cm
Rms momentum spread (SR equilibrium)	$1.64*10^{-4}$
Optical amplifier wave length	2 μm
Optical amplifier bandwidth	10%
Optical gain (amplitude)	90
Delay in the chicane	6 mm
Undulator length / Number of periods	2 m / 20

F. Kärtner, A. Siddiqui



Optical Stochastic Cooling Fundamentals

- The sequence of our consideration
  - 6x6 matrix parameterization
    - Matrix symplecticity is used
  - Damping rates in linear regime
    - Perturbation theory for symplectic motion
  - Damping rates for large amplitudes
    - Cooling range for finite amplitudes
    - Correction factors for the finite amplitude
  - Undulators
  - Gain and power of optical amplifier
    - Final expression for damping rates

Transfer Matrix Parameterization

- Vertical plane is uncoupled and we omit it in further equations
- Matrix from point 1 to point 2

$$\mathbf{M}=\begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p / p \end{bmatrix}$$

$$M_{11}=\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu+\alpha_1\sin\mu\right)$$

$$M_{22}=\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu-\alpha_2\sin\mu\right)$$

$$M_{12}=\sqrt{\beta_1\beta_2}\sin\mu$$

$$M_{21}=\frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu-\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu$$

- M_{16} & M_{26} can be expressed through dispersion

$$\begin{bmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_1' \\ 1 \end{bmatrix}=\begin{bmatrix} D_2 \\ D_2' \\ 1 \end{bmatrix}$$

=>

$$M_{16}=D_2-M_{11}D_1-M_{12}D_1'$$
$$M_{26}=D_2'-M_{21}D_1-M_{22}D_1'$$

- Symplecticity ($\mathbf{M}^T\mathbf{U}\mathbf{M}=\mathbf{U}$) binds up M_{51},M_{52} and M_{16},M_{26} =>

$$M_{51}=M_{21}M_{16}-M_{11}M_{26}$$
$$M_{52}=M_{22}M_{16}-M_{12}M_{26}$$

- M_{56} is related to the partial slip factor, $\eta_{1\to2}$

=> All matrix elements can be expressed through $\beta_k,\alpha_k,D_k,D_k',M_{56}\ ,\quad k=1,2$

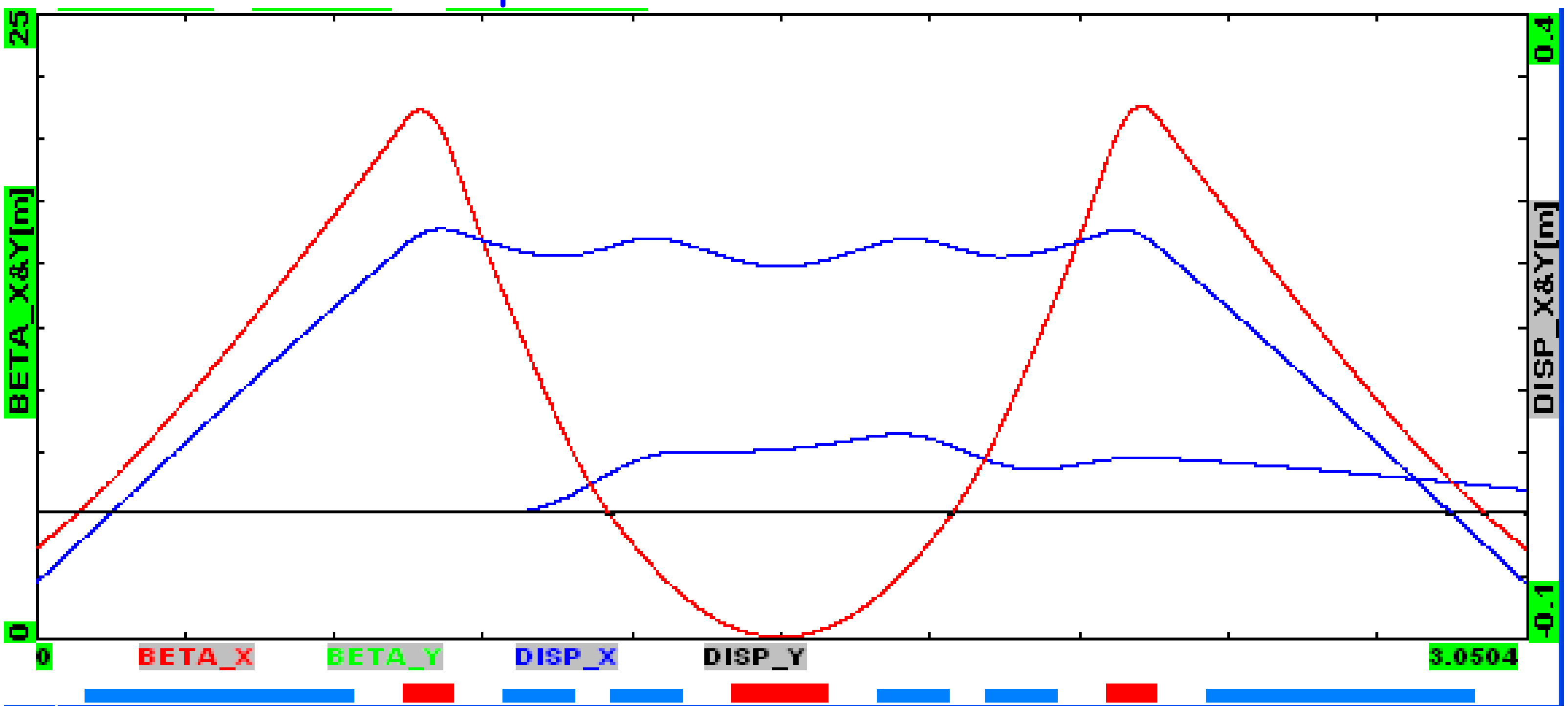
Partial slip factor

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to M_{56}

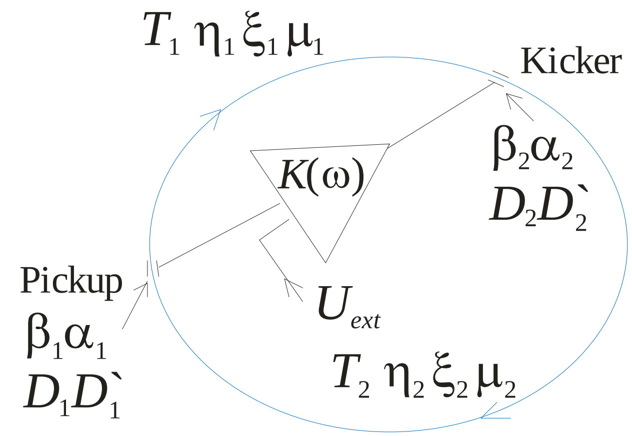
$$\Delta s_{1\to2} \equiv \hat{M}_{56}\frac{\Delta p}{p} = M_{51}D_1\frac{\Delta p}{p} + M_{52}D_1'\frac{\Delta p}{p} + M_{56}\frac{\Delta p}{p} + \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

  - Further we assume that $\mathrm{v}=c$, i.e. $1/\gamma^2=0$ and $\eta_1=-\alpha_{1\to2}$.

- That results in $\boxed{\hat{M}_{56} = M_{51}D_1 + M_{52}D_1' + M_{56}}$
  - Note that M_{56} sign is positive if a particle with positive Δp moves faster than the reference particle



Damping Rates of Optical Stochastic Cooling



- Longitudinal kick

$$\frac{\delta p}{p} = \kappa \Delta s = \kappa \left(M_{151} x_1 + M_{152} \theta_{x\,1} + M_{156} \frac{\Delta p}{p} \right) .$$

- Tune shifts

  Rewriting above Eq. in matrix form, adding the rest of the ring, and using perturbation theory for symplectic motion one obtains the tune shifts

$$\delta Q_k = \frac{1}{4\pi} \mathbf{v}_k^{+} \mathbf{U}\ \mathbf{M}_c\ \mathbf{U}\, \mathbf{M}_1^{T} \mathbf{U}\, \mathbf{v}_k$$

  where U is unit symplectic matrix

M_1 - pickup-to-kicker matrix

M_2 - kicker-to-pickup matrix

M = $M_1 M_2$ – ring matrix

$\mu = \mu_1 + \mu_2$

$$\mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k$$

- Expressing matrix elements and eigen-vectors through Twiss parameters one obtains the cooling rates:

$$\lambda_x = -\frac{\kappa}{2}\left(M_{156} - \hat{M}_{56} \right)$$

$$\lambda_s = -\frac{\kappa}{2} \hat{M}_{156}$$

- That yields that the sum of the decrements is

$$\lambda_x + \lambda_s = -\frac{\kappa}{2} M_{156}$$

Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

$$\sigma_{\Delta s}{}^2 = \int \left(M_{151}x + M_{152}\theta_x + M_{156}\tilde{p}\right)^2 f\left(x,\theta_x,\tilde{p}\right)dx\,d\theta_x\,d\tilde{p}\ , \quad \tilde{p} = \frac{\Delta p}{p}$$

