Computing $K \rightarrow \pi\pi$ and $m_{K_L} - m_{K_S}$ from lattice QCD

Project X Physics Study

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RBRC and UKQCD Collaboration

Outline

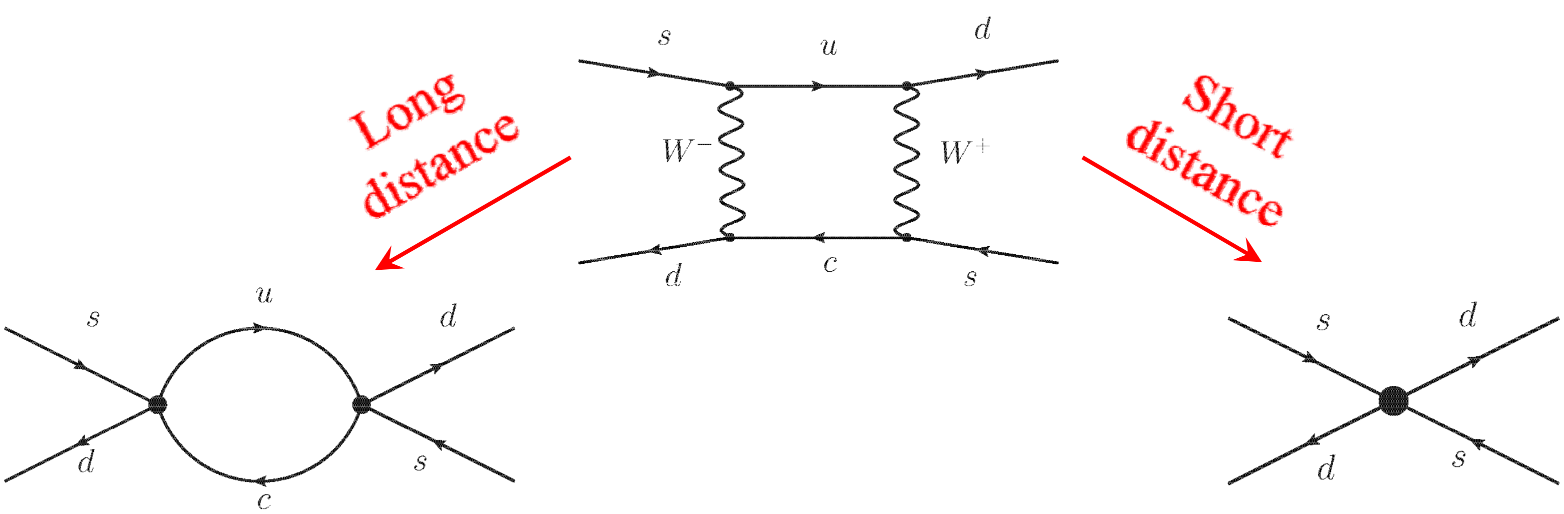
- Overview
- $K \rightarrow \pi\,\pi$
  - $\Delta I = 3/2$ decay
  - $\Delta I = 1/2$ decay
- Second order weak processes
  - $m_{K_L} - m_{K_S}$
  - Next: $K \rightarrow \pi\, l\, \bar{l}$

Overview

- Lattice QCD now a reliable tool – like QED PT
  - Hadron masses
  - Two and three-point functions: $< 0| J |A >$ & $<A|J|B>$
- K and π calculations are easiest:
  - Pion propagator $\propto |S_{quark}|^2 \rightarrow$ constant signal/noise
  - First place to study difficult problems
- $K \rightarrow \pi\pi$ decay computed with controlled errors
  - $\Delta I = 3/2$ solved
  - $\Delta I = 1/2$ the next challenge
- Experiment: Measure ε' more accurately

Overview

- Second order weak processes
  - Long distance:
    - Represent as bi-local operator
    - Δm_K and 5% of ε are good examples
  - Short distance:
    - Represent as $c_W\,(\bar{s}\gamma^{\mu}d)_L\,(\bar{s}\gamma^{\mu}d)_L$
    - $\varepsilon - B_K$ provide a good example



Overview

- Calculate long distance part on the lattice?
  - On-shell intermediate states?
  - Energy integrals with principal parts?
  - Can be controlled using the same finite volume methods as $K \to \pi\,\pi$!
  - Important possible targets:
    - $m_{K_L} - m_{K_S}$
    - Long distance, dispersive contribution to ε
    - $K \to \pi\, l\, \bar{l}$
  - Experiment: New SM predictions for more rare *K* decays

Overview

- Calculate long distance part on the lattice?
  - On-shell intermediate states?
  - Energy integrals with principal parts?

$$M_{0\bar{0}} = \sum_\alpha \mathcal{P} \int_{m_\pi}^{\infty} dE \frac{\langle K^0 | H_W | \alpha(E) \rangle \langle \alpha(E) | H_W | \overline{K}^0 \rangle}{m_K - E}$$

    - $m_{K_L} - m_{K_S}$
    - Long distance, dispersive contribution to ε
    - $K \rightarrow \pi l \bar{l}$
  - Experiment: Reliable SM predictions for more rare *K* decays

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$K \rightarrow \pi\,\pi$ decay

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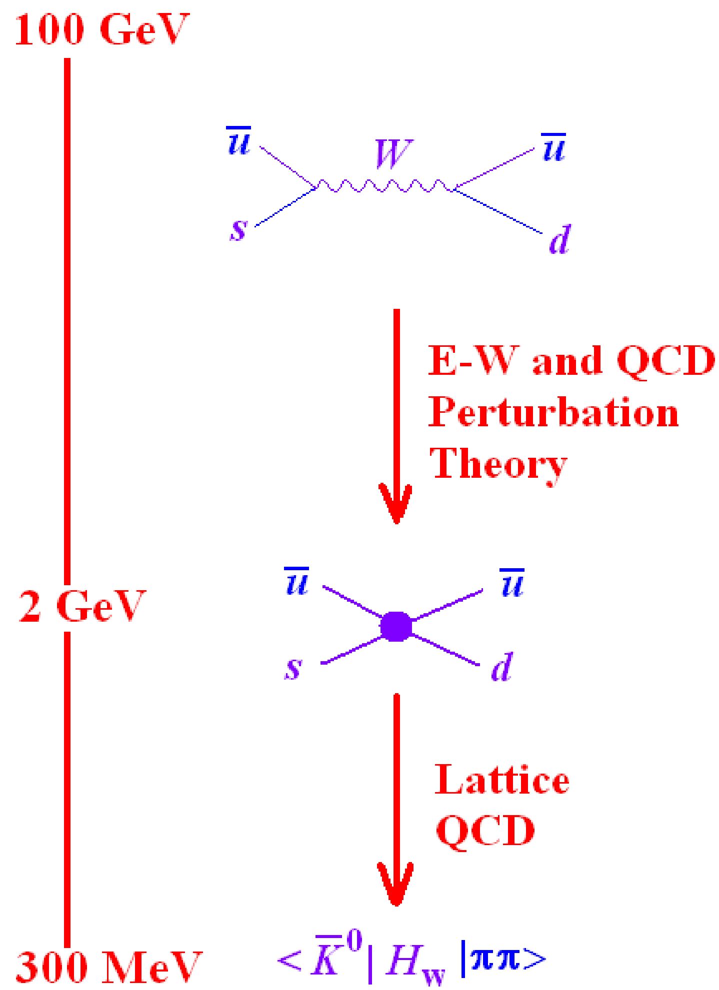
Lattice Methods

Low Energy Effective Theory

- Represent weak interactions by local four-quark Lagrangian

$$\mathcal{H}^{(\Delta S=1)} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \left\{ \sum_{i=1}^{10} \left[z_i(\mu) - \frac{V_{td}}{V_{ud}} \frac{V_{ts}^*}{V_{us}^*} y_i(\mu) \right] Q_i \right\}$$

- $V_{qq'}$ – CKM matrix elements
- z_i and y_i – Wilson Coefficients
- Q_i – four-quark operators



Four quark operators

- **Current-current operators**

$$Q_1 \equiv (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}$$
$$Q_2 \equiv (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A}$$

- **QCD Penguins**

$$Q_3 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_4 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V-A}$$
$$Q_5 \equiv (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_6 \equiv (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V+A}$$

- **Electro-Weak Penguins**

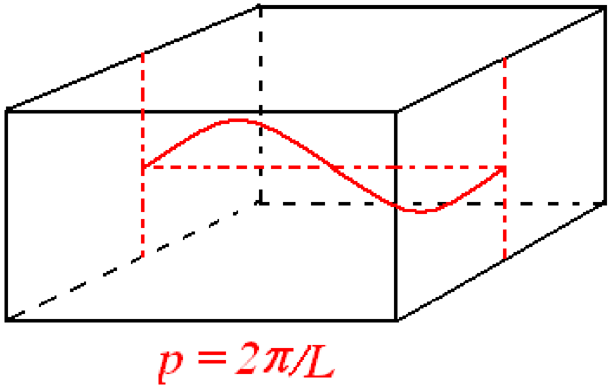
$$Q_7 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V+A}$$
$$Q_8 \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V+A}$$
$$Q_9 \equiv \frac{3}{2}(\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\beta)_{V-A}$$
$$Q_{10} \equiv \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}_\beta q_\alpha)_{V-A}$$

Evaluate $<\pi\pi|H_W|K>$

- Use SU(3) ChPT: $<K|H_W|\pi>$ & $<K|H_W|0>$ → $<K|H_W|\pi\pi>$?
  - m_K too large
  - ~70% errors
- Maiani-Testa theorem (1990):
  - Euclidean space: e^{-Ht} projects onto lowest energy state
  - Gives π - π with zero relative momentum
  - Watson theorem: outgoing $\pi - \pi$ scattering phase requires Minkowski space?

