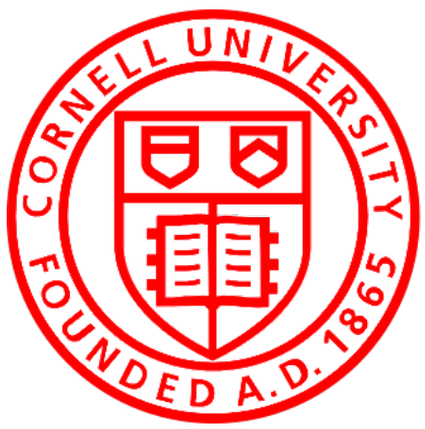
Snowmass Rare Processes and Precision Measurements
Frontier Spring Meeting, May 17th 2022

$K \to \mu^+\mu^-$ as a third kaon *golden mode*

Avital Dery



$K_L \to \pi^0 \nu\bar{\nu}$ $K^+ \to \pi^+ \nu\bar{\nu}$

Theoretically clean, sensitive to functions of $V_{ts}^* V_{td}$, $V_{cs}^* V_{cd}$.

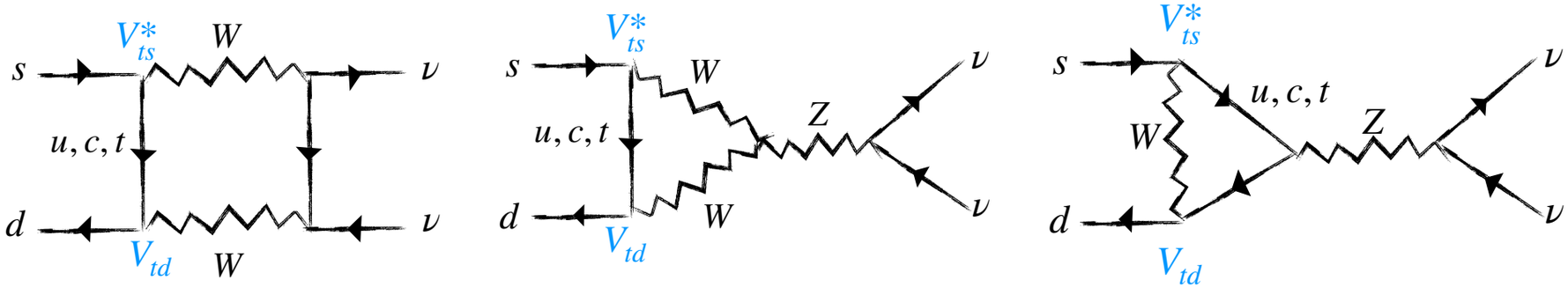
What about $K_{L,S} \to \mu^+\mu^-$?

- 2-body vs. 3-body
- Muons vs. missing energy
- Similar weak Hamiltonian (also sensitive to $V_{ts}^* V_{td}$, $V_{cs}^* V_{cd}$)

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} \sum_{l=e,\mu,\tau} \left(V_{cs}^* V_{cd} X_{NL}^l + V_{ts}^* V_{td} X(x_t) \right) (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A}$$

$K_L \to \pi^0 \nu\bar{\nu}$

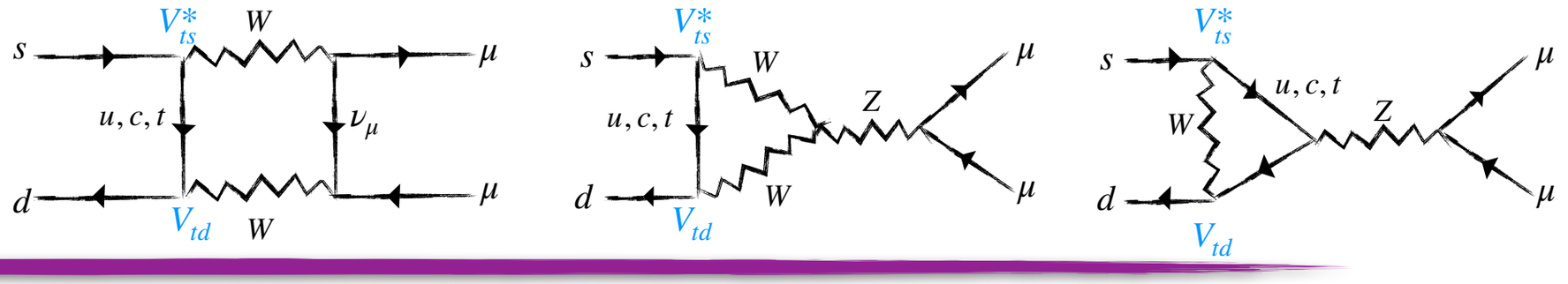
- Within the SM determined entirely from the weak effective Hamiltonian (short-distance physics)
- Purely CP-violating



$K_{L,S} \to \mu^+\mu^-$

- Long-distance effects mediate **CP conserving transitions** which dominate both for $K_L \to \mu^+\mu^-$ and for $K_S \to \mu^+\mu^-$

- The short-distance contribution, coming from the weak Hamiltonian, induces **CP violating transitions**



