Lattice-QCD: an enabling technology for Project X

Tom Blum and Ruth Van de Water

(For the lattice-QCD working group: Christopher Aubin, Tanmoy Bhattacharya, Michael Buchoff, Norman Christ, Saul Cohen, Don Holmgren, Taku Izubuchi, Andreas Kronfeld, Jack Laiho, Paul Mackenzie, Emanuele Mereghetti, Santi Peris, Brad Plaster, Dru Renner, Eigo Shintani)

(Dated: November 16, 2012)

We discuss several key opportunities for lattice-QCD calculations to aid in the interpretation of experimental measurements at Project X. We focus on three general categories of calculations for which the technical issues are different: kaons, nucleons, and the muon anomalous magnetic moment. We summarize the current status of lattice-QCD calculations in these areas; more detailed information can be found in the talks on the Project X Physics Study website and in the references. We also discuss future prospects for lattice-QCD calculations in these areas, focusing on the computational and methodological improvements needed to obtain the precision required by experiments at Project X.

I. INTRODUCTION

The Project X accelerator complex will perform a broad range of high-precision measurements that probe quantum-mechanical loop effects and are sensitive to physics at higher energy scales than are directly explored at the LHC. Through the use of intense sources and sensitive detectors, the various Project X experiments will search for processes that are extremely rare in the Standard Model and look for tiny deviations from Standard-Model expectations. In many cases, the comparison between the measurements and Standard-Model predictions are currently limited by theoretical uncertainties, often from hadronic matrix elements. Lattice QCD provides a first-principles method for calculating these hadronic matrix elements from *ab initio* QCD with reliable and systematically-improvable uncertainties.

Lattice-QCD calculations formulate QCD on a discrete Euclidean spacetime lattice, thereby transforming the infinite-dimensional Quantum Field Theory path integral into a finite-dimensional integral that can be solved numerically with Monte Carlo methods and importance sampling. In practice, lattice-QCD simulations are computationally intensive, and require the use of the world's most powerful computers. The QCD Lagrangian has $1+n_f+1$ parameters: the gauge coupling g^2, the quark masses m_f, and the QCD θ-term. Because measurements of the neutron EDM bound $\bar{\theta} < 10^{-10}$, most lattice-QCD simulations set $\bar{\theta} = 0$. The gauge-coupling and quark masses in lattice-QCD simulations are tuned by calibrating to $1 + n_f$ experimentally-measured quantities (typically hadron masses, decay constants, or mass-splittings). Once the parameters of the lattice action are fixed, everything else is a prediction of QCD.

There are many ways to discretize the QCD action which correspond to different lattice fermion actions, but all of them recover QCD in the continuum limit (*i.e.* when the lattice spacing $a \to 0$). The various fermion formulations in use have different advantages (such as computational speed or exact chiral symmetry) and different sources of systematic uncertainty; hence it is essential to compute quantities with more than one method for independent validation. The time required for numerical simulations increases as the quark mass decreases (due to the need to invert the fermion determinant), so quark masses in lattice simulations are higher than those in the real world. Typical lattice calculations now use pions with masses $m_\pi \lesssim 300$ MeV, while state-of-the art calculations for some quantities use pions at or slightly below the physical mass of $m_\pi \sim 140$ MeV. Ultimately, improvement in algorithms and increases in computing power will render chiral extrapolations unnecessary.

Most lattice-QCD simulations proceed in two steps. First one generates an ensemble of gauge fields with a distribution $\exp[-S_{\rm QCD}]$; this must be done in series and generally requires high-capability machines such as the BlueGenes at Argonne and Livermore or the Cray at Oak Ridge. Next one computes operator expectation values on these gauge fields; these jobs can be run in parallel and are well-suited for high-capacity PC and GPU clusters such as the dedicated lattice-QCD facilities at Fermilab and Jefferson Lab. A major breakthrough in lattice-QCD occurred with the advent of gauge-field ensembles that include the effects of the dynamical u, d, and s quarks in the vacuum. Lattice-QCD simulations now regularly "$N_f = 2+1$" sea quarks in which the light u and d sea-quark masses are degenerate and somewhat heavier than the physical values, and the strange-sea quark mass takes its physical value. Further, "$N_f = 2+1+1$" simulations that include a charm sea quark are now underway; dynamical charm effects are expected to become important as precision for some quantities reaches the percent-level.

The easiest quantities to compute with controlled systematic errors and high precision in lattice-QCD simulations have only hadron in initial state and at most one hadron in final state, where the hadrons are stable under QCD (or narrow and far from threshold). "Gold-plated" lattice quantities Includes meson masses and decay constants, semileptonic and rare decay form factors, and neutral meson mixing parameters, and enable determinations of all CKM matrix elements except $|V_{tb}|$. Many interesting QCD observables are not gold-plated, however, such as resonances like the ρ and K^* mesons, fully hadronic decay matrix elements such as for $K \to \pi\pi$ and $B \to DK$, and long-distance dominated quantities such as D^0-mixing.