Resolved by Lellouch-Luscher

- Use finite-volume quantization.
- Adjust volume so 1st or 2nd excited state has correct *p*.
- Requires extracting signal from non-leading large *t* behavior:

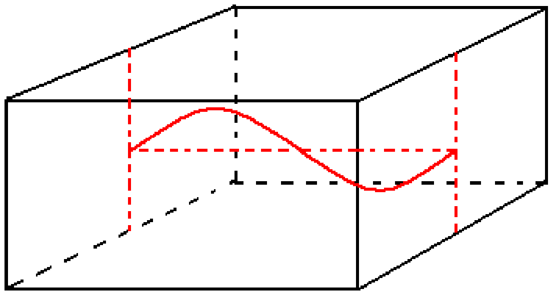


$$G(t) \sim c_0 e^{-E_0 t} + c_1 e^{-E_1 t}$$

- Introduce boundary conditions to remove leading term.

Resolved by Lellouch-Luscher

- Finite volume states correctly include π - π interactions.
- Point-like matrix element couples only to physical *s*-wave π - π state.



- Finite box reduces size of *s*-wave component because of known mixing with higher *l*.

$$|A_i| = \frac{1}{2\pi q_\pi}\sqrt{\frac{\partial\phi}{\partial q_\pi} + \frac{\partial\delta_I}{\partial q_\pi}}\, L^{3/2}\sqrt{m_K}\, E_{\pi\pi}\, |\langle\pi\pi|O_i|K\rangle|$$

Lellouch-Luscher factor

$$\tan(\phi(q)) = -\frac{\pi^{3/2} q}{Z_{00}(1,q^2)} \qquad Z_{00}(s,q^2) = \frac{1}{\sqrt{4\pi}}\sum_{n\in Z^3}\frac{1}{(n^2-q^2)^s} \qquad q_\pi = \frac{k_\pi L}{2\pi}$$

- Use Luscher quantization condition to determine $\delta(k)$

$$\phi(q_\pi) + \delta(k) = n\pi$$

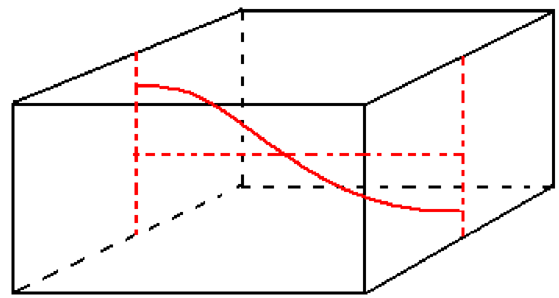
Lattice matrix elements

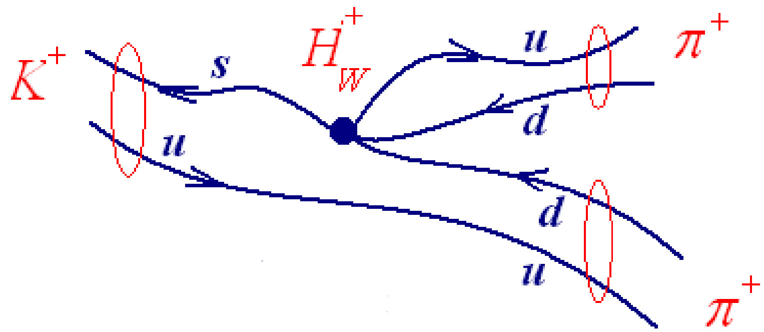
- Use chiral fermions (DWF): good short-distance chiral symmetry controls operator mixing (L_s=16 and 32)
- Use non-perturbative methods to convert lattice operators to regularization invariant (RI) scheme at a scale μ.
- Use a series of finer lattice ensembles to non-perturbatively run μ up to 3 GeV.
- Use continuum perturbation theory to convert RI to $\overline{\mathrm{MS}}$

$$\Delta \mathbf{I} = \mathbf{3/2}$$

$\Delta I = 3/2 \;\; K \rightarrow \pi\pi$

- Calculate Re(A_2) and Im(A_2).
- Three operators contribute $O^{(27,1)}$, $O^{(8,8)}$ and $O^{(8,8)m}$.
- Use isospin to relate to $K^+ \rightarrow \pi^+\pi^+$.
- Use anti-periodic boundary conditions for *d* quark.
  (Changhoan Kim, hep-lat/0210003).
- **Achieve essentially physical kinematics!**
  (63 → 147 configurations)
  - m_π = 142.9(1.1) MeV
  - m_K = 511.3(3.9) MeV
  - $E_{\pi\pi}$ = 492(5.5) MeV

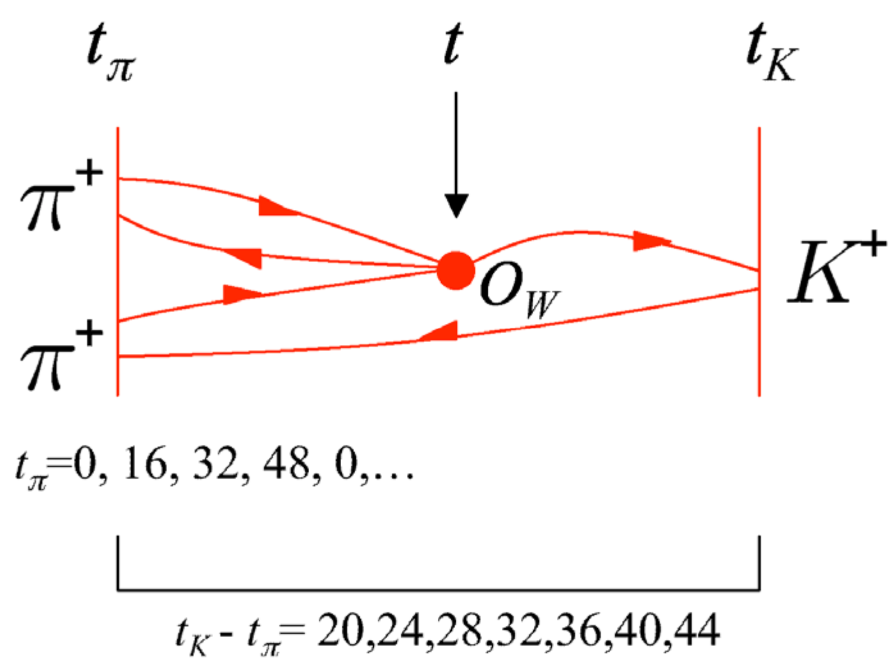




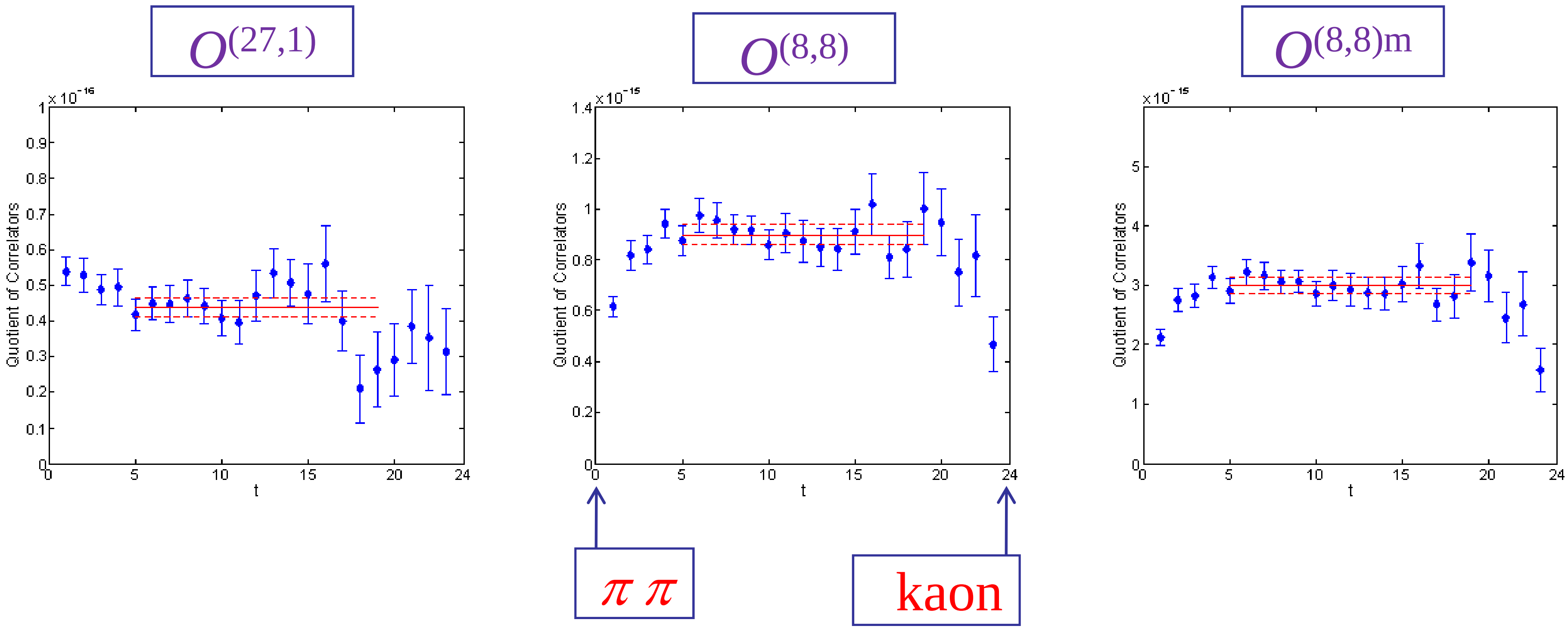
Computational Set-up

(M. Lightman and E. Goode)

- Use anti-periodic boundary conditions for d quark in two directions (average over three choices).
- Fix $\pi - \pi$ source at $t = 0$, vary location of O_W and kaon source.