  - ♦ Performing integration one obtains for Gaussian distribution

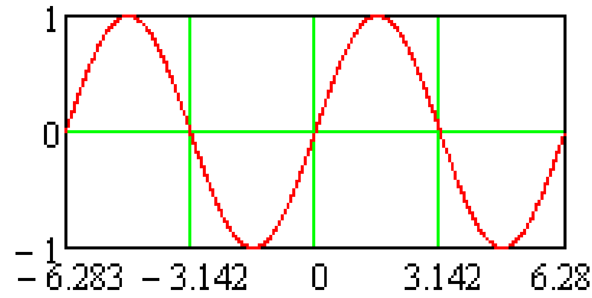
$$\sigma_{\Delta s}{}^2 = \sigma_{\Delta s\varepsilon}{}^2 + \sigma_{\Delta sp}{}^2$$

$$\sigma_{\Delta s\varepsilon}{}^2 = \varepsilon\left(\beta_p M_{151}{}^2 - 2\alpha_p M_{151}M_{152} + \gamma_p M_{152}{}^2\right)$$

$$\sigma_{\Delta sp}{}^2 = \sigma_p{}^2\left(M_{151}D_p + M_{152}D'_p + M_{156}\right)^2$$

  - ♦ Both $\Delta p/p$ and ε contribute to the lengthening

Cooling Range



- The cooling force depends on Δs nonlinearly

$$\frac{\delta p}{p} = \frac{\Delta E_{max}}{E}\sin(k\delta s) = \frac{\Delta E_{max}}{E}\sin\left(a_x \sin(\psi_x) + a_p \sin(\psi_p)\right)$$

where a_x & a_p are the lengthening amplitudes due to ⊥ and L motions measured in units of laser phase ($a = k\,\delta s$)

- The form-factor for damping rate of longitudinal cooling for particle with amplitudes a_x & a_p

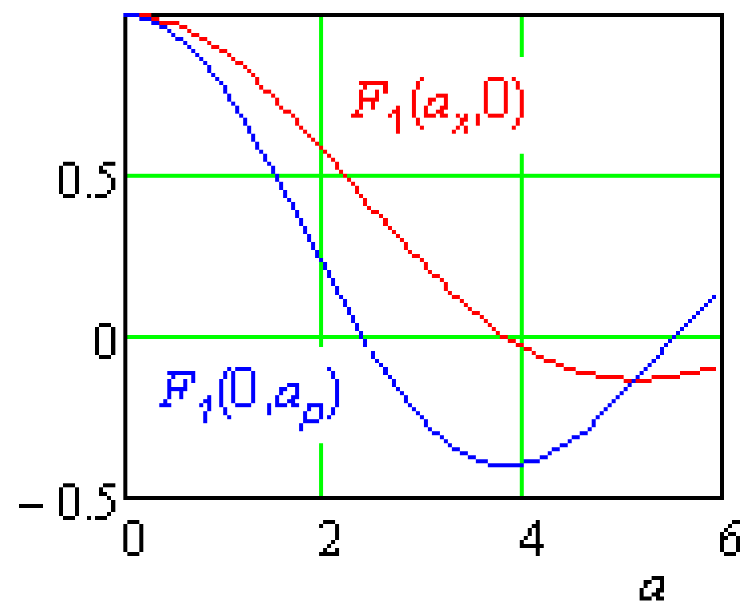
$$F_s(a_x,a_p) = \frac{2}{a_p}\oint \sin\left(a_x \sin\psi_x + a_p \sin\psi_p\right)\sin\psi_p \frac{d\psi_x}{2\pi}\frac{d\psi_p}{2\pi}$$

⇨ $$F_s(a_x,a_p) = \frac{2}{a_p} J_0(a_x) J_1(a_p)$$

- Similar for transverse motion

⇨ $$F_x(a_x,a_p) = \frac{2}{a_x} J_0(a_p) J_1(a_x)$$

- Damping requires both lengthening amplitudes be smaller $\mu_0 \approx 2.405$



Cooling of the Gaussian beam

- Averaging the cooling form-factors for Gaussian distribution can be presented in the following form

$$F_x(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = \frac{1}{2k^2\sigma_{\Delta s\varepsilon}^{\ 2}} \int_0^\infty a_x^{\ 2} F_1(a_x, a_p) \exp\left(-\frac{a_x^{\ 2}}{2k^2\sigma_{\Delta s\varepsilon}^{\ 2}} - \frac{a_p^{\ 2}}{2k^2\sigma_{\Delta sp}^{\ 2}} \right) \frac{a_x da_x a_p da_p}{k^4 \sigma_{\Delta s\varepsilon}^{\ 2} \sigma_{\Delta sp}^{\ 2}}$$

  - Integration yields

$$F_x(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = F_s(k\sigma_{\Delta s\varepsilon}, k\sigma_{\Delta sp}) = \exp\left(-\frac{k^2\sigma_{\Delta sp}^{\ 2}}{2} \right) \exp\left(-\frac{k^2\sigma_{\Delta s\varepsilon}^{\ 2}}{2} \right)$$

- Good beam lifetime requires the cooling force to be positive for large amplitude particles
- Assuming that cooling becomes zero at 4σ for both planes

⇨ $k\,\sigma_{\Delta sp} = k\,\sigma_{\Delta s\varepsilon} = \mu_0/4 \approx 0.6$

⇨ Nonlinearity of cooling force results in the cooling force reduction by factor $F_x(\mu_0/4, \mu_0/4) = F_s(\mu_0/4, \mu_0/4) \approx 0.697$

Longitudinal Kick by E.-M. Wave

- Electric field of flat e.-m. wave focused at z=0 to the rms size $\sigma_\perp$

$$\left|E_x\right|^2 = \frac{8P}{c}\frac{1}{\left|\sigma(z)\right|^2}\exp\left(-\frac{x^2+y^2}{\left|\sigma^2(z)\right|}\right), \quad \left|\sigma^2(z)\right| = 2\varepsilon_w\left(\beta_w^* + \frac{z^2}{\beta_w^*}\right), \quad \varepsilon_w = \frac{1}{2k} \equiv \frac{\lambda_w}{4\pi}, \quad \sigma_\perp{}^2 = 2\varepsilon_w\beta_w^*$$

- The beam is deflected in the x-plane by wiggler magnetic field
  - That results in the beam energy change $\Delta \mathrm{E} = e\int(\mathbf{E}\cdot\mathbf{v})dt$
- Helical dipole suggest √2 times better kicker efficiency
  - Circular polarized light
  - Optical amplifier requires flat wave
- For helical dipole
  - Resonance condition

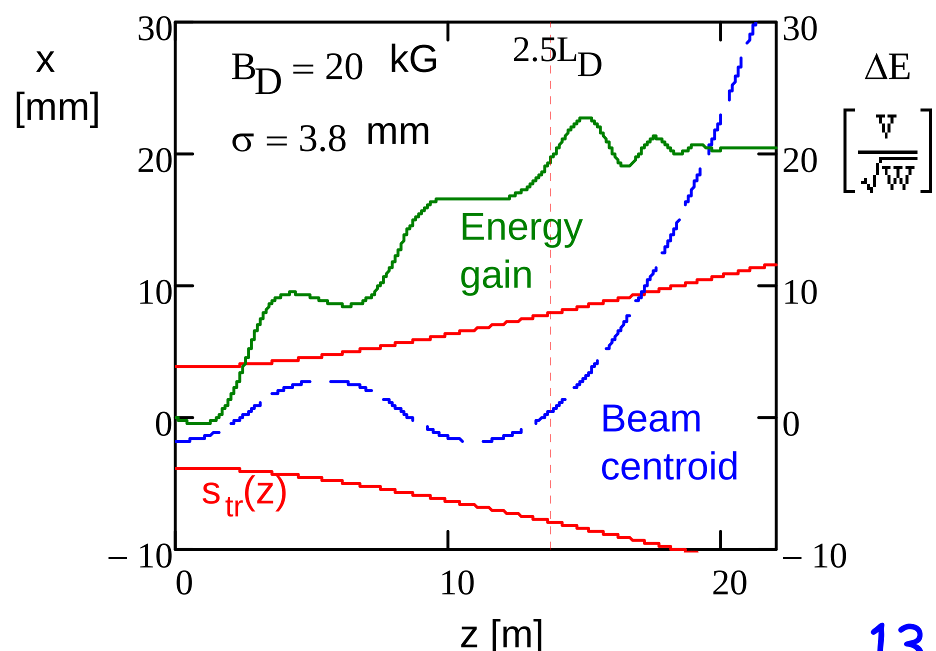
$$k_{wgl} = \frac{k}{2\gamma^2}\left(1+K_U{}^2\right), \quad K_U = \frac{eB}{k_{wgl}mc^2}$$

  - Optimal focusing for helical wiggler

$$\sigma_\perp \approx \sqrt{0.946L\lambda_w}\ , \quad L = n_{wgl}\lambda_{wgl}$$