Many errors in lattice-QCD calculations can be assessed within the framework of effective field theory. Lattice-QCD calculations typically quote the following sources of uncertainty:

- *monte carlo statistics and fitting,*
- *tuning lattice spacing and quark masses* by calibrating to a few experimentally-measured quantities such as m_π, m_K, m_{D_s}, m_{B_s}, and f_π,
- *matching lattice gauge theory to continuum QCD* using fixed-order lattice perturbation theory, step-scaling, or other partly- or fully-nonperturbative methods,
- *chiral and continuum extrapolation* by simulating at a sequence of quark masses and lattice spacings and extrapolating to $m_{\rm lat} \to m_{\rm phys}$ and $a \to 0$ using functional forms derived in chiral and weak-coupling perturbation theory.

The methods for estimating uncertainties can be verified by comparing results for known quantities with experiment. Lattice-QCD calculations successfully reproduce the experimentally-measured low-lying hadron spectrum [1–11]. Lattice-QCD results agree with non-lattice determinations of the quark masses [12–19] and strong coupling constant α_S [13, 20–25], but now surpass the precision obtained by other methods. Further, lattice-QCD calculations correctly predicted the mass of the B_c meson [26, 27], the leptonic decay constants f_D and f_{D_s} [28, 29], and the $D \to K\ell\nu$ semileptonic form factor [30, 31]. These successful predictions and post-dictions demonstrate that the systematic uncertainties in lattice-QCD calculations are under control, and that lattice-QCD results can reliably be used to test the Standard Model and search for new physics via high-precision measurements at Project X and elsewhere.

II. KAONS

The lattice-QCD community has a well-established and successful kaon physics program which was summarized at the workshop by J. Laiho [32]. The matrix elements needed to obtain pion and kaon leptonic decay constants, light-quark masses, the $K \to \pi\ell\nu$ semileptonic form factor, and neutral kaon mixing are all gold-plated on the lattice, and can therefore be computed with lattice QCD to a few percent or better precision. Many lattice-QCD collaborations are attacking these quantities with $N_f = 2+1$ [12–17, 33–38] and now $N_f = 2+1+1$ [39, 40] dynamical gauge-field ensembles, thereby providing independent cross checks and enabling global lattice-QCD averages [41, 42]. A highlight of the lattice-QCD kaon physics effort is the calculation of the neutral-kaon mixing parameter B_K, which enables a constraint on the apex of the CKM unitarity triangle when combined with experimental measurements of indirect CP-violation in the kaon system. Until recently, the unitarity-triangle constraint from ϵ_K was limited by the $\sim$20% uncertainty in the hadronic matrix element B_K [43]. Therefore B_K was flagged as a key goal for lattice flavor physics, and significant theoretical and computational effort has been devoted to its improvement. There are now several independent lattice results for B_K that are in good agreement [14, 17, 44, 45], and the error in the average is $\lesssim 1.5\%$ [46]. In fact, B_K is now a sub-dominant source of uncertainty in the ϵ_K band, below the parametric error from $A^4 \propto |V_{cb}|^4$ and the perturbative truncation errors in the Inami-Lim functions η_{cc} and η_{ct} [47, 48].

The Project X kaon-physics program offers many opportunities for lattice-QCD to provide crucial inputs needed to interpret the experimental measurements as tests of the Standard Model and constraints on new physics. The proposed ORKA experiment at Fermilab, which will begin running before Project X, will measure the CP-conserving rare decay $K^+ \to \pi^+\nu\bar{\nu}$ and collect $\sim$ 200 events/year, assuming the Standard-Model rate. With Stage 1 of Project X this rate will increase to $\sim$ 340 events/year, enabling a measurement of the branching fraction to $\sim$ 3% precision. Stage 2 of Project X will enable a measurement of the branching fraction for the CP-violating rare decay $K_L \to \pi^0\nu\bar{\nu}$ to $\sim$5%, assuming the Standard-Model rate. The Project X kaon-physics experiments will also measure numerous other kaon observables such as $\Gamma(Ke2)/\Gamma(K\mu 2)$, $K^+ \to \pi^+\ell^+\ell^-$, and $K_L \to \pi^0\ell^+\ell^-$. Correlations between these channels will allow discrimination between different new-physics scenarios, provided sufficiently precise theoretical predictions.

A. $K \to \pi\pi$ decay

Given the mature state of lattice-QCD calculations of simple kaon matrix elements, the lattice kaon-physics community is beginning to tackle more challenging quantities. The RBC/UKQCD collaboration, in particular, is the farthest along in this area; N. Christ summarized their progress on computing $K \to \pi\pi$ decay and the K_L-K_S mass difference from lattice QCD [49].