$< \pi\, \pi \,|\, O \,|\, K >$ from 63 configurations

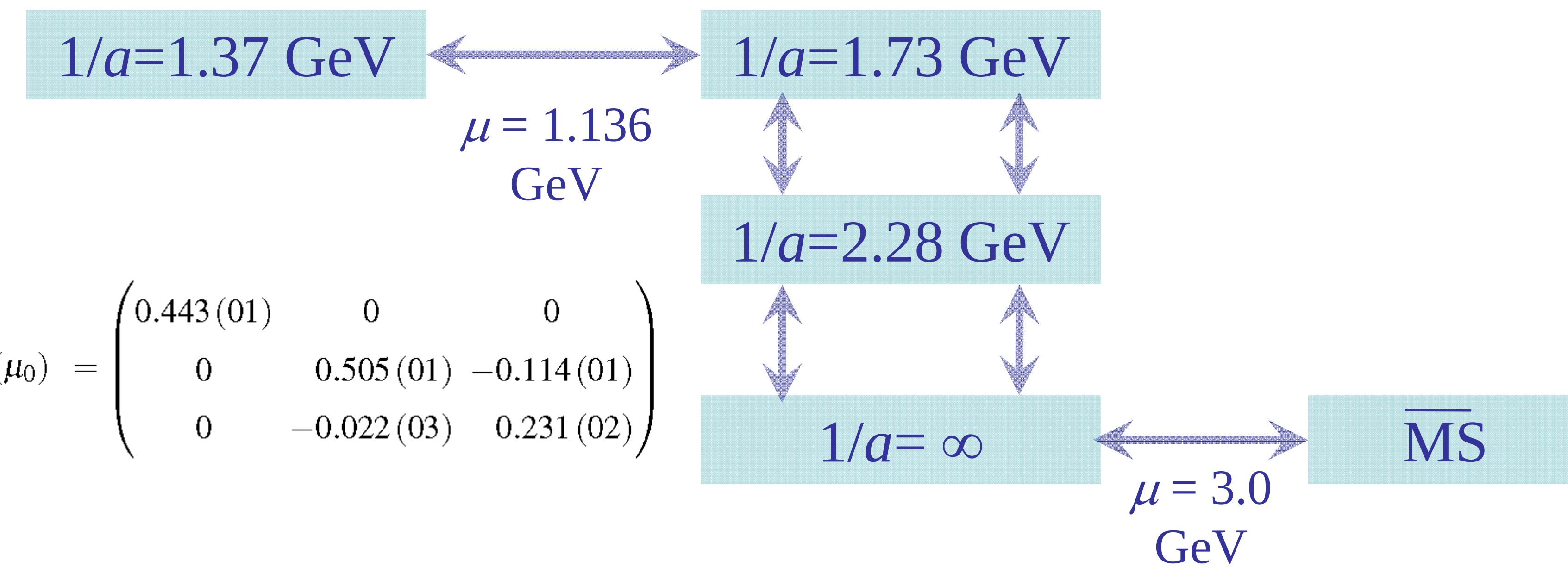


Plot ratio of correlators:

$$\frac{C^{i}_{K\pi\pi}(t)}{C_K(t_K - t)\,C_{\pi\pi}(t)} = \frac{\mathcal{M}_i}{Z_K Z_{\pi\pi}}$$

Relate lattice and continuum operators

- Calculation is performed on $1/a$=1.37 GeV lattice.
- Matching to perturbative $\overline{\text{MS}}$ scheme is unreliable at scale $\mu \sim 1/a$ **!**
- Carry out sequence of NP RI matching steps:



$$Z^{(\gamma^\mu,\gamma^\mu)}(\mu_0) = \begin{pmatrix} 0.443\,(01) & 0 & 0 \\ 0 & 0.505\,(01) & -0.114\,(01) \\ 0 & -0.022\,(03) & 0.231\,(02) \end{pmatrix}$$

Determine physical A_2

- Error estimates:

	ReA_2	ImA_2
lattice artefacts	15%	15%
finite-volume corrections	6.2%	6.8%
partial quenching	3.5%	1.7%
renormalization	1.8%	5.6%
unphysical kinematics	0.4%	0.8%
derivative of the phase shift	0.97%	0.97%
Wilson coefficients	6.6%	6.6%
Total	18%	19%

- $\mathrm{Re}(A_2) = (1.436 \pm 0.063_{\mathrm{stat}} \pm 0.258_{\mathrm{sys}})\ 10^{-8}$ GeV

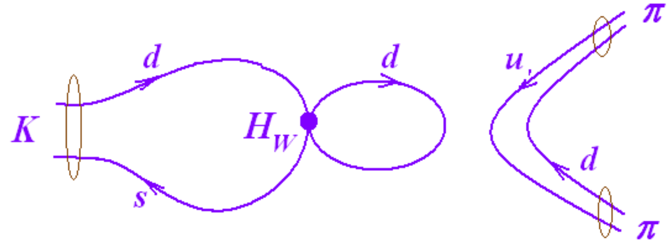
Experiment: $1.479(4)\ 10^{-8}$ GeV

- $\mathrm{Im}(A_2) = -(6.29 \pm 0.46_{\mathrm{stat}} \pm 1.20_{\mathrm{sys}})\ 10^{-13}$ GeV

$$\Delta \mathbf{I} = \mathbf{1/2}$$

$\Delta I = 1/2$ $K \rightarrow \pi\pi$
(Qi Liu)

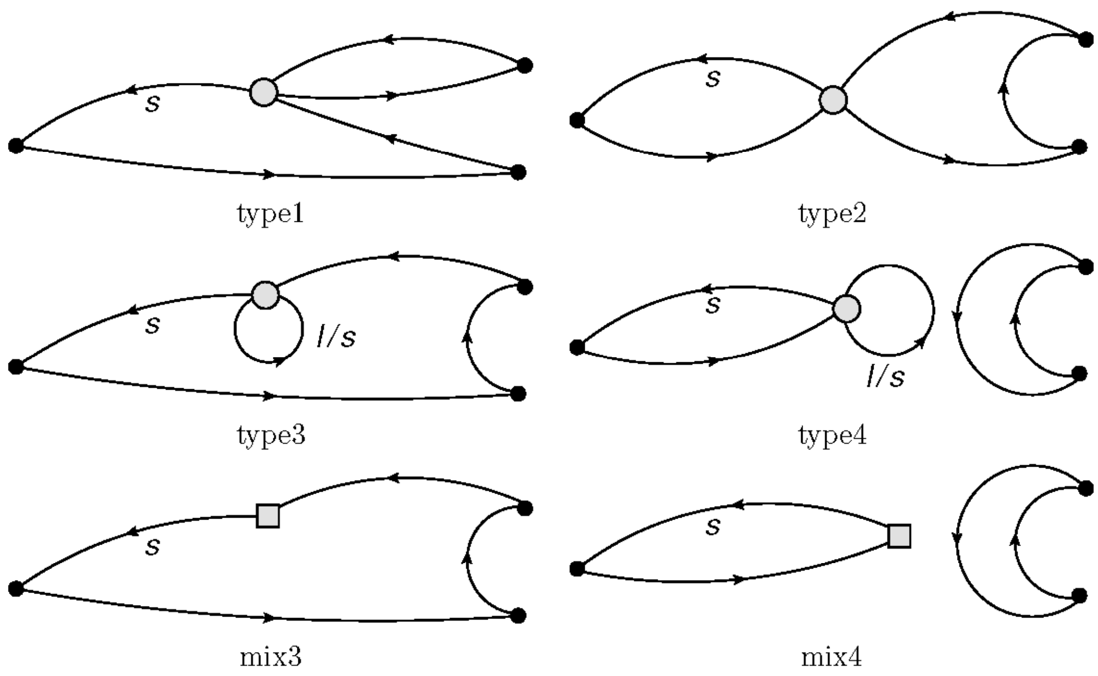
- Made much more difficult by disconnected diagrams:



- $16^3 \times 32$ ensemble (arXiv:1106.2714 [hep-lat])
  - $1/a$ = 1.73 GeV, m_π = 420 MeV, L = 1.8 fm
  - Use 8000 time units, measure every 10 (800 configs.)
- $24^3 \times 64$ ensemble (22 x harder)
  - $1/a$ = 1.73 GeV, m_π = 329 MeV, L = 2.8 fm
  - Use 5520 time units, measure every 40 (138 configs.)
- Adjust valence strange mass for on-shell, threshold kinematics ($\pi\pi$ state is unitary)