L is the total wiggler length

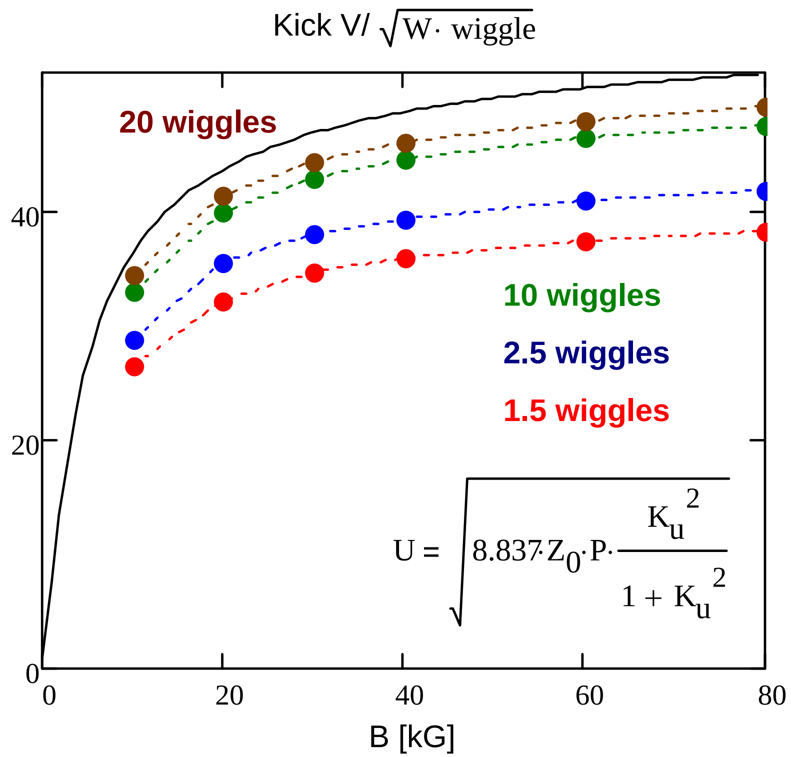
5 dipole wiggler



Longitudinal Kick by E.-M. Wave (continue)

- For helical kicker and large number of periods (n_{wgl} >> 1) the helical kicker strength is (M. Zolotorev)

$$\frac{\Delta E_{max}}{e} \approx \sqrt{8.837 n_{wgl} P Z_0 \frac{K_u^2}{1+K_u^2}}$$



where $K_u = \frac{\lambda_{wgl}}{2\pi} \frac{eB}{mc^2}$, Z_0=377 Ω

- The waist size is growing with kicker length –

  $\sigma_\perp \approx \sqrt{0.946 L \lambda_w}$

  $\beta_w^* \approx 5.944 L$

- The kicker is less effective than formula prediction for small n_{wgl} $\rho_{wgl} \sim \sigma_\perp$ & negative contribution of E_z

Damping rates

- Assembling the above equations one obtains

$$\begin{bmatrix} \lambda_s \\ \lambda_x \end{bmatrix} = \frac{\pi f_0 \Delta E_{wgl} G}{\lambda_w E_0 \left(1+\kappa_u K_U{}^2\right)} F_c \begin{bmatrix} \hat{M}_{56} \\ M_{156} - \hat{M}_{56} \end{bmatrix}$$

Here G is the gain of power amplifier (in amplitude)

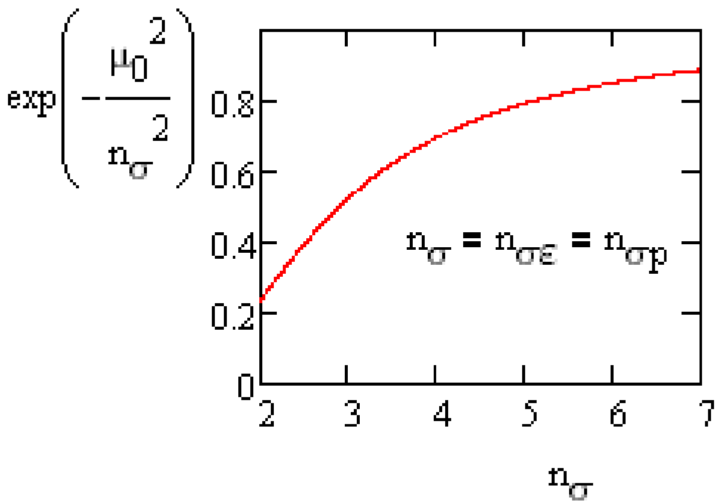
$$\Delta E_{wgl} = \kappa_u \frac{2}{3} r_e{}^2 B^2 \gamma^2 L$$ is the total energy radiated in wiggler

$$K_u = \frac{\lambda_{wgl}}{2\pi} \frac{eB_0}{mc^2}$$ is the undulator parameter

$$F_c = \exp\left(-\frac{k^2\left(\sigma_{\Delta sp}{}^2 + \sigma_{\Delta s\varepsilon}{}^2\right)}{2} \right) = \exp\left(-\frac{\mu_0{}^2}{2} \left(\frac{1}{n_{\sigma\varepsilon}{}^2} + \frac{1}{n_{\sigma\varepsilon}{}^2} \right) \right)$$

$$\kappa_u = \begin{cases} 1, & \text{helical undulator} \\ 1/2, & \text{flat undulator} \end{cases}$$

- Damping rate for flat undulator is about half of helical undulator with the same number of wiggles