With $K \to \pi\pi$ decay in the $I = 2$ channel at physical quark mass and kinematics under control, the RBC/UKQCD collaboration has started to tackle the more difficult $I = 0$ channel. This offers the first realistic possibility to calculate direct CP violation in kaon decays (ϵ') in the Standard Model. New physics in ϵ' is tightly correlated with that in rare kaon decays (see Buras [50] and the talk by U. Haisch [51] at this meeting). Experiments have measured $\mathrm{Re}(\epsilon'/\epsilon)$ to $\sim$10% precision, but the ability to constrain new physics with ϵ' has been handicapped by the uncertainty in the $K \to \pi\pi$ hadronic matrix elements. Thus the payoff of improved lattice-QCD calculations of $K \to \pi\pi$ decays with a precision comparable to experiment will be significant. The new feature for lattice-QCD calculations of the $I = 0$ channel is to use $G-$parity boundary conditions to produce two-pion final states with $I = 0$ and non-zero relative momentum, so that the ground state has physical, not threshold, kinematics. That way one does not need to extract the physical matrix element from an "excited" state. The aid of new computing resources coming online and new methods to reduce statistical errors associated with quark-disconnected diagrams (for example, see the talk by Izubuchi at this workshop [52]), should soon enable a calculation of the $I = 0$ channel at physical kinematics using the same parameters as the $I = 2$ calculation. Results from these calculations may be available within a year and should reveal the method's ultimate effectiveness. The systematic error associated with the nonzero lattice spacing, which was the most significant for the $I = 2$ calculation ($\sim$ 15%), will take longer to control. A determination of ϵ' with a total error at the 20% level, however, may be possible in two years.

Using similar techniques, the RBC/UKQCD collaboration has demonstrated that long-distance contributions to the $K_L - K_S$ mass difference can be calculated using lattice QCD. This should pave the way to computations of long-distance contributions to neutral kaon mixing and rare kaon decays. A first paper describing the methodology and preliminary results with unphysically heavy quarks ($m_\pi \sim$ 420 MeV) is due very soon, and the work was recently presented at the Lattice 2012 conference [53]. A more accurate calculation, including physical light-quark masses, may be finished within one year. As for the decays, systematic errors from the nonzero lattice spacing will take longer to control.

B. Rare kaon decays

The rare kaon decays $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$ are called "golden modes" because the Standard-Model branching ratios are known to a precision unmatched by any other quark flavor-changing-neutral-current process. In particular, the hadronic uncertainties are small because the form factor can be obtained precisely using experimental $K \to \pi\ell\nu$ data combined with chiral perturbation theory. The limiting source of uncertainty in the Standard-Model predictions for $\mathrm{BR}(K^+ \to \pi^+\nu\bar{\nu})$ and $\mathrm{BR}(K_L \to \pi^0\nu\bar{\nu})$ is the parametric error from $A^4 \propto |V_{cb}|^4$ and is approximately $\sim$10% [48, 54]. The CKM matrix element $|V_{cb}|$ can be obtained from exclusive $B \to D^{(*)}\ell\nu$ decays provided calculations of the hadronic form factors. Lattice-QCD results for the $B \to D^*\ell\nu$ form factor now enable the determination of $|V_{cb}|$ to $\sim 2.5\%$ precision [55], and improved lattice-QCD calculations of both the $B \to D\ell\nu$ and $B \to D^*\ell\nu$ form factors are underway [56]. The expected increase in computing resources over the next few years should lead to a reduction in the error on $|V_{cb}|$ to $\lesssim 1.5\%$, and the error on the Standard-Model $K \to \pi\nu\bar{\nu}$ branching fractions to $\lesssim 6\%$ within five years [32]. With this precision, the theoretical uncertainties in the Standard-Model branching fractions will be commensurate with the projected experimental errors in time for Stage 1 of Project X.

Errors from long-distance contributions are subdominant in the Standard Model predictions for $K \to \pi\nu\bar{\nu}$ due to quadratic GIM suppression, but are significant in other rare kaon decays such as $K \to \pi\ell^+\ell^-$. Currently the Standard-Model estimates for the $K \to \pi\ell^+\ell^-$ branching fractions rely on Chiral Perturbation Theory and have large uncertainties that are not competitive with those on $K \to \pi\nu\bar{\nu}$. If they can be brought under theoretical control, however, $K \to \pi\ell^+\ell^-$ may afford experimentalists new search channels that, through correlations with other observables, provide additional handles to distinguish between new-physics scenarios. A method for calculating long-distance contributions to rare kaon decays in lattice QCD that addresses issues concerning renormalization and short distances has been proposed by Isidori, Martinelli, and Turchetti in [57]. The RBC/UKQCD collaboration has begun studying this topic, an outline of which was given by Sachrajda at a recent BNL workshop [58]. The calculation builds upon techniques developed for ΔM_K, but there are new features and systematics that must be addressed. Some of these, like finite volume and UV divergences may be easier to handle than for ΔM_K. There are many disconnected diagrams, however, so new techniques to reduce statistical errors (like those mentioned earlier) will be needed. Lattice-QCD calculations of long-distance contributions to rare decays have not yet begun, so it is premature to forecast time scales for completion or uncertainties obtained.