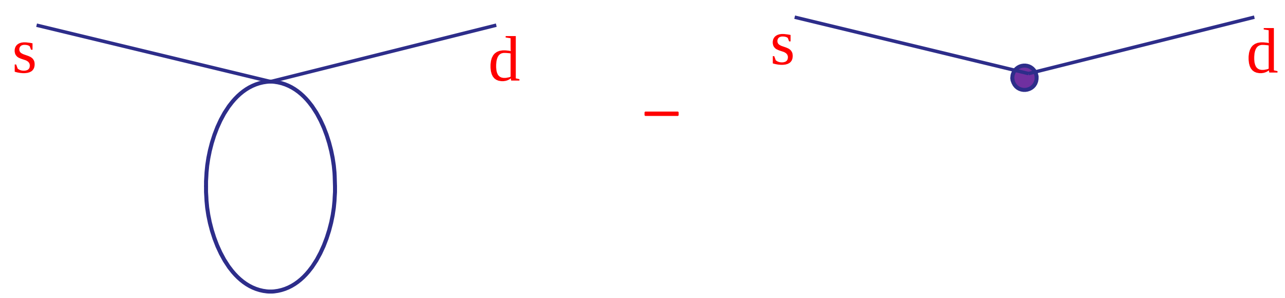
$\Delta I = 1/2$ $K \rightarrow \pi\pi$

- Code 50 different contractions of four types:



Substantially improved methods

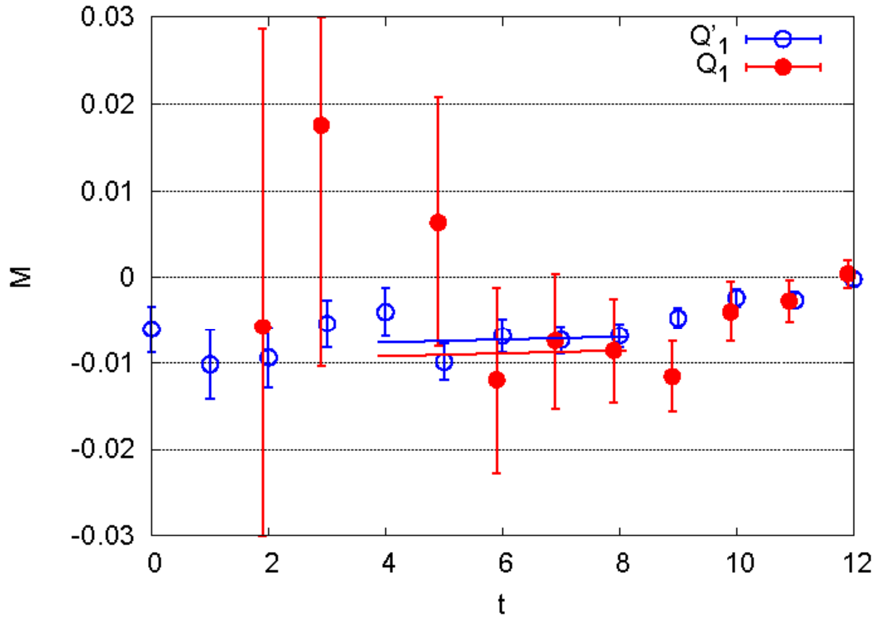
- Improve statistics using sources at each of 32 or 64 times
- Accelerate inversions with low-mode deflation or EigCG
- Reduce vacuum coupling by separating pion sources
- Subtract divergent $\bar{s}d$ and $\bar{s}\,\gamma^5 d$ terms
  - Does not affect on-shell amplitudes
  - Suppress $1/a^2$-enhanced excited state contributions.

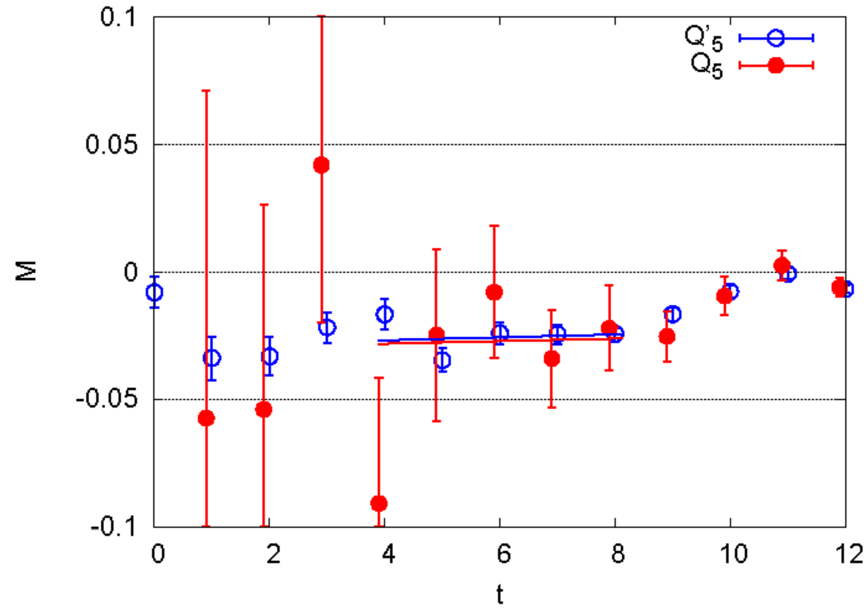


$\Delta I = 1/2$ $K \rightarrow \pi\pi$ 24^3 x 64

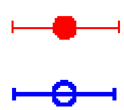
Q2 - largest part of Re(A_0)

Q6 - largest part of Im(A_0)





$\Delta = 12$ $K - \pi\pi$ separation


Full amplitude

(′) Drop disconnected

m_π(MeV)	m_K(MeV)	Re(A_0)	Re(A'_0)	Im(A_0)	Im(A'_0)	Re(A_2)	Im(A_2)
329.3	662.1	31.1(4.5)	27.8(0.8)	-33(15)	-36.3(16)	2.668(14)	-0.6509(34)

Summary

- Physical calculation of complex $\Delta I=2$ amplitude A_2.
  - Performed with a single, large lattice spacing
  - 20% systematic errors
- Exploratory calculation of $\Delta I=0$ amplitude A_0
  - Threshold decay
  - $m_\pi \geq 329$ MeV
  - Disconnected diagrams appear to contribute only noise
  - $\mathrm{Re}(A_0)$ ~ 20% error
  - $\mathrm{Im}(A_0)$ ~ 50% error
  - $\mathrm{Re}(\varepsilon'/\varepsilon) = (2.0 \pm 1.7) \times 10^{-3}$

$\Delta I = 1/2$ $K \rightarrow \pi\pi$: Future

- Goal is a 20% calculation of ε'/ε with all errors controlled
- Repeat $\Delta I = 3/2$ kinematics
  - Use $32^3 \times 64$ volume with $1/a = 1.37$ GeV
  - Achieve $p = 205$ MeV from **G-parity** in 2 directions
- Exploring "all-2-all" propagators (KEK)
  - Provide deflation (explicit use of low modes to reduce condition number)
  - Extract many measurements from one configuration
  - Sum over localized sources – highly suppress vacuum coupling
- Switch to BG/Q for 20 x speedup
- Result hoped for in 2 years

$K_L - K_S$ mass difference

$K^0 - \overline{K^0}$ Mixing

- Time evolution of $K^0 - \overline{K^0}$ system given by familiar Wigner-Weisskopf formula:

$$i\frac{d}{dt}\begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix} = \left\{ \begin{pmatrix} M_{00} & M_{0\bar{0}} \\ M_{\bar{0}0} & M_{\bar{0}\bar{0}} \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \Gamma_{00} & \Gamma_{0\bar{0}} \\ \Gamma_{\bar{0}0} & \Gamma_{\bar{0}\bar{0}} \end{pmatrix} \right\} \begin{pmatrix} K^0 \\ \overline{K}^0 \end{pmatrix}$$

where:

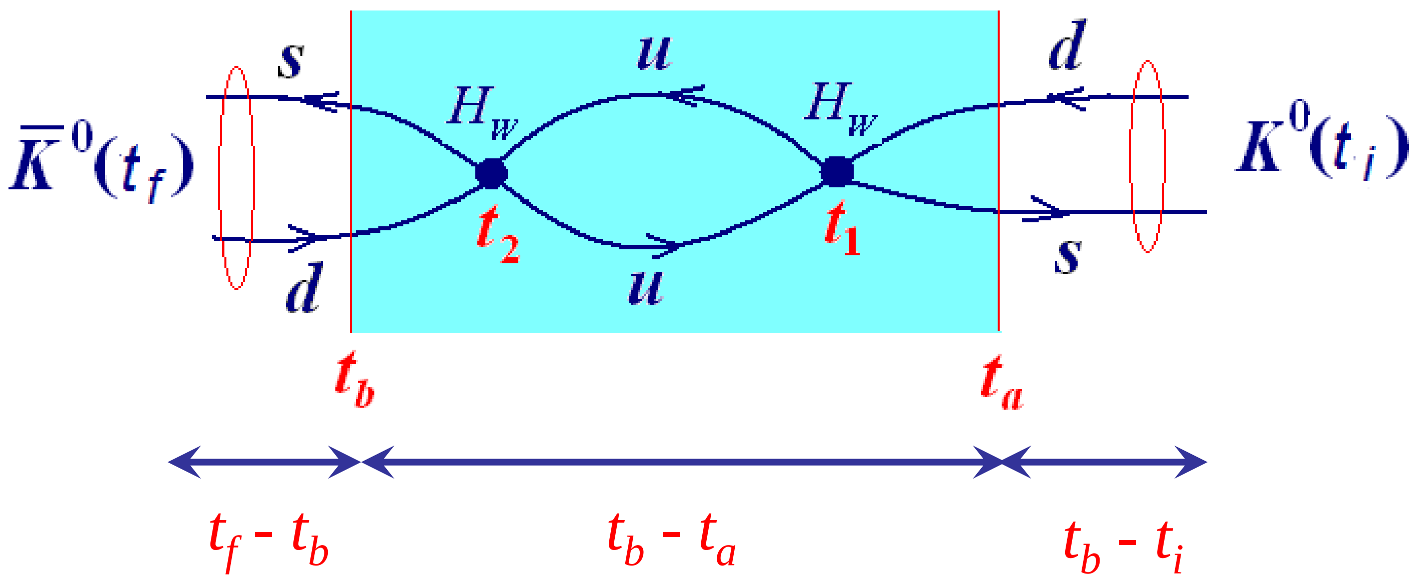
$$\Gamma_{ij} = 2\pi \sum_\alpha \int_{2m_\pi}^{\infty} dE \langle i | H_W | \alpha(E) \rangle \langle \alpha(E) | H_W | j \rangle \delta(E - m_K)$$