Beam optics

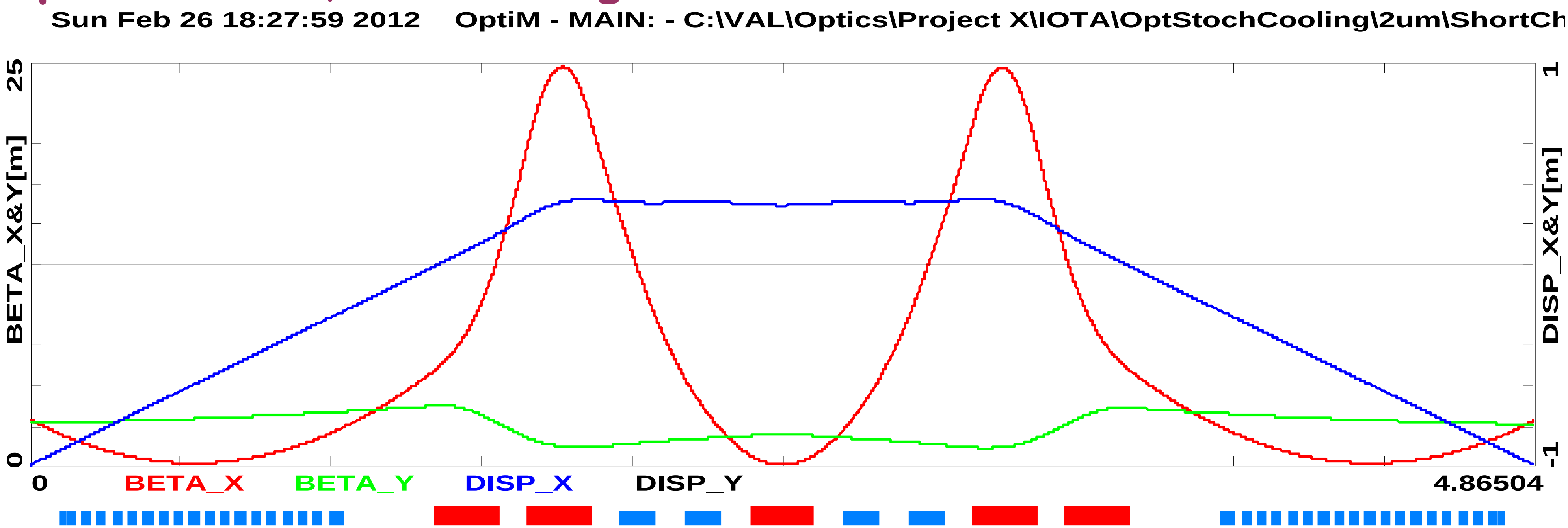
Sequence of optics adjustments

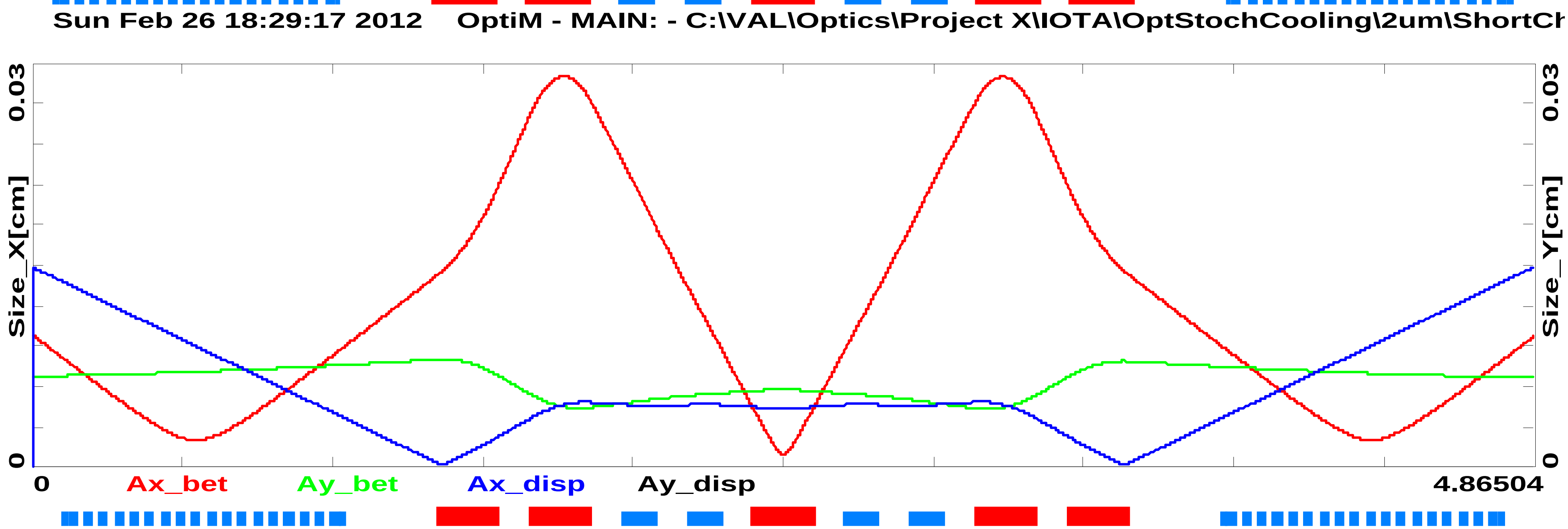
- Set required delay in the chicane, Δs
- Adjust focusing in the chicane center to get desired M_{56}
  - That sets the sum of damping rates
  - In absence of focusing $M_{56} \approx 2\Delta s$
  - Defocusing reduces M_{56}
- Adjust dispersion and dispersion prime to make desired value of partial slip-factor, $\hat{M}_{56}$
  - That determines the ratio of damping rates and the cooling range in momentum
- Adjust beta-function through the chicane to minimize sample lengthening from pickup to kicker
  - For optics symmetrical relative to the chicane center the optimum is achieved when β^* is minimum in the center
  - Larger value at the ends yields larger range of horizontal damping
- Adjust focusing outside of chicane to minimize beam sizes in wigglers
- If necessary iterate to achieve desired parameters

Optics choice for the cooling chicane

- 3 choices were considered
  - Choice 1: $\lambda_w = 2\ \mu m$, equal decrements
    - small delay of ~2 mm, therefore an optical amplifier hardly can be used
    - OSC without amplifier yields an order of magnitude faster damping than SR
  - Choice 2: $\lambda_w = 6\ \mu m$, $\lambda_x \approx 3\lambda_s$,
    - 10 mm delay
    - Reasonable accuracy of beam optics is required
    - Reduced energy (150->86 MeV) if the same undulator is used.
    - Both active and passive coolings are possible
  - Choice 3: MIT-Bates like - $\lambda_w = 2\ \mu m$, $\lambda_x \approx 5\lambda_s$
    - 4 mm delay – tough to squeeze an optical amplifier
    - High sensitivity to optics errors
- All 3 choices can be realized in the same layout and hardware
  - Only strengths of dipoles and quadrupoles and the location of central quadrupole need to be changed