III. NUCLEONS

Although many nucleon matrix elements are gold plated, lattice-QCD calculations involving baryons are generally more challenging than for mesons. They are more computationally demanding because statistical noise in baryon correlation functions grows rapidly with Euclidean time. Further, the extrapolation to physical light-quark masses is more difficult because baryon chiral perturbation theory converges less rapidly.

The most studied nucleon matrix element is that of the axial charge g_A. Because it can be measured precisely in neutron β-decay experiments (see the talk by B. Plaster [59]), g_A provides a benchmark for of the accuracy of lattice-QCD nucleon matrix element calculations. The present quoted errors in lattice calculations of g_A are $\sim$ 6–10% [60–63], but the lattice determinations are all systematically lower than the experimental measurement by about 10%, indicating the presence of an underestimated uncertainty. Possible sources of this discrepancy are errors due to the finite lattice spatial volume or contamination from excited states in the 3-point correlation functions, both of which are under scrutiny. The expected increase in computing power over the next five years (discussed by D. Holmgren [64]) should allow simulations with larger volumes and close-to-physical light quark masses, while new algorithms such as all-mode-averaging (discussed by T. Izubuchi [65]) should greatly reduce the statistical errors. We therefore expect to achieve percent-level lattice-QCD calculations of g_A on this timescale. We note, however, that percent-level precision is not needed for most of the other nucleon matrix element calculations of relevance to Project X, so computing and human resources may be directed elsewhere once the target precision (typically $\sim$ 10 or 20%) for these quantities is reached.

Stage 1 of Project X will feature a spallation target facility optimized for particle physics, enabling world-class ultra-cold neutron and EDM programs. The interpretation of many of these experimental measurements as constraints on TeV-scale or GUT-scale new physics requires knowledge of nucleon matrix elements that can be computed in lattice QCD. Project X will also provide intense neutrino sources and beams that can be used to illuminate nearby detectors at Fermilab or far detectors at other facilities. If the far detector is sufficiently large, and is shielded from cosmic rays either by an underground location or an above-ground veto system, it will enable a proton-decay search that can improve upon the projected reach of current facilities. The interpretation of experimental limits on the proton lifetime as constraints on new-physics models depend upon the expectation values $\langle \pi, K, \eta, \ldots | \mathcal{O}_{\rm BSM} | p \rangle$ of non-Standard Model operators; these can be computed with lattice QCD. In practice, the preferred scenario for LBNE Phase 1 would be a beamline to Homestake with a 10 kton LAr-TPC surface detector, which would not be suitable for a proton-decay search. Thus a proton-decay search with Project X/LBNE would not occur until at least Stage 2.

A. Proton and neutron decay matrix elements

Proton decay is a generic prediction of most Grand Unified Theories. Therefore experimental limits on the proton lifetime can be interpreted as constraints on GUT-scale new-physics models given the expectation values $\langle \pi, K, \eta, \ldots | \mathcal{O}_{\Delta {\rm B}=1} | p \rangle$ of the baryon-number violating operators. Estimates of these matrix elements based on the bag model, sum rules, and the quark model vary by as much as a factor of $\sim$3, and lead to an $\mathcal{O}(10)$ uncertainty in the model predictions for the proton lifetime. Thus lattice-QCD calculations of proton-decay matrix elements with controlled systematic uncertainties of $\sim$ 20% would represent a significant improvement, and be sufficiently precise for constraining GUT theories.

Experimental measurements of neutron β-decay can place constraints on TeV-scale new-physics models, in particular those with scalar or tensor interactions, provided values for the nucleon scalar and tensor charges g_S and g_T. The next generation of neutron β-decay experiments is expected to increase their sensitivity to scalar and tensor interactions by an order of magnitude. Model estimates of g_S and g_T disagree and provide only loose bounds, but lattice-QCD can provide precise results for these quantities. Studies by Bhattacharya *et al.* have shown that lattice-QCD calculations of g_S and g_T with $\sim$10–20% precision will be needed to fully exploit the increased experimental sensitivity [66].