$K \to \mu^+\mu^-$

$\mathcal{B}(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \cdot 10^{-9}$

$\mathcal{B}(K_S \to \mu^+\mu^-) < 2.1 \cdot 10^{-10}$

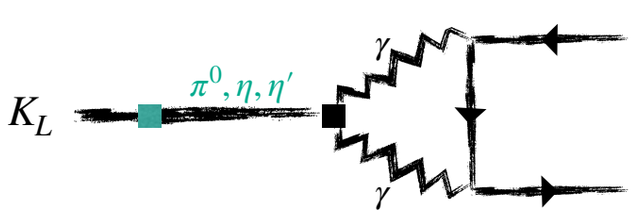
LD SD

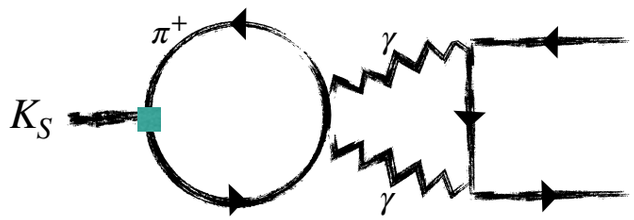
$$\mathcal{B}(K_L \to \mu^+\mu^-)_{\rm SM} = \begin{cases} (6.85 \pm 0.80 \pm 0.06) \times 10^{-9} \, (+) \\ (8.11 \pm 1.49 \pm 0.13) \times 10^{-9} \, (-) \end{cases}$$

LD SD

$$\mathcal{B}(K_S \to \mu^+\mu^-)_{\rm SM} = (5.18 \pm 1.50 \pm 0.02) \times 10^{-12}$$

The key: CPV comes from the weak (SD) diagrams





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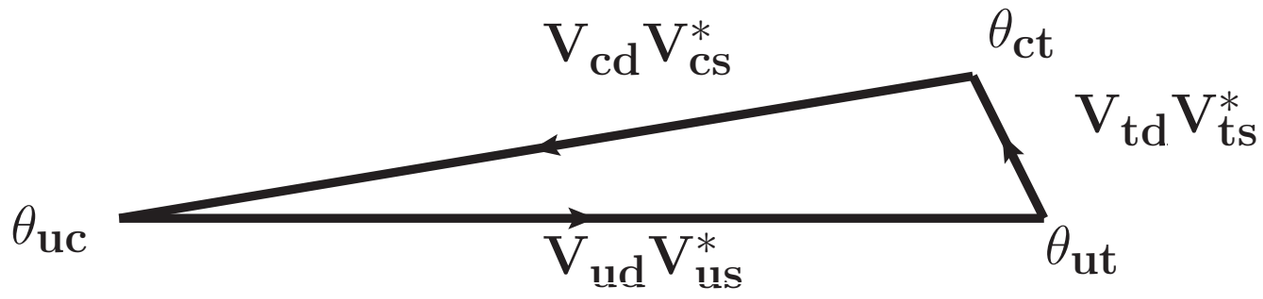

$$\theta_{uc} = arg\left(-\frac{V_{cd}V_{cs}^*}{V_{ud}V_{us}^*}\right) \sim \lambda^4$$

CP Analysis of $K \to \mu^+\mu^-$

Initial state: kaon mass eigenstates are approximately also CP eigenstates,

$$\underbrace{K_L}_{\text{CP-odd}}, \qquad \underbrace{K_S}_{\text{CP-even}}$$

Final state: since the kaon has $J = 0$, the dimuon state can have either $S = 0, \ell = 0$ or $S = 1, \ell = 1$

corresponding to final states:

$$\underbrace{(\bar{\mu}\mu)_{\ell=0}}_{\text{CP-odd}}, \qquad \underbrace{(\bar{\mu}\mu)_{\ell=1}}_{\text{CP-even}}$$

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$$\underset{\text{CP-odd}}{(\bar{\mu}\mu)_{\ell=0}}, \qquad \underset{\text{CP-even}}{(\bar{\mu}\mu)_{\ell=1}}$$

In practice, we measure the incoherent sums,

$$\Gamma(K_S \to \mu^+\mu^-)_{meas.} = \Gamma(K_S \to (\mu^+\mu^-)_{\ell=0}) + \Gamma(K_S \to (\mu^+\mu^-)_{\ell=1})$$

$$\Gamma(K_L \to \mu^+\mu^-)_{meas.} = \Gamma(K_L \to (\mu^+\mu^-)_{\ell=0}) + \Gamma(K_L \to (\mu^+\mu^-)_{\ell=1})$$

If we could extract the CPV modes, we would have a similar situation (theoretically) to $K_L \to \pi^0\nu\bar{\nu}$

K_S-K_L Interference

G. D'Ambrosio and T. Kitahara [arXiv:1707.06999]
AD, M. Ghosh, Y. Grossman, S. Schacht [arXiv:2104.06427]

Time dependent rate:

$$\left(\frac{d\Gamma}{dt}\right) = N_f\, f(t)\,, \qquad f(t) = C_L\, e^{-\Gamma_L t} + C_S\, e^{-\Gamma_S t} + 2\left[C_{\cos}\,\cos(\Delta m t) + C_{\sin}\,\sin(\Delta m t)\right] e^{-\Gamma t}\,,$$