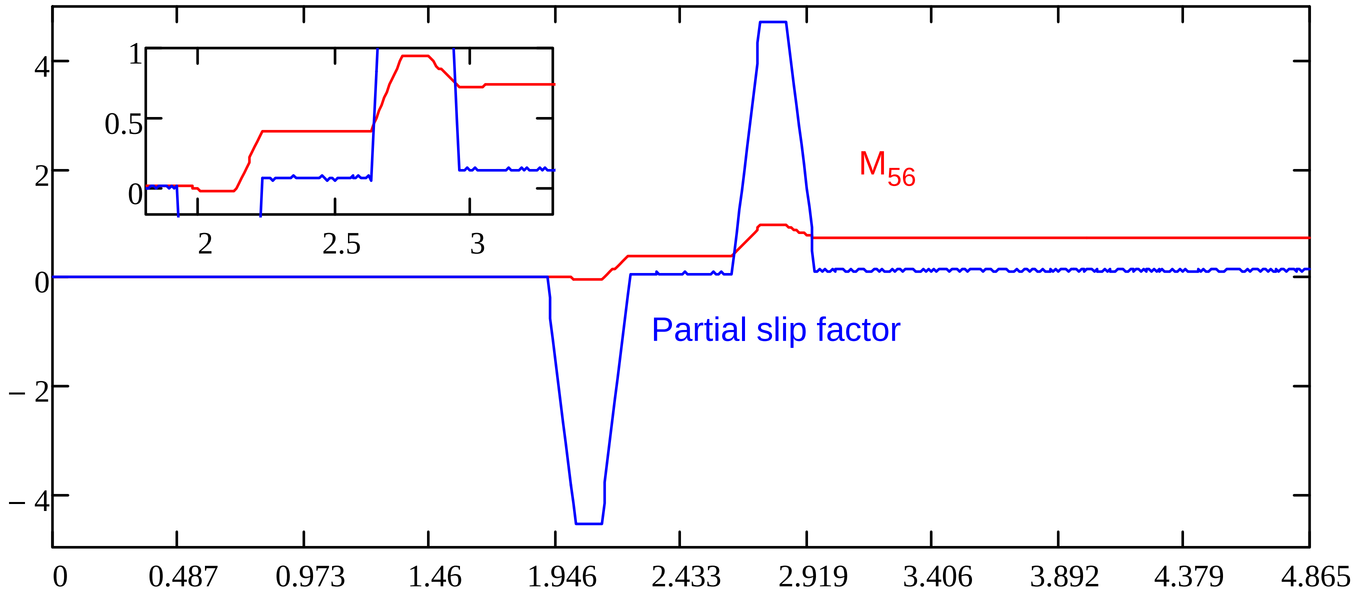
Optics for 2 μm wavelength and Δs=4 mm

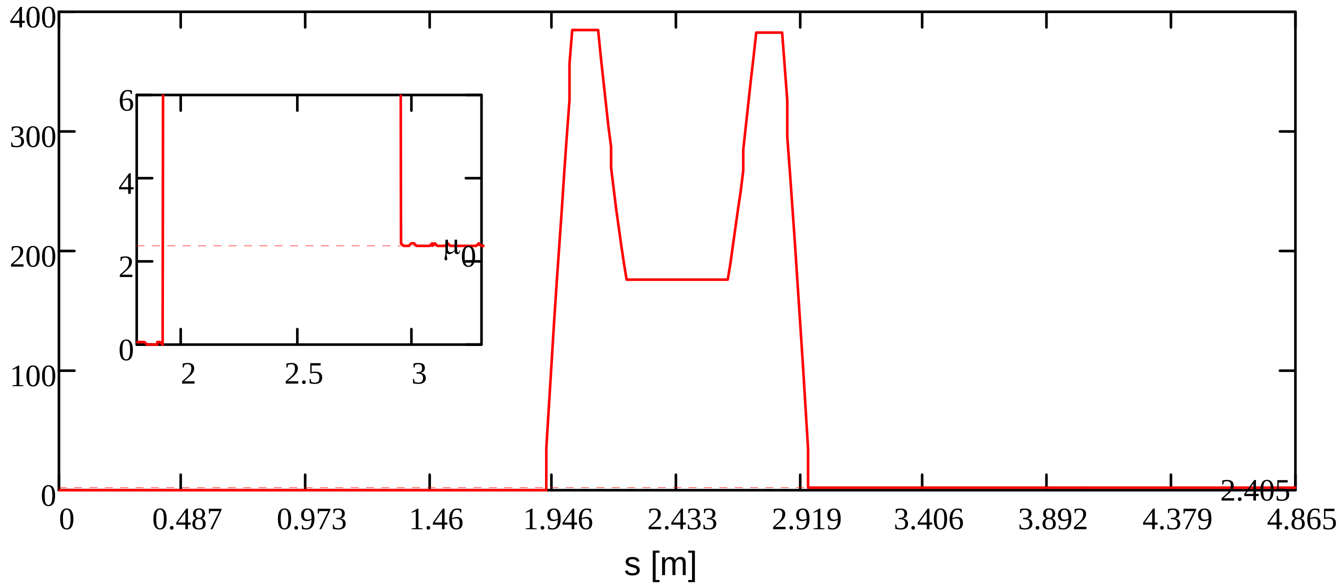




Twiss parameters (top) and rms beam sizes through OSC section

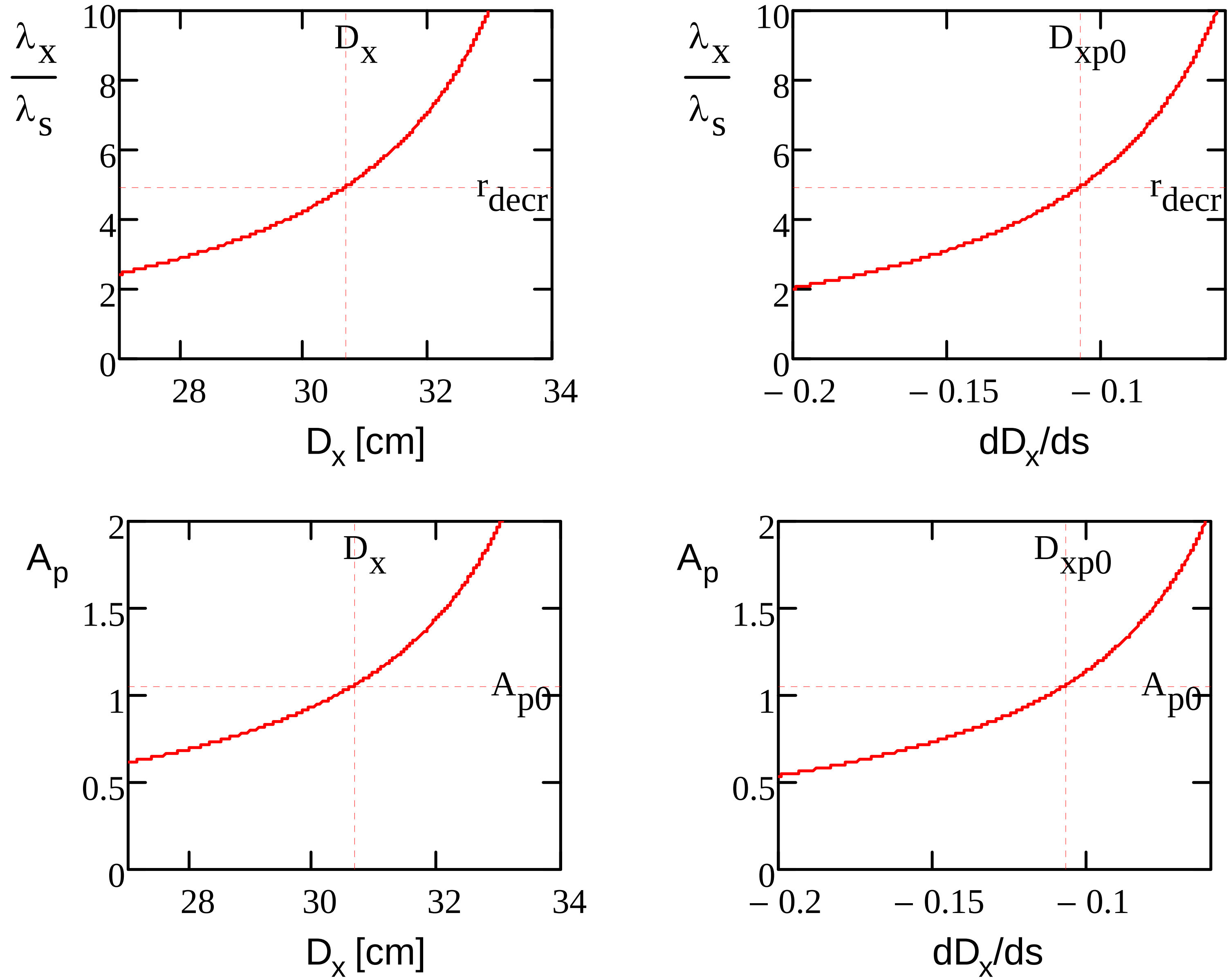
Optics for 2 μm wavelength and Δs=4 mm (continue)





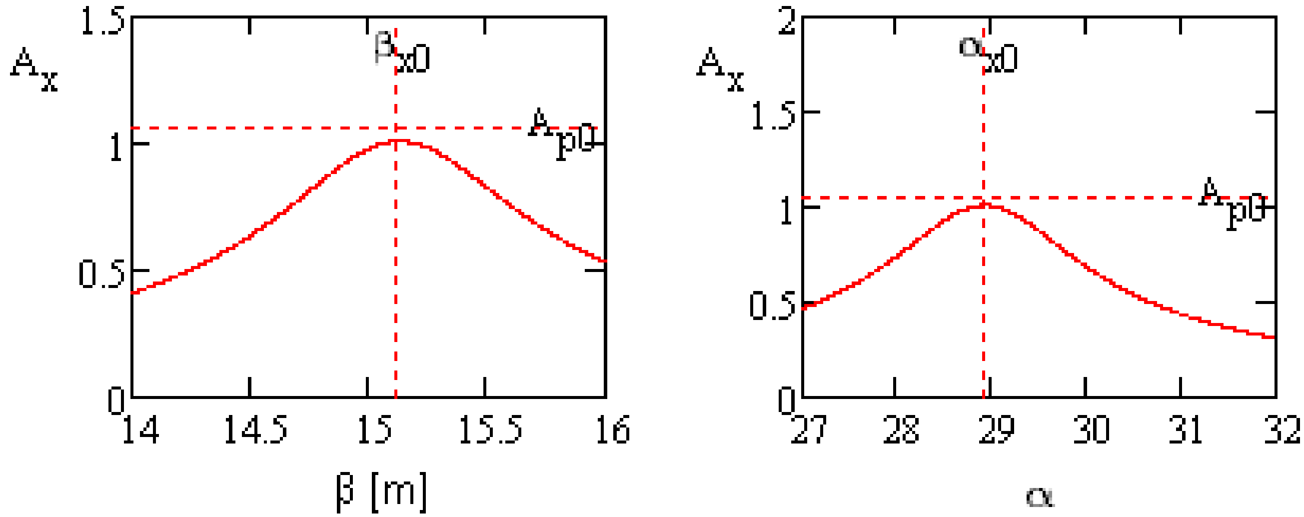
M_{56} and $\hat{M}_{56}$ (top) and sample lengthening due to betatron motion, a_x, (bottom) through OSC section

Optics for 2 μm wavelength and Δs=4 mm (continue)

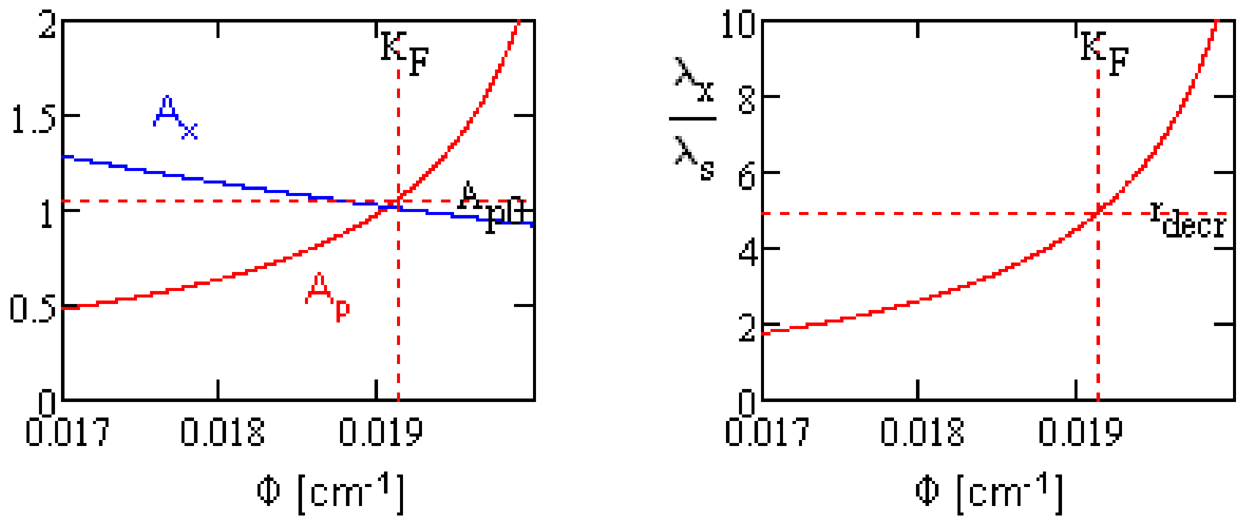


Dependence of ratio of decrements and longitudinal cooling acceptance (expressed in units of $4\sigma_p$) on dispersion and its derivative at the chicane entrance

Optics for 2 μm wavelength and Δs=4 mm (continue)



Dependence of transverse cooling acceptance (expressed in units of $4\sigma_x$) on beta- and alpha-functions at the chicane entrance



Dependence of transverse and longitudinal cooling acceptances (expressed in units of $4\sigma_{x,p}$) on focusing strength of quadrupole located in the chicane center

Optics structure of OS cooling insertion

N	Name	S[cm]	L[cm]	B[kG]	G[kG/cm]	type			
1	oq	10	10			drift			
2	bWGLm	99.376	89.376			9 per. wiggler			
3	oL	131.252	30			drift			
4	qChF1	151.252	20		-0.5	quad			
5	oq	161.252	10			drift			
6	qChD	181.252	20		0.8475	quad			
7	oq	191.252	10			drift			
8	bChp	202.252	11	6.9	0	0	0	0	8.66202
9	od	212.252	10			drift			
10	bChm	223.252	11	-6.9	0	0	0	0	-8.66202
11	oq	233.252	10			drift			
12	qChF	253.252	20		-0.184	quad			
13	oq	263.252	10			drift			
14	bChm	274.252	11	-6.9	0	0	0	0	-8.66202
15	od	284.252	10			drift			
16	bChp	295.252	11	6.9	0	0	0	0	8.66202
17	oq	305.252	10			drift			
18	qChD	325.252	20		0.8475	quad			
19	oq	335.252	10			drift			
20	qChF1	355.252	20		-0.5	quad			
21	oL	385.252	30			drift			
22	bWGLhp	386.502	89.376			9 per. wiggler			
23	oq	486.504	10			drift			