Lattice-QCD calculations of proton and neutron decay matrix elements are similar to that of the nucleon axial charge, but with different $\Delta B = 1$ operators and external hadron states. Hence they are generically expected to have comparable uncertainties. T. Izubuchi presented preliminary results for proton-decay matrix elements on behalf of E. Shintani and the RBC/UKQCD Collaboration [67]. These are the first determinations of proton-decay matrix elements from $N_f = 2+1$ lattice QCD, and the uncertainties are still quite large, with 20% statistical errors and 30% chiral extrapolation errors. The use of all-mode averaging and ensembles with close-to-physical pion masses should, however, enable determinations of these matrix elements with $\sim 10\%$ accuracy in several years' time. S. Cohen presented preliminary results for neutron-decay matrix elements on behalf of the PNDME Collaboration [68]. They obtain the most precise lattice-QCD determination of g_T and the only lattice-QCD determination of g_S with statistical errors of approximately $\sim$11% for g_S and $\sim$4% for g_T. In the future, use of data at close to the physical pion mass and nonperturbative renormalization should enable determinations of g_S and g_T with total errors of a few percent in the next five years.

In summary, for both proton- and neutron-decay matrix elements, we expect lattice-QCD calculations to obtain uncertainties needed for experiments in time for Stage 1 of Project X.

B. Neutron-antineutron oscillations

A neutron-antineutron oscillation experiment at Project X could improve the limit on the n-$\bar{n}$ transition rate by a factor of ~ 1000. This sensitivity could enable one to rule out many new-physics theories, such as GUT models with quark-lepton unification or models of sphaleron baryogenesis, given values of the hadronic matrix elements for the non-Standard Model $\Delta B = 2$ six-quark operators. An estimate from the MIT bag model indicates that the sizes of these matrix elements is consistent with dimensional expectations [69], while one from the chiral bag model predicts an exponential suppression of these matrix elements [70, 71] that would render n-$\bar{n}$ oscillations unobservable by proposed experiments. Lattice-QCD can provide calculations of these matrix elements from first-principles QCD with controlled uncertainties.

M. Buchoff presented the first lattice-QCD results for the complete set of n-$\bar{n}$ transition matrix elements [72]. The calculation is in a very early stage, so the results should be taken *cum grano salis*, but their order-of-magnitude is consistent with dimensional expectations. It is also premature to make predictions for future total uncertainties, but even calculations of these matrix elements with reliable errors below 50% would provide valuable guidance for new-physics model predictions, and are certainly achievable on the timescale of Project X Stage 1.

C. Proton and neutron EDMs

Nucleon electric dipole moments can arise both in the Standard Model (due to the QCD-θ term) and in theories beyond the Standard Model. Experimental limits on the size of the neutron EDM (d_N) constrain the size of $|\theta| \lesssim 10^{-10}$, but this this constraint is not known precisely because of uncertainties in model estimates for d_N/θ. Lattice-QCD can provide first-principles QCD calculations of d_N/θ with improved precision and controlled uncertainties. Experimental limits on the neutron EDM also place constraints on TeV-scale new physics models with new sources of CP-violation (and even rule out parts of parameter space in some SUSY models). In some cases the model predictions require hadronic matrix elements that can also be calculated in lattice QCD. Finally if a neutron EDM is observed, a measurement of the proton EDM (along with a corresponding lattice-QCD calculation) will be needed to distinguish between contributions from QCD and contributions from non-Standard Model CP-violating interactions.

E. Shintani presented the status of nucleon EDM calculations on the lattice [73]. Results have been obtained with $N_f = 2$ dynamical quark flavors using two methods: (i) calculating the energy difference between two spin states of the nucleon in an external electric field [74], and (ii) computing the form factor of the electromagnetic current [75, 76]. Currently the statistical errors are still $\sim$30%, both because of the general property that nucleon correlation functions have large statistical errors and because the calculation involves correlations with the topological charge density, which introduces substantial statistical noise. Shintani also outlined the future plans of the

RBC and UKQCD Collaborations: they will address the large statistical errors by exploiting all-mode averaging, and they will analyze the planned $N_f = 2+1$ RBC-UKQCD gauge-field ensemble with a close-to-physical pion mass to control the chiral extrapolation. Given these improvements they expect to obtain the nucleon EDM with ∼10% accuracy in the next five years. T. Bhattacharya outlined a method for determining contributions to the nucleon EDM from dimension 6 operators that arise in beyond-the-Standard Model theories [77]. The calculation of the contribution from the quark electric dipole moment is in progress, and should be comparable in difficulty to that of the contribution from the topological charge. Thus we expect that the quark-connected contribution will be calculated with ∼20% precision soon, and that a ∼10% full lattice-QCD calculation of the quark EDM should be possible in the next five years. The calculation of the contribution from the chromoelectric dipole moment is more challenging because it involves a lattice four-point function and the quark chromoelectric moment operator mixes with the topological charge. Thus the precision achievable in the next few years is difficult to predict. Even a lattice-QCD calculation of the chromo-EDM with reliable errors below 50%, however, would provide valuable guidance for new-physics model predictions, and this seems attainable for the onset of Stage 1 of Project X.