4 Experimental parameters

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4 Experimental parameters

Theory parameters: 4 amplitudes, 2 are CPC transitions and 2 are CPV,

SD $|A(K_S \to (\mu^+\mu^-)_{\ell=0})|$
$|A(K_S \to (\mu^+\mu^-)_{\ell=1})|$
$|A(K_L \to (\mu^+\mu^-)_{\ell=0})|$
SD $|A(K_L \to (\mu^+\mu^-)_{\ell=1})|$

$\varphi_0 \equiv \arg(A(K_S)_0^* A(K_L)_0)$
$\varphi_1 \equiv \arg(A(K_S)_1^* A(K_L)_1)$

A priori, 6 theory parameters

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$|A(K_S \to (\mu^+\mu^-)_{\ell=1})|$

$|A(K_L \to (\mu^+\mu^-)_{\ell=0})|$

SD ~~$|A(K_L \to (\mu^+\mu^-)_{\ell=1})|$~~

$\varphi_0 \equiv \arg(A(K_S)_0^* A(K_L)_0)$

~~$\varphi_1 \equiv \arg(A(K_S)_1^* A(K_L)_1)$~~

The SM weak leptonic current, $(\bar{\mu}_L \gamma^\mu \mu_L)$, creates only the CP-odd $(\bar{\mu}\mu)_{\ell=0}$ state

(corrections at $\mathcal{O}\left(m_K^2/m_W^2\right)$)

A priori, ~~6~~ theory parameters

4, 1 of which is pure SD

SM prediction

G. D'Ambrosio and T. Kitahara [arXiv:1707.06999]

AD, M. Ghosh, Y. Grossman, S. Schacht [arXiv:2104.06427]

Only hadronic parameter, $\mathcal{O}(1\%)$ uncertainty from isospin breaking

$$|A(K_S)_0|^2 = \left| \frac{G_F}{2} \frac{2\alpha_{em} m_K m_\mu Y(x_t)}{\pi \sin^2\theta_W} \times f_K \times V_{cs}V_{cd}\, \mathrm{Im}\left(\frac{V_{ts}^* V_{td}}{V_{cs}^* V_{cd}}\right) \right|^2$$

$$\overline{\eta}_{ds} = A^2\lambda^4\overline{\eta} + \mathcal{O}(\lambda^7)$$

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$$C_L = |A(K_L)_0|^2$$

$$C_S = |A(K_S)_0|^2 + \beta_\mu^2\,|A(K_S)_1|^2$$

$$C_{\cos} = D\,|A(K_S)_0 A(K_L)_0|\cos(\varphi_0)$$

$$C_{\sin} = D\,|A(K_S)_0 A(K_L)_0|\sin(\varphi_0)$$

$$D = \frac{N_{K^0} - N_{\overline{K}^0}}{N_{K^0} + N_{\overline{K}^0}}$$

$$\boxed{\frac{1}{D^2}\,\frac{C_{\sin}^2 + C_{\cos}^2}{C_L} = |A(K_S)_0|^2}$$

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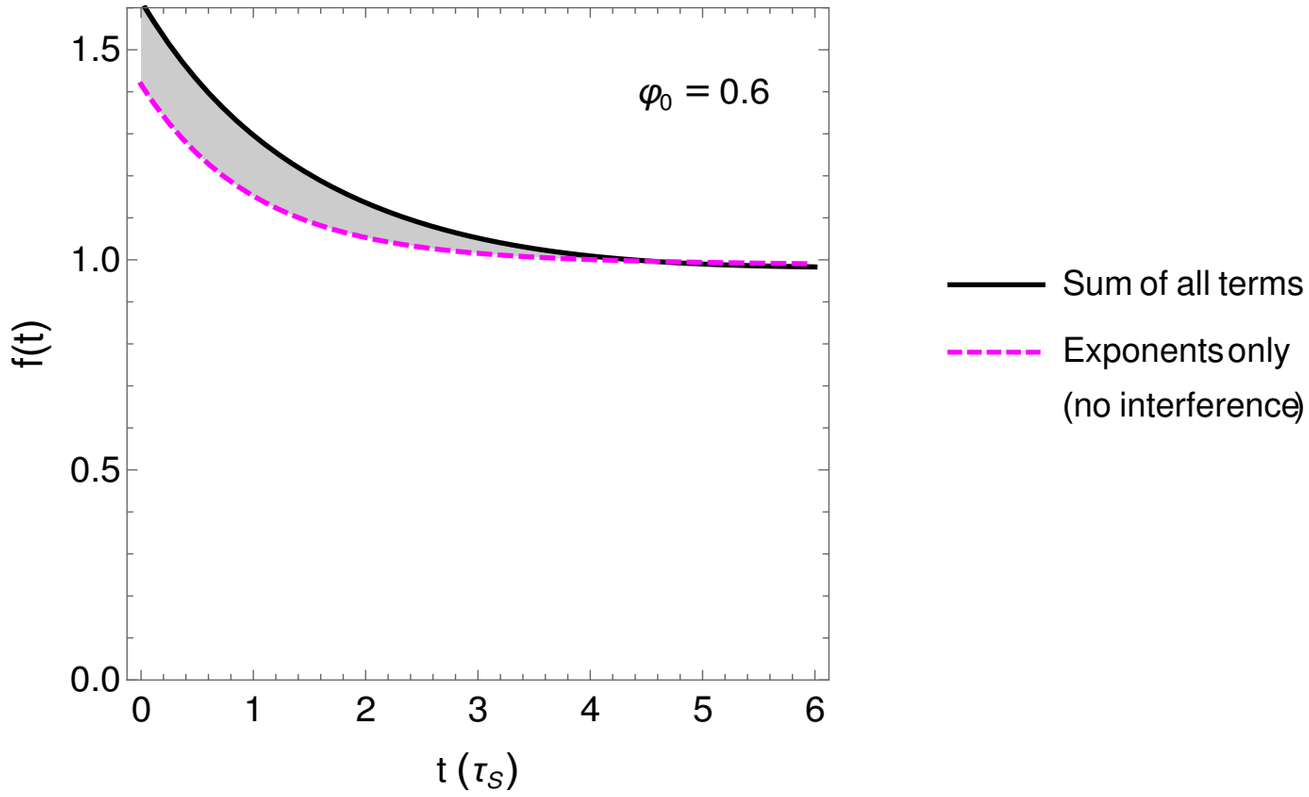