Cooling parameters

	Choice 2	Choice 3	MIT-Bates
Beam energy, MeV	86	150	300
SR transverse damping rate, s^{-1}, τ_x	0.29	3.7	0.2
Machine circumference, m	37.4		190.2
Number of particles per bunch	$3 \cdot 10^8$		10^8
Number on bunches	1		12
Rms horizontal emittance (SR equilibrium), cm	$1.21 \cdot 10^{-7}$	$3.4 \cdot 10^{-7}$	$98 \cdot 10^{-7}$
Rms momentum spread (SR equilibrium)	$0.857 \cdot 10^{-4}$	$1.48 \cdot 10^{-4}$	$1.64 \cdot 10^{-4}$
Rms bunch length (SR equilibrium), cm	11	11	-
Optical amplifier wave length, μm	6	2	2
Delay in the chicane, mm	10	4	6
Electron beam offset in the chicane, mm	50	32	98
Undulator length [m] / Number of periods	1 / 10		2 / 20
Undulator type	flat		flat
Undulator parameter, K_u	2.2		3.5
Ratio of decrements, λ_s/λ_x	3	5	~7
Cooling range in σ	6	4	2.8 ?
Cooling rates with gain equal to 1, s^{-1}, λ_s/λ_x	10/32	12/62	-
Optical amplifier bandwidth	~10%	~10%	10%
Optical gain (amplitude)	15	10	90
Optical amplifier power, mW	30	30	-
Cooling rates with optical amplifier, s^{-1}, λ_s/λ_x	160/500	110/550	-

6 μm versus 2 μm

- 6 μm looks as much more attractive choice
  - A lot of flexibilities in every important parameter
    - Optical amplifier
    - Optics sensitivity to errors
    - cooling range
  - Optical amplifier needs to be investigated
- 2 μm looks attractive
  - Looks very attractive without optical amplifier
    - 2 mm pass length difference reduces optics problems
    - Order of magnitude gain in damping rates(relative to SR)
      - Helical undulators increase gain by ~2 times
  - However a possibility of its use with optical amplifier needs to be investigated
    - Is delay of 4 mm sufficient?
    - An increase of delay above 4 mm may be possible
      but it increases the ratio of decrements and sensitivity of optics to errors,
      and increases difficulties of matching OSC section to a ring lattice

Discussion

- Optical stochastic cooling looks realistic with IOTA parameters
- Wave length of ~6 μm is preferable
  - It has considerable freedom in cooling parameters
- 2 μm choice requires an amplifier with ≤4 mm delay
  - This possibility requires additional investigation

Backup Viewgraphs

Damping Rates of Optical Stochastic Cooling

Transfer Matrix Parameterization

- Vertical degree of freedom is uncoupled and we will omit it in further consideration

$$\mathbf{M}=\begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{X}=\begin{bmatrix} x \\ \theta_x \\ s \\ \Delta p/p \end{bmatrix}$$

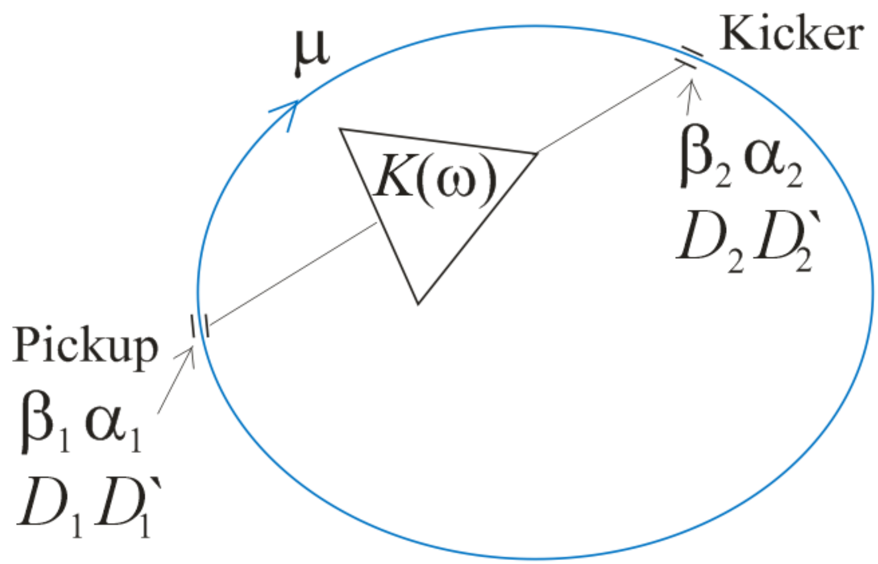
- M_{16} & M_{26} can be expressed through dispersion

$$\begin{bmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} D_1 \\ D_1' \\ 1 \end{bmatrix}=\begin{bmatrix} D_2 \\ D_2' \\ 1 \end{bmatrix}$$

That yields

$$M_{16}=D_2-M_{11}D_1-M_{12}D_1'$$
$$M_{26}=D_2'-M_{21}D_1-M_{22}D_1'$$

$$M_{11}=\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu+\alpha_1\sin\mu\right)$$
$$M_{22}=\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu-\alpha_2\sin\mu\right)$$
$$M_{12}=\sqrt{\beta_1\beta_2}\,\sin\mu$$
$$M_{21}=\frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu-\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu$$



Transfer Matrix Parameterization (continue)

- Symplecticity ($\mathbf{M}^T \mathbf{U} \mathbf{M} = \mathbf{U}$) binds up M_{51}, M_{52} and M_{16}, M_{26}
- That yields

$$M_{51} = M_{21} M_{16} - M_{11} M_{26}$$
$$M_{52} = M_{22} M_{16} - M_{12} M_{26}$$

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- Finally one can write

$$M_{16} = D_2 - D_1 \sqrt{\frac{\beta_2}{\beta_1}} \left(\cos\mu + \alpha_1 \sin\mu \right) - D_1' \sqrt{\beta_1 \beta_2} \sin\mu$$

$$M_{26} = D_1 \left(\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin\mu + \frac{\alpha_2 - \alpha_1}{\sqrt{\beta_1 \beta_2}} \cos\mu \right) + D_1' - D_1' \sqrt{\frac{\beta_1}{\beta_2}} \left(\cos\mu - \alpha_2 \sin\mu \right)$$

$$M_{51} = -D_2 \left(\frac{1 + \alpha_1 \alpha_2}{\sqrt{\beta_1 \beta_2}} \sin\mu + \frac{\alpha_2 - \alpha_1}{\sqrt{\beta_1 \beta_2}} \cos\mu \right) + D_2' - D_2' \sqrt{\frac{\beta_2}{\beta_1}} \left(\cos\mu + \alpha_1 \sin\mu \right)$$

$$M_{52} = -D_1 + D_2 \sqrt{\frac{\beta_1}{\beta_2}} \left(\cos\mu - \alpha_2 \sin\mu \right) - D_2' \sqrt{\beta_1 \beta_2} \sin\mu$$

- In the first order the orbit lengthening due to betatron motion is equal to zero if $D_1 = D`_1 = D_2 = D`_2 = 0$

Transfer Matrix Parameterization (continue)

- Partial momentum compaction and slip factor (from point 1 to point 2) are related to M_{56}

$$\Delta s_{1\to 2} \equiv 2\pi R\eta_1 \frac{\Delta p}{p} = M_{51}D_1\frac{\Delta p}{p} + M_{52}D_1'\frac{\Delta p}{p} + M_{56}\frac{\Delta p}{p} + \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

  - Further we assume that $v=c$, v=c, i.e. $1/\gamma^2=0$ and $\eta_1=\alpha_{1\to 2}$.

- That results in $\eta_1 = \dfrac{M_{51}D_1 + M_{52}D_1' + M_{56}}{2\pi R}$ or

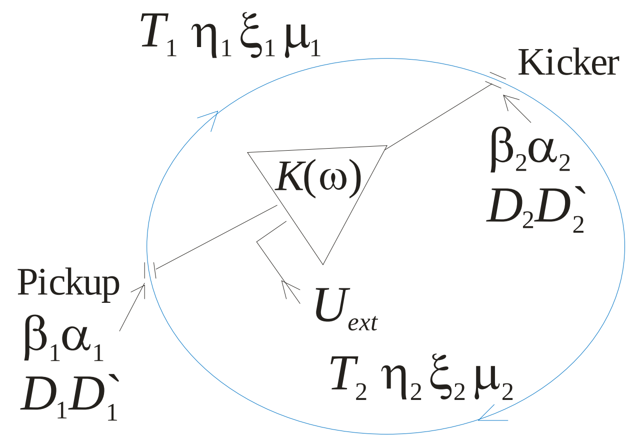
$$M_{56} = 2\pi R\eta_1 + D_1D_2\left(\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu + \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}}\cos\mu\right) + D_1D_2'\sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu + \alpha_1\sin\mu\right)$$
$$- D_1'D_2\sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu - \alpha_2\sin\mu\right) + D_1'D_2'\sqrt{\beta_1\beta_2}\sin\mu$$

- Thus, the entire transfer matrix from a point 1 to a point 2 can be expressed through the β-functions, dispersions and their derivatives at these points and the partial slip factor

Damping Rates of Optical Stochastic Cooling

Longitudinal kick

$$\frac{\delta p}{p} = \kappa \Delta L = \kappa \left(M_{151} x_1 + M_{152} \theta_{x1} + M_{156} \frac{\Delta p}{p} \right)$$