IV. MUON $g-2$

The muon anomalous magnetic moment a_μ is an extremely sensitive probe of heavy mass scales in the several hundred GeV range. Different new-physics scenarios predict a wide range of contributions to a_μ, so experimental measurements combined with sufficiently precise theoretical predictions can: (i) rule out numerous new-physics scenarios, (2) distinguish between models with similar LHC signatures, and (3) help determine the parameters of the TeV-scale theory that is realized in Nature.

The muon $(g-2)$ has been measured experimentally to 0.54 ppm, and disagrees with the Standard-Model prediction by more than 3σ. Therefore both experimental and theoretical efforts are underway to investigate this discrepancy. The New $(g-2)$ Experiment at Fermilab, which will run before Project X, will reduce the experimental error by a factor of four to approximately 0.14 ppm. The quoted error in the Standard-Model prediction for a_μ is 0.42 ppm, and has several sources. The QED and EW contributions to a_μ are known very precisely (to four loops and two loops in perturbation theory, respectively), but the QCD contributions to a_μ are currently under less theoretical control. The current theoretical estimate of the hadronic light-by-light contribution a_μ^{HLbL}, in particular, comes from a range of QCD model predictions and must be taken with caution. Lattice QCD can provide calculations of the hadronic vacuum polarization and hadronic light-by-light contributions to the muon $(g-2)$ from QCD first principles with reliable uncertainties and, ultimately, greater precision than currently available.

Although the currently planned $(g-2)$ experiment is not part of Project X, the measurement of $(g-2)$ could benefit from the Project X accelerator upgrade. Thus a second-generation muon $(g-2)$ experiment would be possible with Project X if it seemed warranted based on improvements in the theoretical calculation and the evolution of the discrepancy with respect to the Standard Model.

A. Hadronic vacuum polarization

The dominant source of uncertainty in the Standard-Model prediction for the muon $(g-2)$ is from the hadronic vacuum polarization contribution (0.36 ppm). Currently it is obtained from combining experimental data for $e^+e^- \to$ hadrons with a dispersion relation for the correlation function. The hadronic vacuum polarization contribution to $(g-2)$ can also be obtained from data for $\tau \to$ hadrons, but this method for extracting $a_\mu^{\rm HVP}$ disagrees with the standard approach and reduces the discrepancy with the Standard Model to a bit below 3σ. Interpreting these experimental measurements as determinations of $a_\mu^{\rm HVP}$ requires applying radiative corrections in the case of e^+e^- data and isospin corrections in the case of τ data. A direct lattice-QCD calculation of the hadronic vacuum polarization with $\sim 1\%$ precision may help shed light on the apparent discrepancy between e^+e^- and τ data. Ultimately, a lattice-QCD calculation of $a_\mu^{\rm HVP}$ with sub-1% precision can circumvent these concerns by supplanting the determination of $a_\mu^{\rm HVP}$ from experiment with one from first-principles QCD.

The HVP contribution to the muon anomalous magnetic moment (see Fig. 1) has been computed in lattice QCD by several groups [78–81], with attempts to quantify systematic errors by only some of them. Quoted errors range from 3% to about 10%, where the low end is a statistical error only.

C. Aubin presented a calculation of $a_\mu^{\rm HVP}$ using staggered lattice fermions [82]. The HVP contribution to $(g-2)$ is obtained by an weighted integral over Euclidean momentum-squared (Q^2) of the renormalized hadronic vacuum polarization function $\Pi(Q^2)$, where $\Pi(Q^2)$ is computed nonperturbatively in lattice QCD. The weighting function vanishes as $1/(Q^2)^2$ for large Q^2, so the integral is dominated by the low-momentum region. The momenta in lattice simulations are restricted to nonzero values due to the finite lattice four-volume; further, lattice-QCD data in the extreme low momenta region is both sparse and relatively noisy. As Aubin pointed out, the integral is actually dominated by values of Q^2 below those for which lattice-QCD data is available; therefore the value obtained for $a_\mu^{\rm HVP}$ is quite sensitive to the functional form used for the Q^2-extrapolation [83]. Aubin and his collaborators have introduced model-independent Padé-approximant (PA) fits [83]. The result for $a_\mu^{\rm HVP}$ is stable with the addition of higher poles in the correlated PA fit, but disagrees with earlier vector meson model fits. Work is in progress to reduce the statistical errors using all-mode-averaging (AMA) (see the talk by T. Izubuchi [52]), and preliminary results presented at the Lattice 2012 conference (after the workshop) appear promising [84]. Once statistical errors are under control, reduction of systematic errors due to fitting will follow. In the future Aubin and collaborators plan to fill in the low-momentum region using twisted boundary conditions, an idea already under investigation by the Mainz group and others.