Approximate SM prediction, $D = 1$:

$(C_L)_{\rm SM} \equiv 1$

$(C_S)_{\rm SM} \approx 0.43$

$(C_{Int.})_{\rm SM} = \sqrt{C_{\cos}^2 + C_{\sin}^2} \approx 0.12$

SM prediction

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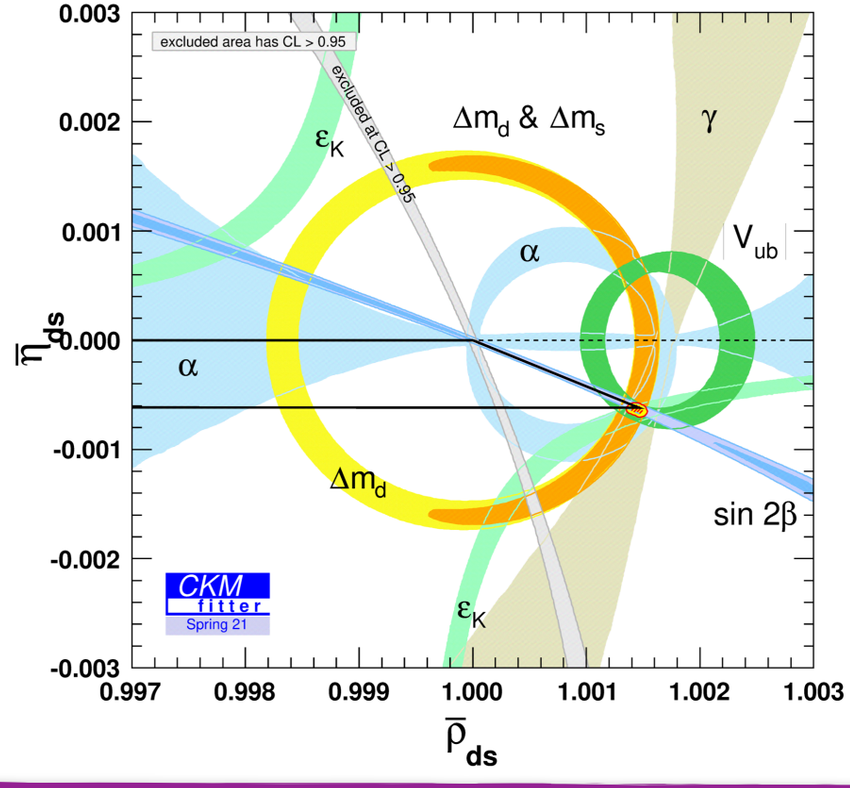
Only hadronic parameter, $\mathcal{O}(1\%)$ uncertainty from isospin breaking

$$|A(K_S)_0|^2 = \left| \frac{G_F}{2} \frac{2\alpha_{em} m_K m_\mu Y(x_t)}{\pi \sin^2\theta_W} \times f_K \times V_{cs}V_{cd}\, \mathrm{Im}\left(\frac{V_{ts}^* V_{td}}{V_{cs}^* V_{cd}} \right) \right|^2$$

$$\bar{\eta}_{ds} = A^2\lambda^4\bar{\eta} + \mathcal{O}(\lambda^7)$$

Current error on $\bar{\eta}_{ds}$ from B physics is $\mathcal{O}(5\%)$



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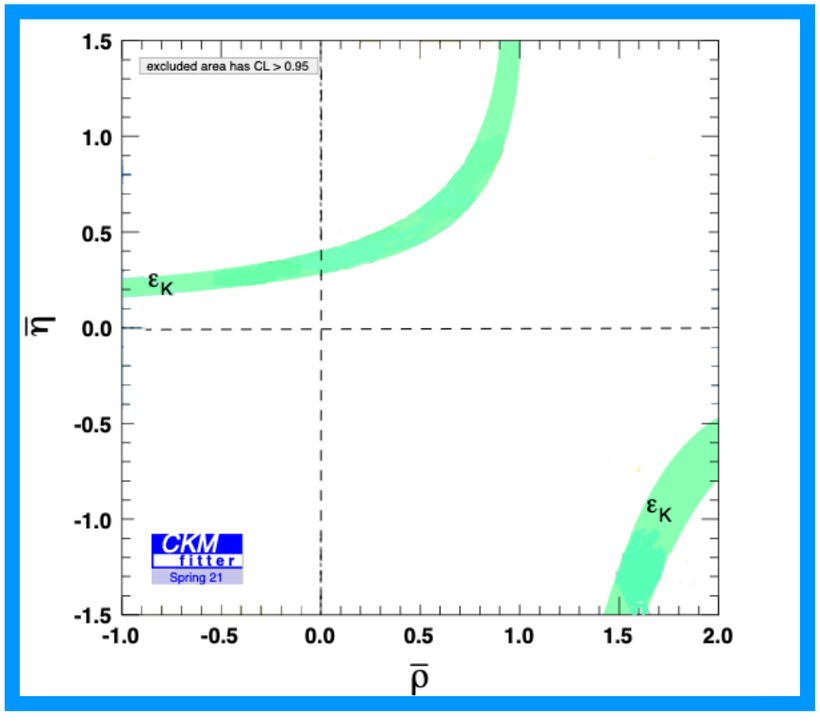
Current picture from kaon physics:



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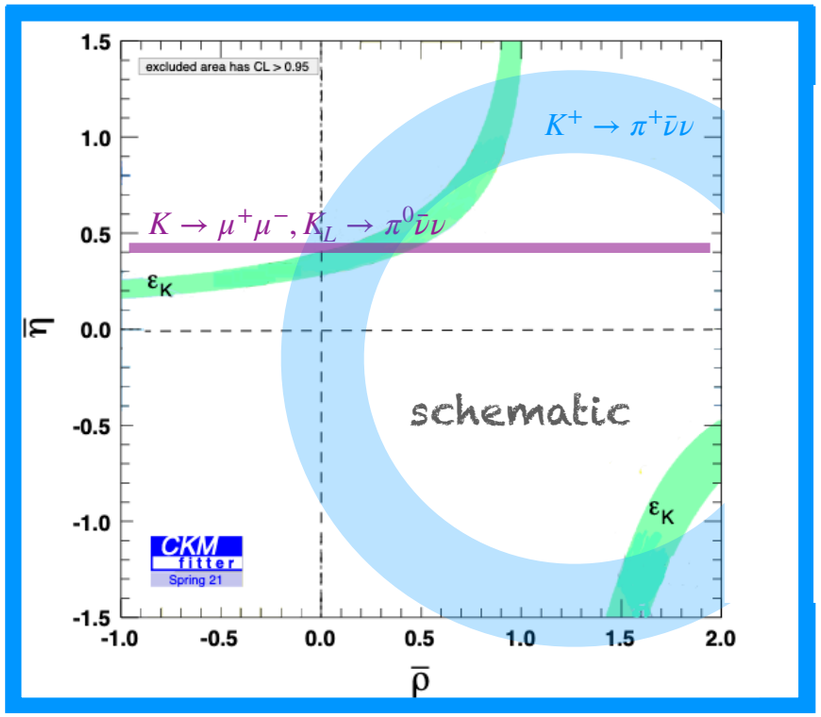
Future picture from kaon physics:



Time dependence in $K \to \mu^+\mu^-$ would make it a *golden mode*, cleanly sensitive to $\overline{\eta}_{ds} \approx A^2\lambda^4\overline{\eta}$

The key is in K_S-K_L interference

Pilot study

Experimental feasibility

Neutral beam (KLEVER-like)

+

charged particles detector
(NA62-like)

close to the target

Experimental feasibility

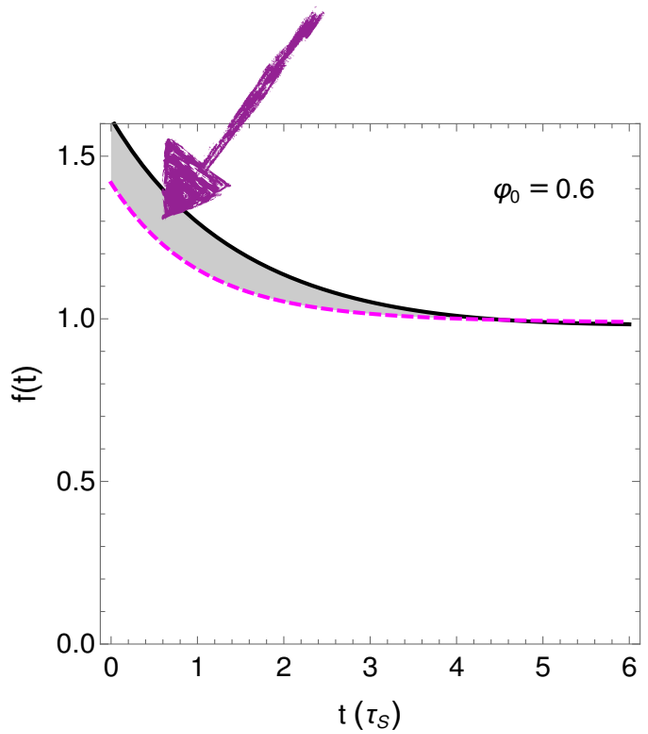
Pilot study

Courtesy of Rado Marchevski

Signal yield for 10^{19} POT/year

- Yield for interference events can`t reliably be computed
  - Depends heavily on the beam setup (incident angle + collimation) and the strong phase
  - Expected number of interference decays in $0 - 6\tau_S$ ~ 500 – 2000 events/year (no selection)
  - Signal efficiency ~ 15% ⟶ 75 – 300 events/year (after full selection)
  - Optimization of the beam line essential to determine if the sensitivity is sufficient