M$_1$ - pickup-to-kicker matrix

M$_2$ - kicker-to-pickup matrix

M = M$_1$M$_2$ – ring matrix

$\mu = \mu_1 + \mu_2$

$$\mathbf{M v}_k = \lambda_k \mathbf{v}_k$$

Or in the matrix form: $\boldsymbol{\delta}\mathbf{X} = \mathbf{M}_c \mathbf{X}_1$

$$\mathbf{M}_c = \kappa \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{151} & M_{152} & 0 & M_{156} \end{bmatrix}$$

Total ring matrix related to kicker (Ring&RF&damper)

$$\mathbf{M}_{tot}\mathbf{X}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{X}_2 + \boldsymbol{\delta}\mathbf{X}_2 = \mathbf{M}_1\mathbf{M}_2\mathbf{X}_2 + \mathbf{M}_c\mathbf{X}_1 = \left(\mathbf{M}_1\mathbf{M}_2 + \mathbf{M}_c\mathbf{M}_2\right)\mathbf{X}_2$$

⇨ $\mathbf{M}_{tot} = \mathbf{M} + \Delta\mathbf{M}_c$ where $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\ , \quad \Delta\mathbf{M} = \mathbf{M}_c\mathbf{M}_2$

Damping Rates of Optical Stochastic Cooling (continue)

Perturbation theory yields that the eigen-value correction is [HB2008]:

$$\delta\lambda_k = \frac{i}{2}\mathbf{v}_k^{+}\,\mathbf{U}\,\Delta\mathbf{M}\,\mathbf{v}_k = \frac{i}{2}\mathbf{v}_k^{+}\mathbf{U}\,\mathbf{M}_c\,\mathbf{M}_1^{-1}\left(\mathbf{M}_1\mathbf{M}_2\right)\mathbf{v}_k = \frac{i}{2}\lambda_k\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\,\mathbf{M}_1^{-1}\,\mathbf{v}_k$$

Corresponding tune shift is: $\delta Q_k = \frac{i}{2\pi}\frac{\delta\lambda_k}{\lambda_k} = -\frac{1}{4\pi}\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\,\mathbf{M}_1^{-1}\,\mathbf{v}_k$

Symplecticity relates the transfer matrix and its inverse:

$$\mathbf{M}_1^{-1} = -\mathbf{U}\,\mathbf{M}_1^{T}\,\mathbf{U}$$

⇨ $$\delta Q_k = \frac{1}{4\pi}\mathbf{v}_k^{+}\mathbf{U}\;\mathbf{M}_c\;\mathbf{U}\,\mathbf{M}_1^{T}\mathbf{U}\,\mathbf{v}_k$$

Performing matrix multiplication and taking into account that symplecticity binds up M_{51},M_{52} and M_{16},M_{26} one finally obtains:

$$\delta Q_k = \frac{\kappa}{4\pi}\mathbf{v}_k^{+}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{v}_k$$

Eigen-vectors and Damping Decrements (Mode 1)

- There are two eigen-vectors
  - One related to the betatron motion $\mathbf{v}_1$
  - And one related to the synchrotron motion $\mathbf{v}_2$
- They are normalized as: $\mathbf{v}_k^{\ +}\,\mathbf{U}\,\mathbf{v}_k = -2i$
- If the synchrotron tune and dispersion in RF cavities are small the effect of RF can be neglected in the computation of $\mathbf{v}_1$
  - In this case $\lambda_1 = e^{-i\mu}$ and the eigen-vector related to the kicker position is

$$\mathbf{v}_1 = \begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ \mathrm{v}_{13} \\ 0 \end{bmatrix}, \quad \mathbf{M}\mathbf{v}_k = \lambda_k \mathbf{v}_k\ , \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{16} \\ M_{21} & M_{22} & 0 & M_{26} \\ M_{51} & M_{52} & 1 & M_{56} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first 2 components are the same as for uncoupled case.
The third component has to be found from the third equation

=> $$\mathrm{v}_{13} = -\frac{iD_2(1-i\alpha_2)+D_2'\beta_2}{\sqrt{\beta_2}}$$

- Corresponding damping rate is

$$\lambda_1 = -2\pi \operatorname{Im} \delta Q_1$$

$$= -\frac{\kappa}{2} \operatorname{Im}\left(\begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ \mathrm{v}_{1_3} \\ 0 \end{bmatrix}^{+} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta_2} \\ -(i+\alpha_2)/\sqrt{\beta_2} \\ \mathrm{v}_{1_3} \\ 0 \end{bmatrix} \right)$$

$$= -\frac{\kappa}{2}\left(D_2 M_{1_{2,6}} - D_2' M_{1_{1,6}} \right)$$

That yields

$$\lambda_1 = -\frac{\kappa}{2}\left[D_1 D_2 \frac{(1+\alpha_1\alpha_2)\sin\mu_1 + (\alpha_2-\alpha_1)\cos\mu_1}{\sqrt{\beta_1\beta_2}} - D_1' D_2 \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\mu_1 - \alpha_2\cos\mu_1\right) \right.$$
$$\left. + D_1 D_2' \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\mu_1 + \alpha_1 \sin\mu_1\right) + D_1' D_2' \sqrt{\beta_1\beta_2}\sin\mu_1 \right]$$

Expressing it through the partial slip factor one gets

$$\lambda_1 = -\frac{\kappa}{2}\left(M_{56} - 2\pi R \eta_1 \right)$$

Eigen-vectors and Damping Decrements (Mode 2)

- To find the second eigen-vector we will ignore the second order effects of betatron motion on the longitudinal dynamics
  - The linerazed RF kick is

$$\frac{\delta p}{p} = -\Phi_s s$$

  - Simple calculations yield for the eigen value $\lambda_1 = e^{-i\mu_s}$
    where the synchrotron tune $\mu_s = \sqrt{2\pi R \eta \Phi_s}$
  - Corresponding eigen-vector related to the kicker position is

$$\mathbf{v}_1 = \begin{bmatrix} -iD_2 / \sqrt{\beta_s} \\ -iD_2' / \sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i / \sqrt{\beta_s} \end{bmatrix}$$

    where the longitudinal beta-function $\beta_s = 2\pi R \eta / \mu_s$

- Corresponding damping rate is

$$\lambda_2 = -2\pi \,\mathrm{Im}\,\delta Q_2$$

$$= -\frac{\kappa}{2}\mathrm{Im}\left(\begin{bmatrix} -iD_2/\sqrt{\beta_s} \\ -iD_2'/\sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i/\sqrt{\beta_s} \end{bmatrix}^{+} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{1_{26}} & -M_{1_{16}} & 0 & M_{1_{56}} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -iD_2/\sqrt{\beta_s} \\ -iD_2'/\sqrt{\beta_s} \\ \sqrt{\beta_s} \\ -i/\sqrt{\beta_s} \end{bmatrix}\right)$$

$$= -\frac{\kappa}{2}\left(M_{1_{56}} - D_2 M_{1_{26}} + D_2' M_{1_{16}}\right)$$

Expressing the matrix elements through Twiss parameters one obtains

$$\boxed{\lambda_2 = -\frac{\kappa}{2}M_{1_{56}} - \lambda_1 = -\pi\kappa R \eta_1}$$

The last expression can be directly obtained from the definition of the partial slip factor

- The above equation yields the sum of the decrements is

$$\boxed{\lambda_1 + \lambda_2 = -\frac{\kappa}{2}M_{1_{56}}}$$

Damping Rates for Smooth Lattice Approximation

- For zero derivatives of beta-function and dispersion at pickup and kicker one obtains

$$\lambda_1 = -\frac{\kappa}{2}\frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1$$

$$\lambda_2 = -\frac{\kappa}{2}\left[M_{156} - \frac{D_1 D_2}{\sqrt{\beta_1\beta_2}}\sin\mu_1\right]$$

- Smooth lattice approximation additionally yields

$$\beta = \frac{R}{\nu},\quad D = \frac{R}{\nu^2},\quad \mu_1 = \nu\frac{L_{pk}}{R}\quad \eta_1 = -\frac{L_{pk}}{2\pi\nu^2 R},\quad M_{156} = -\frac{L_{pk}}{\nu^2} + \frac{R}{\nu^3}\sin\left(\nu\frac{L_{pk}}{R}\right),$$

where L_{pk} is the pickup-to-kicker path length, and ν is the betatron tune

$$\Rightarrow\quad \lambda_1 = -\frac{\kappa}{2}\frac{R}{\nu^3}\sin\left(\nu\frac{L_{pk}}{R}\right)$$

$$\lambda_2 = \frac{\kappa}{2}\frac{L_{pk}}{\nu^2}$$

Comparison to Zholents-Zolotorev result

PRST-AB, v.7,p.12801 (2004)

Eqs. (A9) and (A11) in the paper Appendix can be rewritten in the following simplified form

$$\lambda_1 = \frac{\kappa}{2}\left(D_2 M^{-1}_{1_{51}} + D'_2 M^{-1}_{1_{52}}\right)$$

$$\lambda_2 = -\frac{\kappa}{2}\left(D_2 M^{-1}_{1_{51}} + D'_2 M^{-1}_{1_{52}} + M^{-1}_{1_{56}}\right)$$