D. Renner presented a competing calculation of $a_\mu^{\rm HVP}$ using twisted-mass lattice fermions [85]. He focused primarily on the method that the ETMC collaboration has developed to reduce the dependence of $a_\mu^{\rm HVP}$ on the light-quark masses, thereby lowering the uncertainty due to the extrapolation to physical light-quark masses and the continuum. The method modifies the observable used to extract the HVP contribution by rescaling the integrand according to a ratio of hadronic scales, one computed on the lattice and the corresponding one taken from the continuum. This dramatically decreases the apparent quark mass dependence of the HVP contribution to $(g-2)$, such that the behavior is linear in the range of quark masses available. With this approach Renner and collaborators obtain a result for $a_\mu^{\rm HVP}$ with a $\sim$3% statistical error in $N_f = 2$ QCD. It was pointed out at the PXPS workshop that the modified observable method is not expected to help significantly near the physical quark mass point where the ratio is close to unity, and there is no reason to expect the linear ansatz to hold through the two-pion threshold as the quark mass is lowered. Further, the method and its associated systematics is divorced from the Q^2 problem discussed by Aubin. Renner also showed preliminary results from ETMC's new $N_f = 2+1+1$ flavor calculation. The charm quark contribution is expected from the dispersive calculation to be roughly the same size as the hadronic light-by-light (HLbL) scattering contribution. ETMC finds a charm quark contribution about $2\times$ the size of the HLbL one.

While there is still some way to go, the necessary ingredients for a 1% lattice QCD calculation of $a_\mu^{\rm HVP}$ appear to be in hand. State-of-the-art simulations are now being done with physical quark masses in large boxes. Statistical error reduction techniques are being developed, like AMA, which appear promising. Within three years lattice QCD calculations should reach this level. Beyond this precision, significant challenges await: isospin breaking effects and the related disconnected diagrams (right side of Fig. 1).

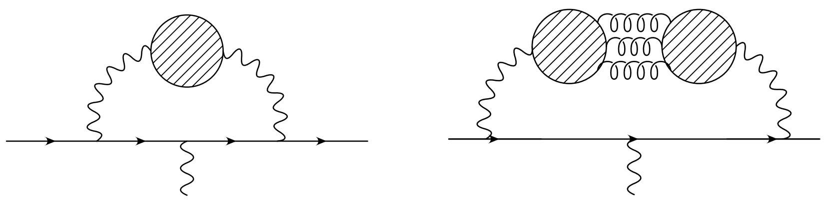

FIG. 1. Hadronic vacuum polarization diagrams contributing to the SM muon anomaly. The horizontal lines represent the muon. Left panel: the blob formed by the quark-antiquark loop represents all possible hadronic intermediate states. Right panel: disconnected quark line contribution. The quark loops are connected by gluons.

B. Hadronic light-by-light

The quoted error in the hadronic light-by-light contribution to the muon $(g-2)$ is approximately a factor of two smaller than the error in the hadronic vacuum polarization contribution (0.22 ppm). The current determination of $a_\mu^{\rm HLbL}$ is obtained from various models such as QCD in the large N_c limit, vector-meson dominance, and chiral perturbation theory. The model results, however, are not consistent, and the error in $a_\mu^{\rm HLbL}$ is estimated from the spread of values; thus the claimed precision must be taken *cum grano salis*. A calculation of $a_\mu^{\rm HLbL}$ with $\sim$ 10–15% precision and a reliable uncertainty estimate is crucial to match the target experimental precision and to test the Standard Model at the 0.14 ppm level.

Lattice-QCD can provide a calculation of $a_\mu^{\rm HLbL}$ from QCD first principles with controlled uncertainties that are systematically improvable. The importance of this calculation to the experimental program is well-known to the lattice-QCD community, and research-and-development efforts are underway by several independent collaborations. The lattice-QCD calculation of the hadronic light-by-light contribution is challenging, however, and still in early stages. Thus, while achieving a lattice-QCD calculation of $a_\mu^{\rm HLbL}$ in the next five years with $\sim$ 10–15% precision is certainly possible, it is not guaranteed. Here we describe various strategies to attack $a_\mu^{\rm HLbL}$ currently being studied. New theoretical developments are impossible to predict, however, and may be needed to match the target experimental precision.