- The fiducial volume of the experiment is larger than the first $6\tau_S$ (FV ~ 60m)
- Large number of K_S, K_L and Λ decays in the 60m FV (setup from previous slide):
  - K_L ~ 4×10^{13} decays/year
  - K_S ~ 3×10^{13} decays/year
  - Λ ~ 1×10^{13} decays/year



Experimental feasibility

Pilot study

Courtesy of Rado Marchevski

- Main experimental challenges
  - ✭ Unprecedented amount of statistics is needed for a few % measurement: O(10^{15}) neutral kaon decays
  - ✭ Incredibly high event rates in the detectors: next generation detector technology is needed
  - ✭ Very intensive mixed neutral beam line (short + long beam): radiation protection issues
- The experiment will be a great challenge but it might be possible in the near future

Experimental feasibility

Pilot study

Other decay modes: Conservative estimate with **10% of the full intensity**

Courtesy of Rado Marchevski

Decays in the general FV 8mrad (6.5 – 65.5m)

Process	P_{decay} (in FV)	Acc	BR	Events/year	Largest sample
$K_S \to \pi^0 e^+ e^-$	0.09	0.07*	3×10^{-9}	4177	7***
$K_S \to \pi^0 \mu^+ \mu^-$	0.09	0.08*	3×10^{-9}	4774	6
$K_S \to \pi^0 \gamma\gamma_{(z>0.2)}$	0.09	0.076	1.3×10^{-6}	2.0×10^{6}	2558
$K_S \to \gamma\gamma$	0.09	0.05	2.1×10^{-6}	2.1×10^{6}	7461
$K_S \to e^+ e^-$	0.09	0.04**	2.1×10^{-14}	0.02	UL(9×10^{-9})(KLOE)
$K_L \to \pi^0 e^+ e^-$	0.05	0.04****	3×10^{-11}	13	UL(2.8×10^{-10}) (KTeV)
$K_L \to \pi^0 \mu^+ \mu^-$	0.05	0.04****	1.5×10^{-11}	7	UL(3.8×10^{-10}) (KTeV)
$K_L \to \mu^+ \mu^-$	0.05	0.04**	6.8×10^{-9}	3023	6210 (E871)

* – acceptances assumed the same as in NA48/1
** – same safety factor 3 (previous slide)
*** – $m_{ee}>0.165$ GeV/c^2 phase space used
**** – Assume factor 2 lower than K_S due to additionally suppressing the Greenley BG

Summary and Conclusions

- The time-dependent rate in $K \to \mu^+\mu^-$ is an additional golden mode, allowing for a clean independent determination of CKM parameters from kaon physics.

- The theoretical uncertainty is of at most $\mathcal{O}(1\%)$

- The relevant CKM parameter is $\bar{\eta}_{ds} \approx A^2\lambda^4\bar{\eta}$, the same as for $K_L \to \pi^0\bar{\nu}\nu$.

- The ratio $\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}$ / $\mathcal{B}(K_L \to \pi^0\bar{\nu}\nu)$ is an extremely clean SM observable, dependent only on the parameters λ, m_t .

$$R_{\rm SL} = \frac{\mathcal{B}(K_S \to \mu^+\mu^-)_{\rm SD}}{\mathcal{B}(K_L \to \pi^0\nu\bar{\nu})} = 1.55 \times 10^{-2} \left[\frac{\lambda}{0.225}\right]^2 \left[\frac{Y(x_t)}{X(x_t)}\right]^2$$

[A. J. Buras and E. Venturini, [arXiv:2109.11032]]

Summary and Conclusions

✦ The same $K \to \mu^+\mu^-$ observable is also a sensitive probe of NP

[**AD**, M. Ghosh [arXiv:2112.05801]]

✦ The experimental requirements are very challenging and require more investigation, but no show stopper for a future machine.

✦ Such a machine would additionally be sensitive to many kaon and hyperon decay modes – Exciting opportunity to understand what interesting physics can be done with these unprecedented yields

- ✓ $\mathcal{O}(10^{13})$ K_L decays
- ✓ $\mathcal{O}(10^{13})$ K_S decays
- ✓ $\mathcal{O}(10^{13})$ Λ decays

V_{us} ?

Hidden sectors ?

Hyperon decays?

Thank you

Backup

Experimental feasibility

Pilot study

Background contamination

Courtesy of Rado Marchevski

- $K_S \rightarrow \pi^+\pi^-$ decays
  - BR ~ 0.7
  - PID suppression ~ 10^{-4}/track (calo + muon detector)
  - Kinematic suppression (wrong mass assignment + nongaussian tails) ~ 10^{-5}
  - Expected S/B ~ 10
- $K_S \rightarrow \pi^+\pi^- \rightarrow \mu^+\mu^-$ decays
  - BR_{eff} ~ $1x10^{-4}$ (2x pion decays)
  - Kinematic suppression ~ 10^{-5}
  - CDA + extrapolating the K_S to the primary target ~ $3x10^{-3}$
  - Expected S/B ~ 2
- $K_L \rightarrow \mu^+\mu^-\gamma$ (BR ~ $3.6x10^{-7}$): TODO
  - Kinematics (missing momentum)
  - Photon rejection
- Accidental muon pairs: TODO