$$M_{ij} = \sum_\alpha \mathcal{P} \int_{m_\pi}^{\infty} dE \frac{\langle i | H_W | \alpha(E) \rangle \langle \alpha(E) | H_W | j \rangle}{m_K - E}$$

Lattice Version
(Jianglei Yu)

- Evaluate standard, Euclidean, 2nd order $K^0 - \overline{K^0}$ amplitude:

$$\mathcal{A} = \langle 0|T\left(K^0(t_f)\frac{1}{2}\int_{t_a}^{t_b} dt_2 \int_{t_a}^{t_b} dt_1 H_W(t_2) H_W(t_1) K^{0^\dagger}(t_i)\right)|0\rangle$$



Interpret Lattice Result

$$\mathcal{A} = N_K^2 e^{-M_K(t_f-t_i)}\Bigg\{\sum_{n\neq n_0}\frac{\langle\overline{K}^0|H_W|n\rangle\langle n|H_W|K^0\rangle}{M_K-E_n}\Bigg(\underbrace{-(t_b-t_a)}_{1.}\underbrace{-\frac{1}{M_K-E_n}}_{2.}+\underbrace{\frac{e^{(M_K-E_n)(t_b-t_a)}}{M_K-E_n}}_{3.}\Bigg)+\underbrace{\frac{1}{2}\langle\overline{K}^0|H_W|n_0\rangle\langle n_0|H_W|K^0\rangle(t_b-t_a)^2}_{4.}\Bigg\}$$

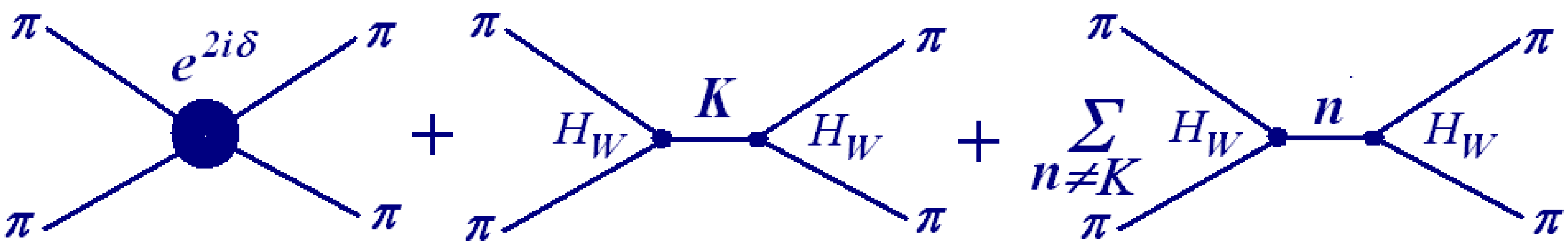
1. $\Delta m_K{}^{\mathrm{FV}}$
2. Uninteresting constant
3. Growing or decreasing exponential:
   $E_n > m_K$ must be removed!
4. Degenerate $E_{\pi\pi} = m_K$ state

Remove finite volume effects

- Use Luscher condition to relate finite and infinite volume energies to 2nd order in H_W :

$$\phi\left(\frac{kL}{2\pi}\right)+\delta_0(k)+\delta_W(k) = n\pi$$

  - Find $k = \sqrt{E_{\pm}^2/4 - m_\pi^2}$ from finite volume energy.
  - $\delta_W(k)$ is 2nd order $\pi - \pi$ weak scattering amplitude, including the K pole with its infinite volume energy shift.



Infinite-finite volume relations

- Expand to 1st order in H_W:

$$\Gamma = 2\frac{\partial}{\partial E}(\phi + \delta_0)\,|\langle \pi\pi(E)|H_W|K_S\rangle|^2$$

- Expand to 2nd order in H_W :

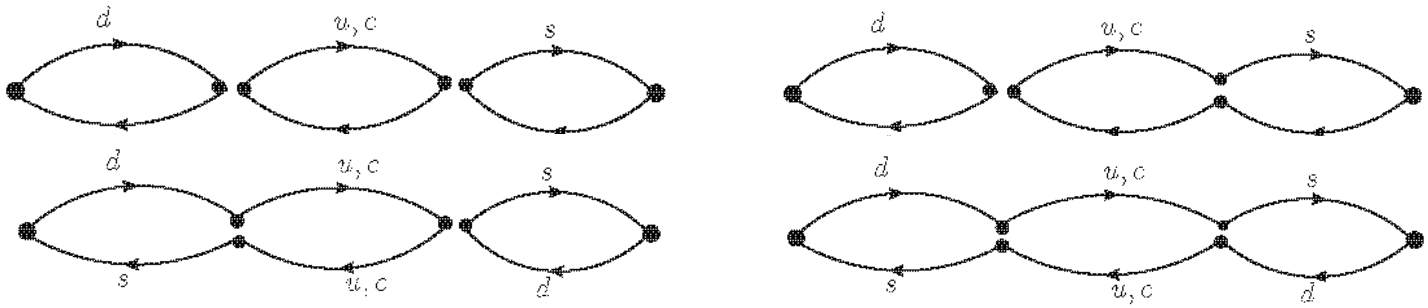
$$\Delta m_K \;=\; \Delta m_K^{\mathrm{FV}} + \frac{1}{\frac{\partial(\phi+\delta_0)}{\partial E}}\Big[\frac{1}{2}\frac{\partial^2(\phi+\delta_0)}{\partial E^2}\,|\langle n_0|H_W|K_S\rangle|^2 \\ -\frac{\partial}{\partial E'}\left\{\frac{\partial(\phi+\delta_0)}{\partial E}|\langle K_S|H_W|n_0\rangle|^2\right\}_{E=m_K}\Big]$$

- An identical formula gives LD part of ε_K.

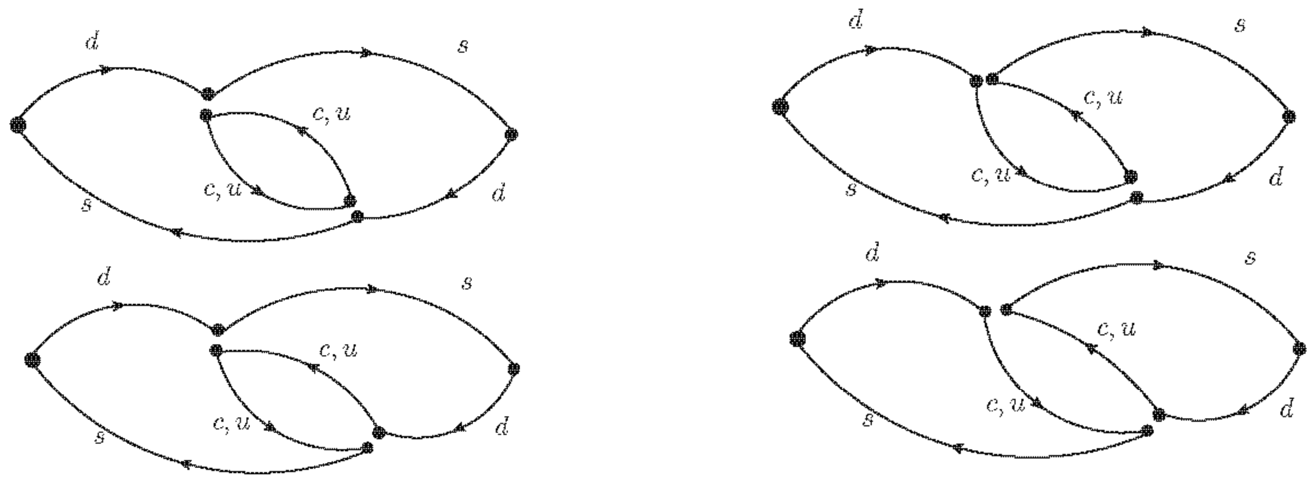
Lattice setup

- $N_f = 2+1$ and $2+1+1$, $16^3 \times 32$, $m_\pi = 420$ MeV
- Include type 1 and type 2 graphs:

Type 1



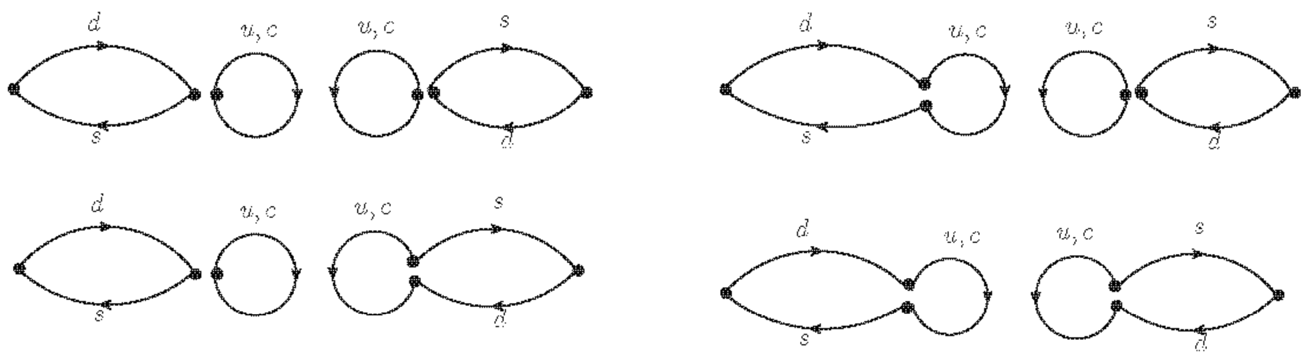
Type 2



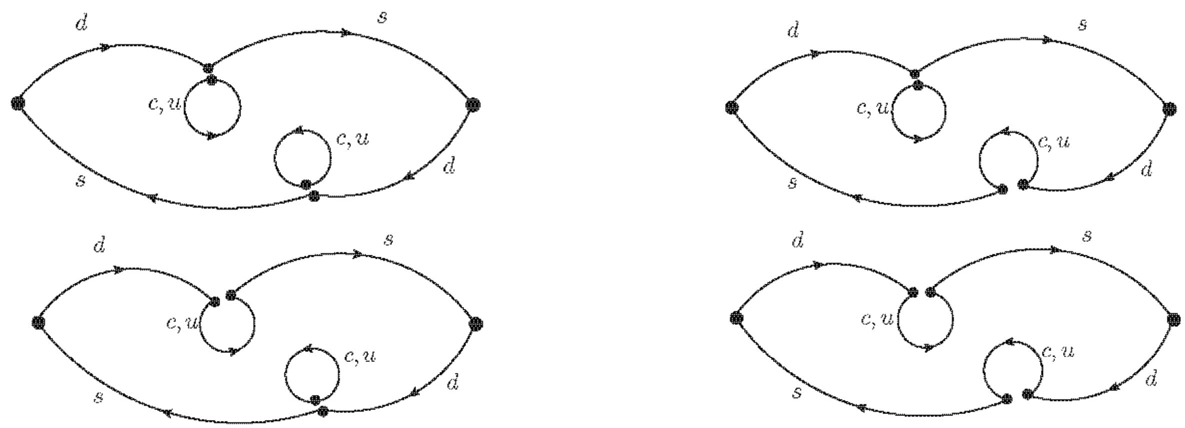
Lattice setup

- Exclude type 3 and type 4 graphs (code and new methods now being tested):

Type 3



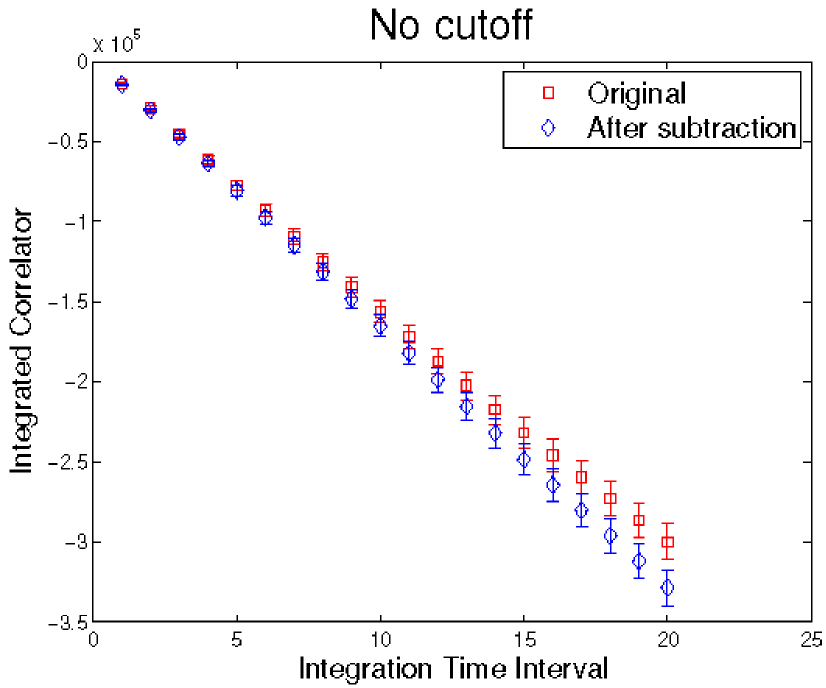
Type 4

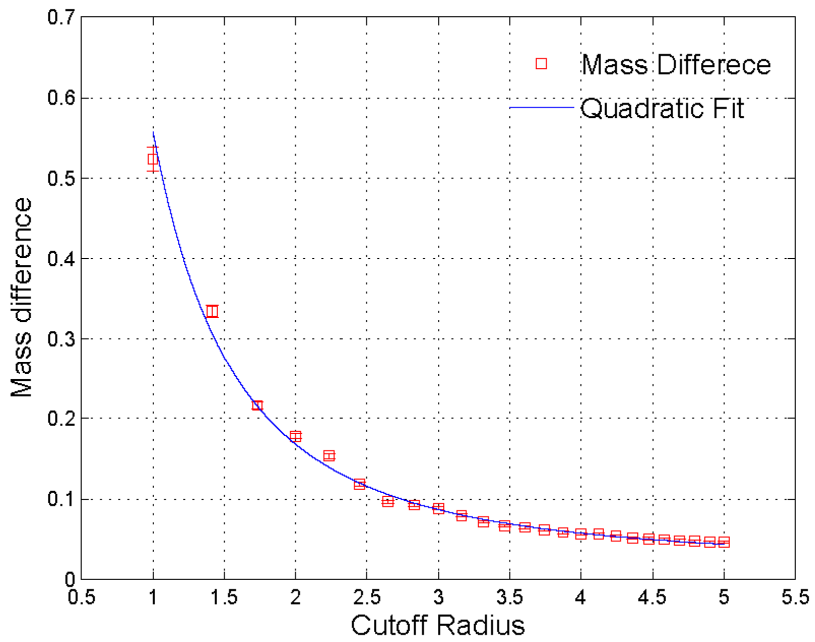


Lattice results

(Jianglei Yu)

- N_f=2+1, 16^3 x 32, m_π = 420 MeV



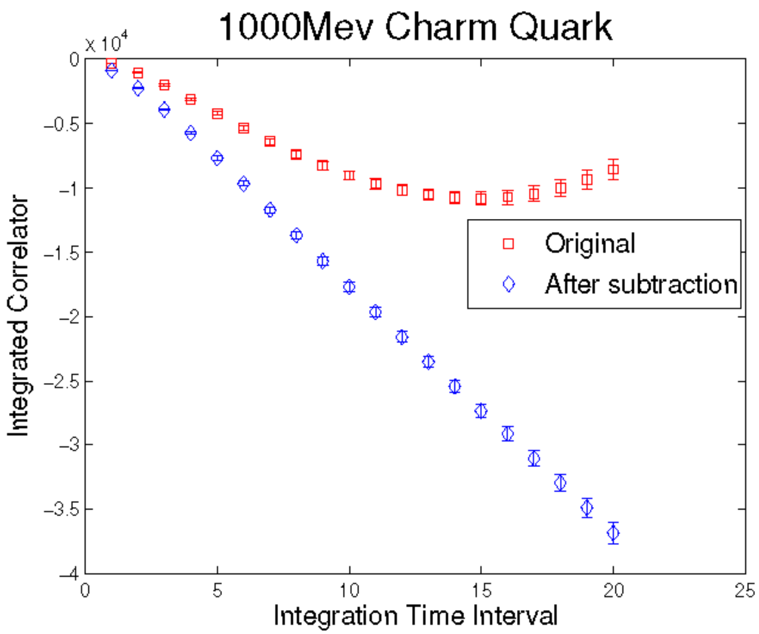


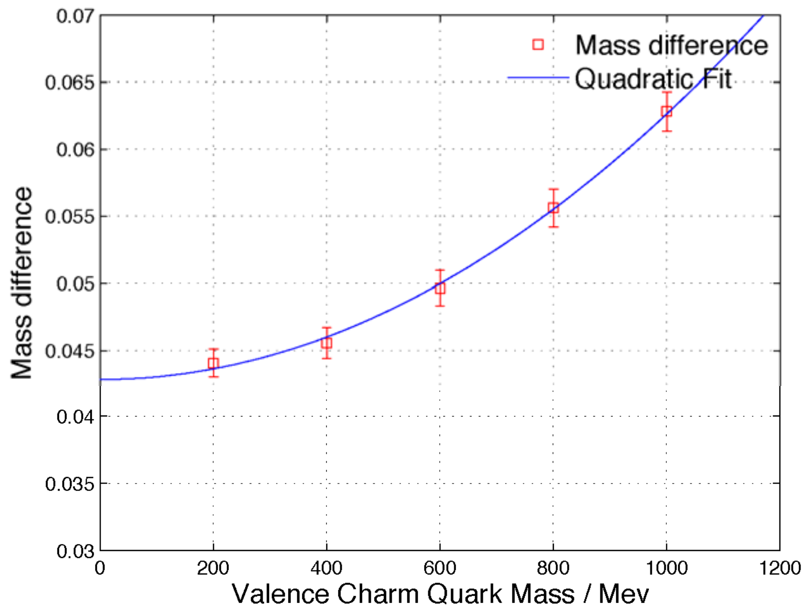
- Linear *t* behavior easy to find.
- Depends quadratically on cutoff radius