The inverse of the matrix is

$$\mathbf{M}_1^{-1} = -\mathbf{U}\,\mathbf{M}_1^{T}\,\mathbf{U} = \begin{bmatrix} M_{1_{22}} & -M_{1_{12}} & 0 & M_{1_{52}} \\ -M_{1_{21}} & M_{1_{11}} & 0 & M_{1_{51}} \\ M_{1_{26}} & M_{1_{16}} & 1 & -M_{1_{56}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Substituting expressions for matrix elements into above Eqs. for decrements one arrives to the same results

Sample Lengthening on Pickup-to-Kicker Travel

- Zero length sample lengthens on its way from pickup-to-kicker

$$\sigma_{\Delta L}{}^2 = \int \left(M_{51} x + M_{52}\theta_x + M_{56}\tilde{p} \right)^2 f(x,\theta_x,\tilde{p})\, dx\, d\theta_x\, d\tilde{p} \, , \quad \tilde{p} = \frac{\Delta p}{p}$$

where for Gaussian distribution

$$f(x,\theta_x,\tilde{p}) = \frac{\exp\left(-\frac{\gamma_p (x - D_p\tilde{p})^2 + 2\alpha_p(\theta_x - D'_p\tilde{p})(x - D_p\tilde{p}) + \beta_p (x - D_p\tilde{p})}{2\varepsilon} - \frac{\tilde{p}^2}{2\sigma_p{}^2} \right)}{\sqrt{2\pi}\, 2\pi\sigma_p\varepsilon} \, , \quad \gamma_p = \frac{1+\alpha_p{}^2}{\beta_p}$$

- ♦ Performing integration one obtains

$$\boxed{\sigma_{\Delta L}{}^2 = \varepsilon\left(\beta_p M_{51}{}^2 - 2\alpha_p M_{51} M_{52} + \gamma_p M_{52}{}^2 \right) + \sigma_p{}^2 \left(M_{51} D_p + M_{52} D'_p + M_{56} \right)^2}$$

- Expressing matrix elements through Twiss parameters yields

$$\sigma_{\Delta L}{}^2 = \varepsilon F_\varepsilon + \sigma_p{}^2 \left(2\pi R \alpha_{1\to 2} \right)^2$$

$$F_\varepsilon = D_p{}^2\gamma_p + D_k{}^2\gamma_k - \frac{2D_p D_k}{\sqrt{\beta_p\beta_k}} \left((1+\alpha_p\alpha_k)\cos\mu_1 + (\alpha_p - \alpha_k)\sin\mu_1 \right) + D'^2_p\beta_p + D'^2_k\beta_k + 2D_p D'_p\alpha_p +$$

$$2D_p D'_p\alpha_p + 2D_p D'_k\sqrt{\frac{\beta_k}{\beta_p}}\left(\sin\mu_1 - \alpha_p\cos\mu_1\right) - 2D_k D'_p\sqrt{\frac{\beta_p}{\beta_k}}\left(\sin\mu_1 + \alpha_k\cos\mu_1\right) - 2D'_k D'_p\sqrt{\beta_p\beta_k}\cos\mu_1$$

- For zero derivatives it yields

$$\sigma_{\Delta L}{}^2 = \varepsilon\left(\frac{D_k{}^2}{\beta_k} + \frac{D_p{}^2}{\beta_p} - \frac{2D_k D_p}{\sqrt{\beta_k \beta_p}}\cos\mu_1\right) + \sigma_p{}^2\left(M_{1_{56}} - \frac{D_k D_p}{\sqrt{\beta_k \beta_p}}\sin\mu_1\right)$$

Longitudinal Kick by E.-M. Wave

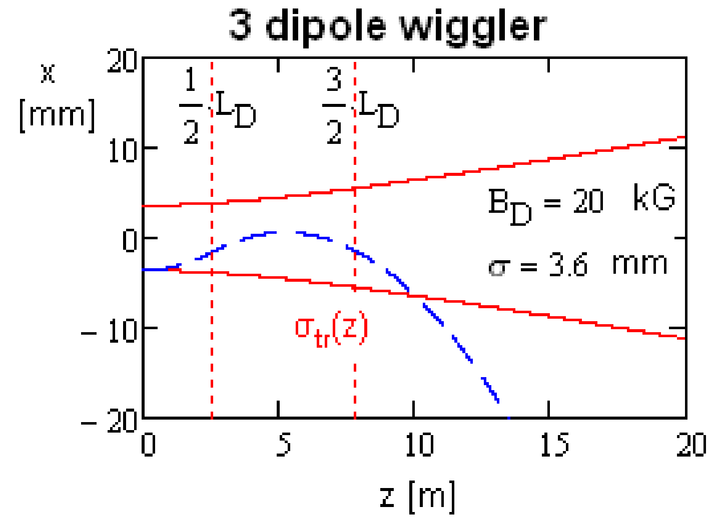
- Electric field of e.-m. wave focused at z=0 to the rms size $\sigma_\perp$

$$E_x(x,y,z,t) = \mathrm{Re}\left(E_0 e^{i(\omega t - kz)} \frac{\sigma_\perp^{\,2}}{\sigma^2(z)} \exp\left(-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} \right) \right)$$

$$E_y(x,y,z,t) = 0$$

$$E_z(x,y,z,t) = \mathrm{Re}\left(iE_0 e^{i(\omega t - kz)} \frac{\sigma_\perp^{\,2} x}{k\sigma^4(z)} \exp\left(-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} \right) \right)$$

$$E_0 = \sqrt{\frac{8P}{c\sigma_\perp^{\,2}}}, \quad \sigma^2(z) = \sigma_\perp^{\,2} - i\frac{z}{k}, \quad k = \frac{2\pi}{\lambda_w}$$



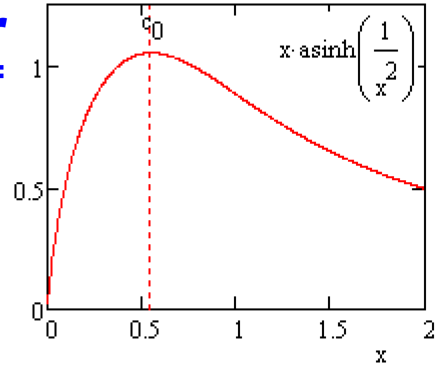
- The beam is deflected in the x-plane by wiggler magnetic field
  - That results in the beam energy change $\Delta E = e\int (\mathbf{E}\cdot\mathbf{v})dt$

⇨
$$\Delta E = eE_0 \int \mathrm{Re}\left\{ \left(\frac{dx}{dz}\frac{\sigma_\perp^{\,2}}{\sigma^2(z)} + \frac{i\sigma_\perp^{\,2} x}{k\sigma^4(z)} \right) \exp\left[-\frac{1}{2}\frac{x^2+y^2}{\sigma^2(z)} + ik\left(\frac{z}{2\gamma^2} + \frac{1}{2}\int_0^z \left(\frac{dx}{dz'}\right)^2 dz' \right) + i\psi \right] \right\} dz$$

where ψ is the accelerating phase (ΔE = 0 for ψ = 0)

and $\frac{1}{2}\int_0^z \left(\frac{dx}{dz'}\right)^2 dz'$ represents the path length difference between light and beam introduced by wiggler (relative to wiggler center)

Estimate of Energy Kick in Helical Wiggler



- Assuming that $\rho_\perp$ << $\sigma_\perp$ the kick amplitude is

$$\frac{\Delta E}{e} = \sqrt{\frac{4P}{c\sigma_\perp^2}}\theta_0 2\int_0^{L/2}\frac{\sigma_\perp^2 dz}{\left|\sigma_\perp^2 - iz/k\right|} = 4\sqrt{\frac{P}{c}}\theta_0 k\sigma_\perp a\sinh\left(\frac{L}{2k\sigma_\perp^2}\right)$$

- The function $x\sinh\left(1/x^2\right)$ achieves its maximum at $x = c_0 \approx 0.54884$

⇨ Maximum kick of $\left.\frac{\Delta E}{e}\right|_{opt} = \frac{4c_0}{\sqrt{2}}\sinh\left(\frac{1}{c_0^2}\right)\sqrt{\frac{P}{c}}\theta_0\sqrt{kL}$ is achieved at $\sigma_\perp = \sqrt{\frac{c_0^2}{2}\frac{L}{k}}$

- Taking into account that $4\pi/c = Z_0$ and $kL = 2\pi n_{wgl}$ we obtain

$$\left.\frac{\Delta E}{e}\right|_{opt} = 2c_0\sinh\left(\frac{1}{c_0^2}\right)\theta_0\sqrt{PZ_0 n_{wgl}}$$

- The condition of resonance is: $k\left(1/(2\gamma^2) + \theta_0^2/2\right) = k_{wgl}$, where the particle angle (relative to wave direction) is $\theta_0 = \frac{1}{k_{wgl}R_L}$, $R_L = \frac{pc}{eB_0}$

- That yields

$$\left.\frac{\Delta E}{e}\right|_{opt} = c_0\sinh\left(\frac{1}{c_0^2}\right)\sqrt{\frac{8PZ_0 n_{wgl}K_u^2}{1+K_u^2}} \approx \sqrt{\frac{8.837PZ_0 n_{wgl}K_u^2}{1+K_u^2}}, \quad K_u = \frac{eB_0}{pck_{wgl}}$$

References

HB2008 - V. Lebedev, A. Burov, “Coupling and its Effects on Beam Dynamics”, HB-2008