Eliminating $|V_{cb}|$- related uncertainty

A. J. Buras and E. Venturini
〔arXiv:2109.11032〕

Use four basic CKM parameters:

$$\lambda = |V_{us}|, \qquad |V_{cb}|, \qquad \beta, \qquad \gamma$$

Tension between inclusive and exclusive determinations of $|V_{cb}|$:

$$|V_{cb}|_{B\to X_c} = (42.16 \pm 0.50)\cdot 10^{-3}$$
$$|V_{cb}|_{B\to D^{(*)}\ell\nu} = (39.36 \pm 0.68)\cdot 10^{-3}$$

$$R_{\rm SL} = \frac{\mathcal{B}(K_S \to \mu^+\mu^-)_{\rm SD}}{\mathcal{B}(K_L \to \pi^0\nu\bar{\nu})} = 1.55\times 10^{-2}\left[\frac{\lambda}{0.225}\right]^2\left[\frac{Y(x_t)}{X(x_t)}\right]^2$$

Independent of any SM parameters other than λ, m_t .

$K \to \mu^+\mu^-$ beyond the SM

AD, M. Ghosh, JHEP 03 (2022) 048, [arXiv:2112.05801]

$$R(K_S \to \mu^+\mu^-)_{\ell=0} \equiv \frac{\mathcal{B}(K_S \to \mu^+\mu^-)_{\ell=0}}{\mathcal{B}(K_S \to \mu^+\mu^-)^{\rm SM}_{\ell=0}} \lesssim \underline{1280} \quad \text{[LHCb]}$$

NP operators

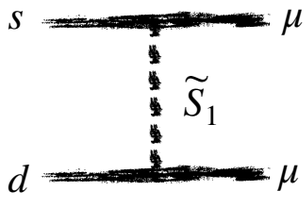
$$O_{VLL} = (\bar{Q}_L\gamma^\mu Q_L)(\bar{L}_L\gamma_\mu L_L)\,; \qquad O_{VLR} = (\bar{Q}_L\gamma^\mu Q_L)(\bar{e}_R\gamma_\mu e_R)\,,$$
$$O_{VRL} = (\bar{d}_R\gamma^\mu d_R)(\bar{L}_L\gamma_\mu L_L)\,; \qquad O_{VRR} = (\bar{d}_R\gamma^\mu d_R)(\bar{e}_R\gamma_\mu e_R)\,,$$

$K_L \to \pi^0\bar{\nu}\nu$

$$O_{SLR} = (\bar{Q}_L d_R)(\bar{e}_R L_L)\,,$$
$$O_{SRL} = (\bar{d}_R Q_L)(\bar{L}_L e_R)\,.$$

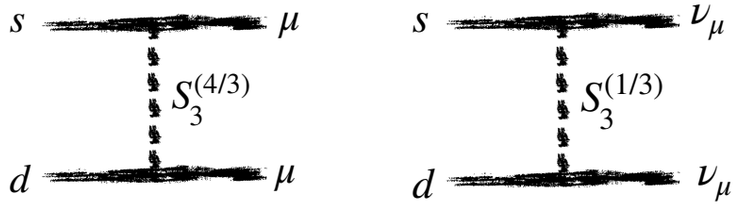
Example models:

A. Scalar Leptoquark $\widetilde{S}_1 \sim (\bar{3},1)_{4/3}$



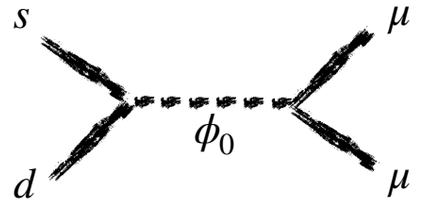
Can saturate the current bound while satisfying all existing constraints

B. Scalar Leptoquark $S_3 \sim (\bar{3},3)_{1/3}$



$R(K_S \to \mu^+\mu^-)_{\ell=0} \lesssim 26$, bounded by the GN bound on $R(K_L \to \pi^0\bar{\nu}\nu)$

C. 2HDM $\Phi \sim (1,2)_{1/2} = \begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}$



Can saturate the current bound while satisfying all existing constraints

$K_L \to \pi^0 \nu\bar{\nu}$

$[K_L]_{\text{CP-odd}} \to [\pi^0 \nu\bar{\nu}]_{\text{CP-even}}$

CP violating transition

- Within the SM determined entirely from the weak effective Hamiltonian (short-distance physics)

- The leptonic current, $(\bar{\nu}_L \gamma^\mu \nu_L)$, creates a CP-odd $(\bar{\nu}\nu)$ state