Lattice results

- Introduce GIM cancellation

$$\Delta M_K(m_c) = a\ m_c^2 + b$$



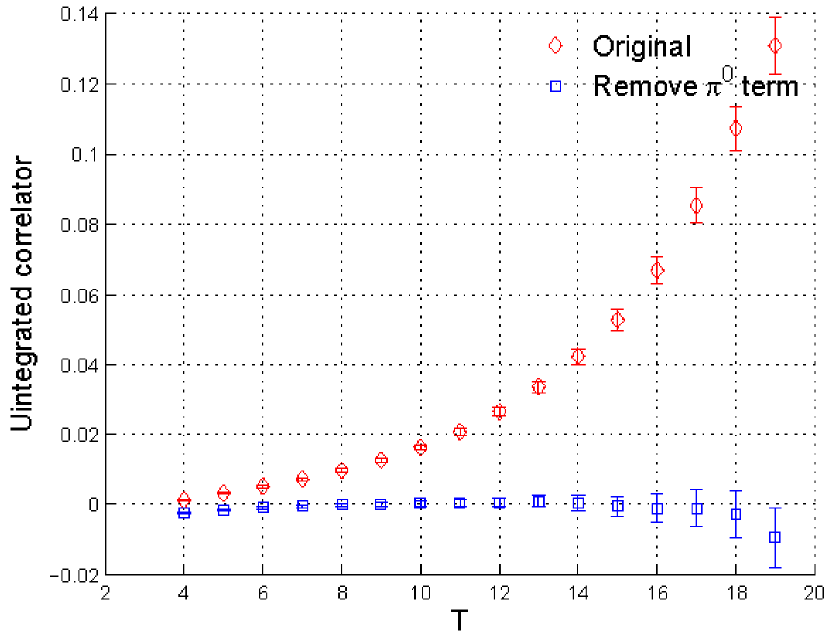


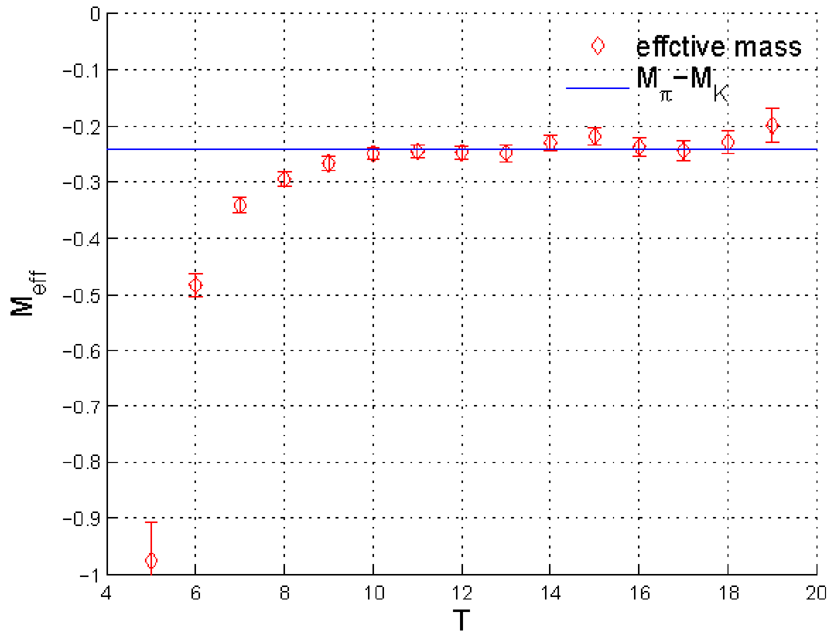
- Accurate m_c^2 behavior → m_c ~ 1 GeV OK for 1/a=1.73?
- Note $p \leq m_c$

Examine long distance part

- Vary separation $T = t_2 - t_1$ between $H_W(t_2)$ and $H_W(t_1)$

Parity odd channel



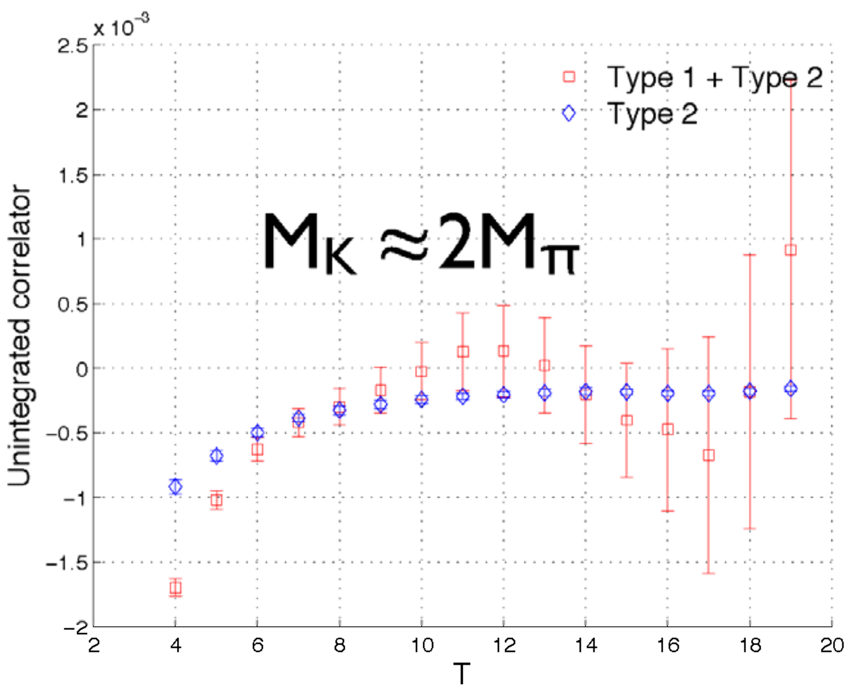


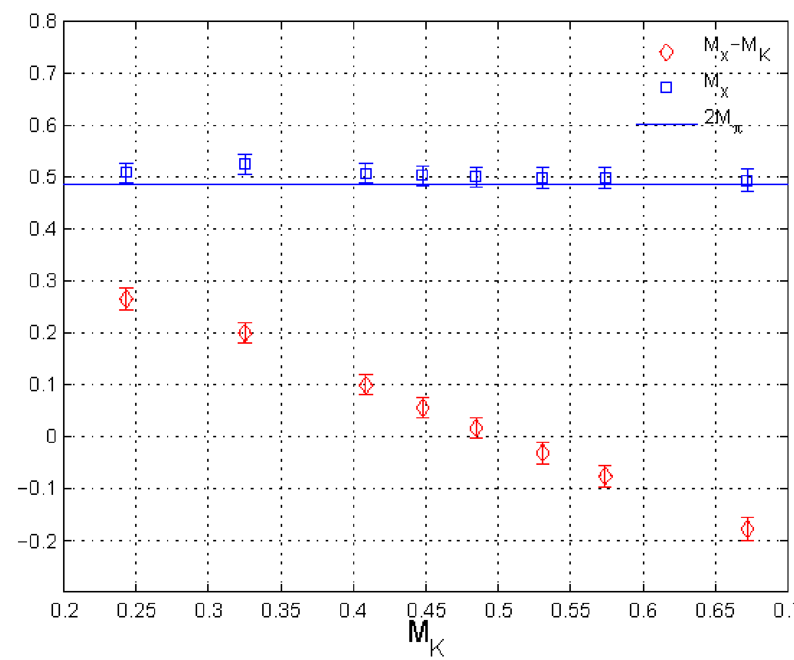
Q_1 x Q_1 product

Examine long distance part

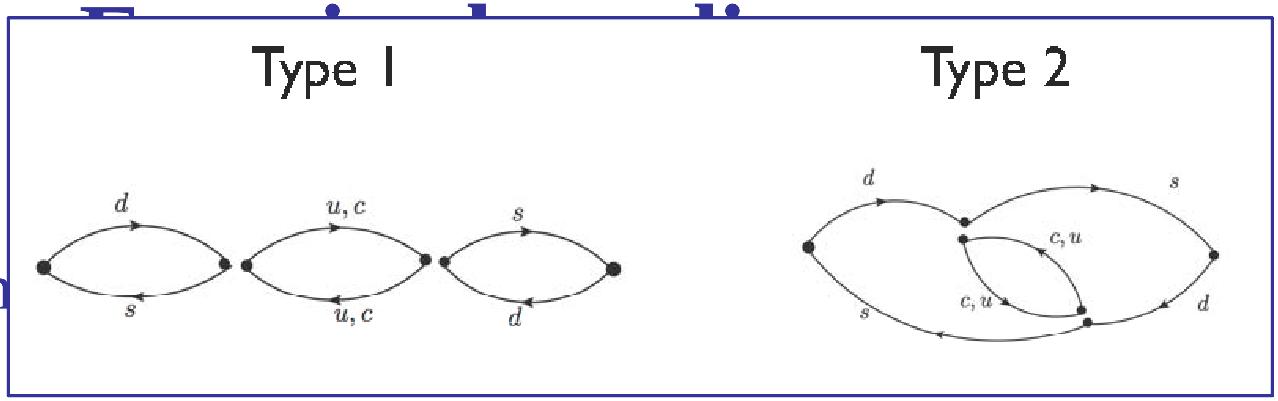
- Examine type 2 diagrams:

<u>Parity even channel</u>

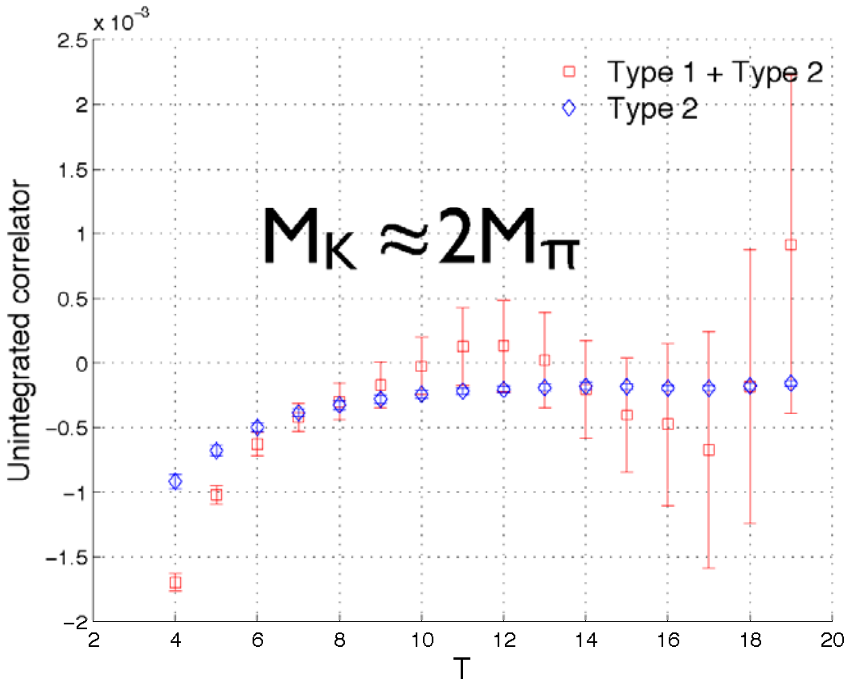


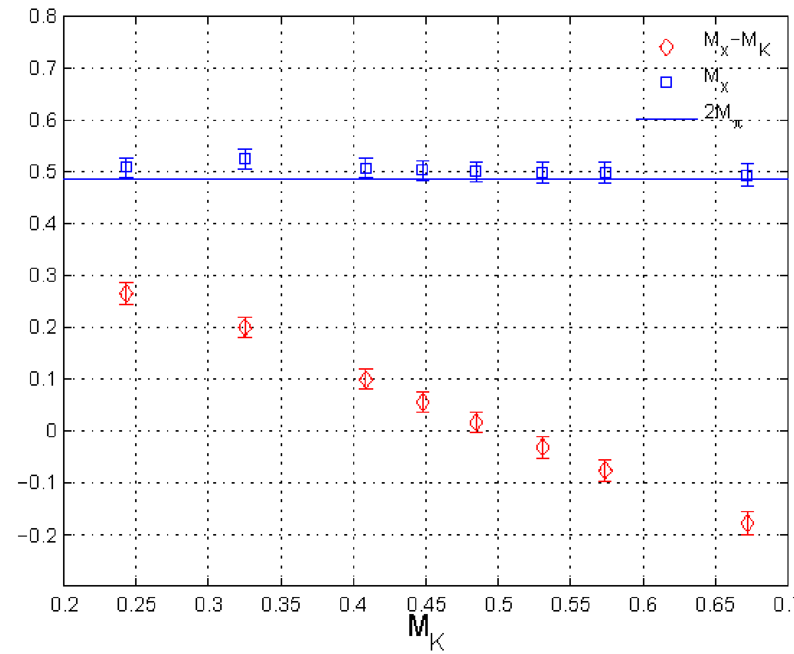


- Exan



Parity even channel

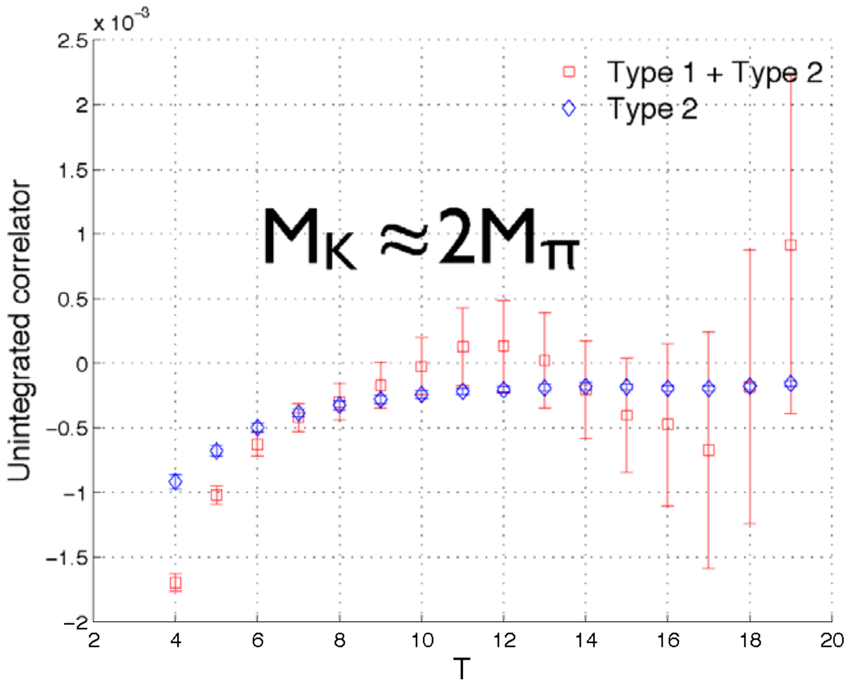


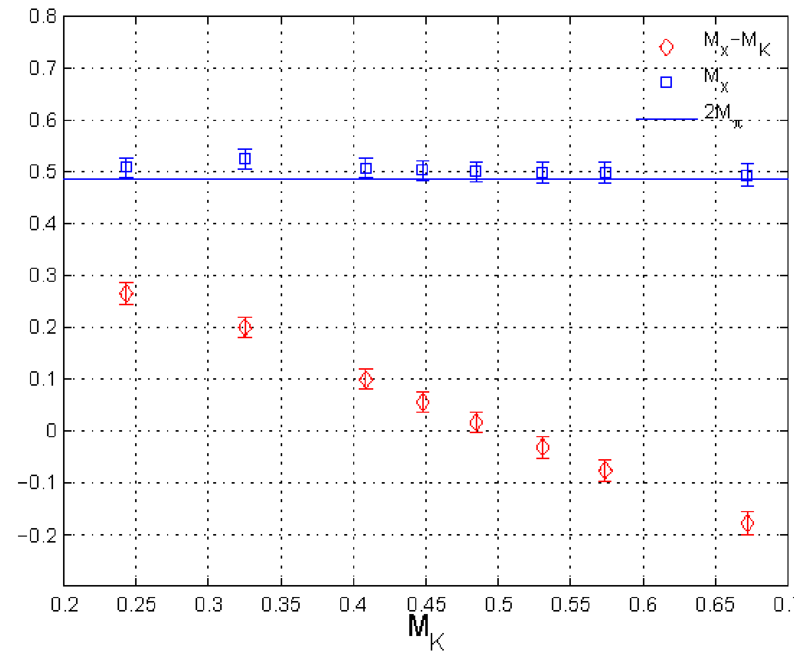


Examine long distance part

- Examine type 2 diagrams:

Parity even channel





Results

M_K	ΔM_K^{11}	ΔM_K^{12}	ΔM_K^{22}	ΔM_K ($\times 10^{-12}$ MeV)
0.3252(7)	6.38(14)	-2.64(14)	1.47(8)	5.52(24)
0.4087(7)	8.90(21)	-2.96(23)	2.10(12)	7.38(37)
0.4480(7)	10.63(27)	-3.18(30)	2.48(15)	8.61(49)
0.4848(8)	12.56(34)	-3.62(40)	2.89(20)	9.93(65)

- $\Delta m_K^{\mathrm{expt}} = 3.483(6)\ 10^{-12}$ MeV
- Unphysical kinematics, $m_\pi = 421$ MeV
- Disconnected diagrams dropped
- Active charm but $m_c\, a = 0.7$
- Improvements underway!

Outlook

- The new generation of supercomputers allows us to work at physical quark masses.
- Chiral fermions and NPR give continuum-like control of operator normalization and mixing.
- Theoretical advances allow rescattering effects to be correctly computed in Euclidean space.
- This opens new areas to precise lattice computation:
  - $K \rightarrow \pi\pi$, ΔI=3/2 and 1/2
  - $m_{K_L} - m_{K_S}$
  - $K \rightarrow \pi l \bar{l}$ ← *e.g.* LD part of $K^+ \rightarrow \pi^+ + l + \bar{l}$ (C. Sachrajda)