$K_{L,S} \to \mu^+\mu^-$

$[K_S]_{\text{CP-even}} \to [\mu^+\mu^-]_{\text{CP-odd}}$

$[K_S]_{\text{CP-even}} \to [\mu^+\mu^-]_{\text{CP-even}}$

$[K_L]_{\text{CP-odd}} \to [\mu^+\mu^-]_{\text{CP-odd}}$

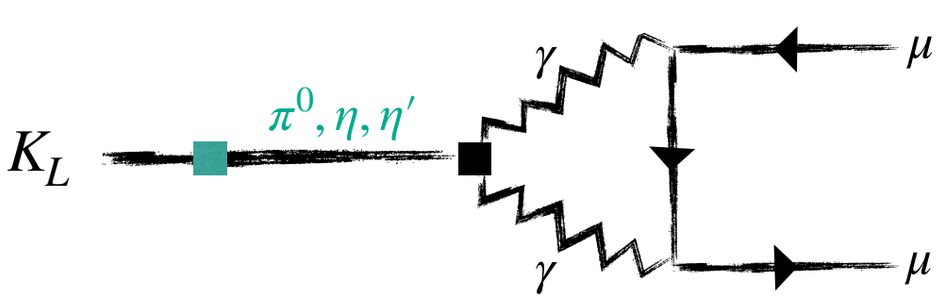
$[K_L]_{\text{CP-odd}} \to [\mu^+\mu^-]_{\text{CP-even}}$

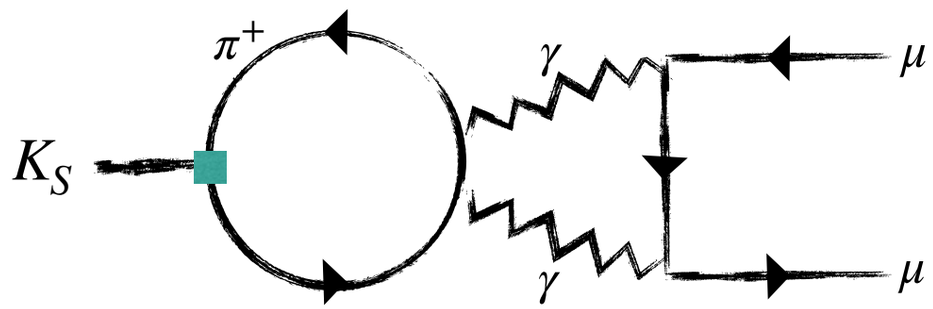
- Long-distance effects mediate **CP conserving transitions** which dominate both for $K_L \to \mu^+\mu^-$ and for $K_S \to \mu^+\mu^-$

- The short-distance contribution, coming from the weak Hamiltonian, induces **CP violating transitions**

$K \to \mu^+\mu^-$

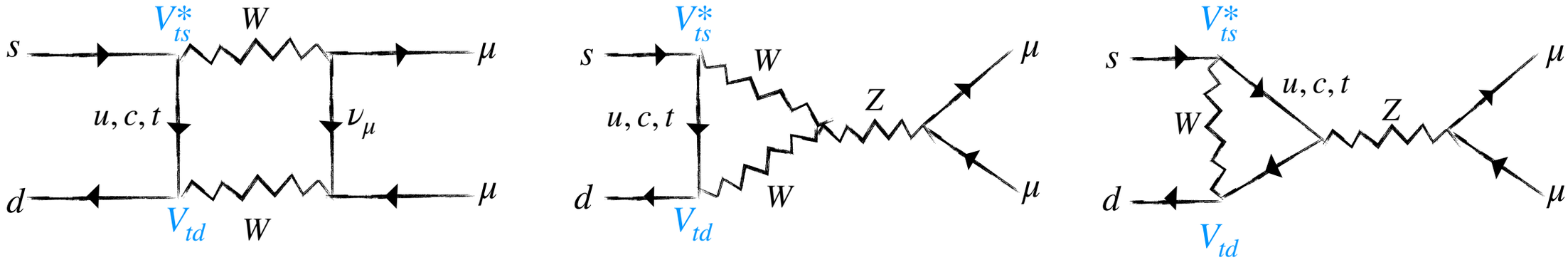
Long-distance (LD) contributions





$K \to \mu^+\mu^-$

Short-distance (SD) diagrams



K_S-K_L Interference

G. D'Ambrosio and T. Kitahara [arXiv:1707.06999]
AD, M. Ghosh, Y. Grossman, S. Schacht [arXiv:2104.06427]

Time dependent rate:

$$\left(\frac{d\Gamma}{dt}\right) = N_f\, f(t)\,, \qquad f(t) = C_L\, e^{-\Gamma_L t} + C_S\, e^{-\Gamma_S t} + 2\left[C_{\cos}\,\cos(\Delta m t) + C_{\sin}\,\sin(\Delta m t)\right] e^{-\Gamma t}\,,$$

$$C_L = |A(K_L)_0|^2$$
$$C_S = |A(K_S)_0|^2 + \beta_\mu^2\,|A(K_S)_1|^2$$
$$C_{\cos} = D\,|A(K_S)_0 A(K_L)_0|\cos(\varphi_0)$$
$$C_{\sin} = D\,|A(K_S)_0 A(K_L)_0|\sin(\varphi_0)$$

$$D = \frac{N_{K^0} - N_{\overline{K}^0}}{N_{K^0} + N_{\overline{K}^0}}$$

$$\mathscr{B}(K_S \to \mu^+\mu^-)_{\ell=0} = \mathscr{B}(K_L \to \mu^+\mu^-) \times \frac{1}{D^2} \times \frac{\tau_S}{\tau_L} \times \left(\frac{C_{int}}{C_L}\right)^2,$$

$$C_{int}^2 = C_{\cos}^2 + C_{\sin}^2.$$