Lattice QCD+QED calculations of the hadronic light-by-light scattering amplitude, where the muon and photons are treated nonperturbatively along with the quarks and gluons [86], were presented by Blum [87]. These early results appear promising. The diagram on the left in Fig. 2 has been computed with light quarks corresponding to $m_\pi = 329$ MeV and muon mass $m_\mu \approx 190$ MeV for the two lowest values of external photon momentum accessible on a 24^3, $a^{-1} = 1.73$ GeV lattice. A stable signal in the ballpark of model estimates may be emerging for this diagram, though statistical errors are still large. It will be easy to reduce the muon mass to its physical value, but many other systematic errors still need to be addressed: finite volume, quark mass dependence, non-zero lattice spacing, and so on. Perhaps the biggest challenge will be to compute the disconnected diagram shown on the right in Fig. 2. Ideas put forward but not yet attempted include brute force, analogous to the previous diagram; QED reweighting to include the virtual sea quark contributions like the one on the right-hand-side of the righthand panel of Fig. 2. ; and calculation of such virtual sea-quark effects by direct inclusion of photons in the gauge field ensemble generation. All of the methods discussed so far require a nonperturbative $O(\alpha^2)$ subtraction [86]. An alternative presented by Izubuchi [52] is to again calculate the entire HLbL amplitude in lattice QCD+QED, but only the desired $O(\alpha^3)$ piece by using a perturbative treatment of the fermion propagators with respect to QED, the Aslash Sequential Source method. Besides being useful to calculate the disconnected piece, the new method can also be used as an independent check on the original QCD+QED calculation. The statistical error from the new method is expected to be smaller, or at least no larger, than the original. At the PXPS workshop Bardeen proposed an additional check of the lattice-QCD+QED methodology for the HLbL calculation by comparing the result for $O(\alpha^2)$ contribution before the subtraction with non-lattice determinations.

Besides calculating the whole amplitude, one can also follow a path similar to the HVP calculation by calculating the four-point vector current correlation function in pure QCD which can then be inserted into the continuum two-loop QED integral. In principle this requires the computation of four-volume-squared number of quark propagators per gauge field configuration, one for each independent momentum in the two-loop integrals. One could start with the four-point function at a few momentum values, however, to check the model calculations. An even simpler step was presented by S. Cohen [88], who described methods to compute $\pi^0 \to \gamma^*\gamma^*$. This intermediate step in the HLbL calculation can be used to check form factors used in some model calculations. A result for on-shell $\pi^0 \to \gamma\gamma$ decay with controlled errors by the JLQCD Collaboration has recently appeared [89].

S. Peris presented new non-perturbative results for the anomaly triangle [90]. An observable of interest here and in model calculations of the HLbL amplitude is the so called magnetic susceptibility which is poorly known but could be calculated using lattice QCD.

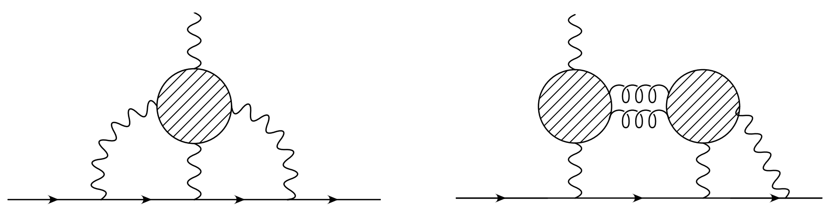

FIG. 2. Hadronic light-by-light scattering diagrams contributing to the SM muon anomaly. The horizontal lines represent the muon. Left panel: the blob formed by the quark loop represents all possible hadronic intermediate states. Right panel: one of the disconnected quark line contributions. The quark loops are connected by gluons.

[1] C. Aubin *et al.* (MILC), Phys.Rev. **D70**, 094505 (2004), arXiv:hep-lat/0402030 [hep-lat]

[2] A. Bazavov *et al.*, Rev.Mod.Phys. **82**, 1349 (2010), arXiv:0903.3598 [hep-lat]

[3] S. Aoki *et al.* (PACS-CS), Phys.Rev. **D79**, 034503 (2009), arXiv:0807.1661 [hep-lat]

[4] S. Dürr *et al.* (BMW), Science **322**, 1224 (2008), arXiv:0906.3599 [hep-lat]

[5] W. Bietenholz *et al.*, Phys.Rev. **D84**, 054509 (2011), arXiv:1102.5300 [hep-lat]

[6] N. Christ *et al.* (RBC and UKQCD), Phys.Rev.Lett. **105**, 241601 (2010), arXiv:1002.2999 [hep-lat]

[7] J. J. Dudek *et al.*, Phys.Rev. **D83**, 111502 (2011), arXiv:1102.4299 [hep-lat]

[8] E. B. Gregory, A. C. Irving, C. M. Richards, and C. McNeil(2011), arXiv:1112.4384 [hep-lat]

[9] C. Bernard *et al.* (Fermilab Lattice and MILC), Phys.Rev. **D83**, 034503 (2011), arXiv:1003.1937 [hep-lat]

[10] E. B. Gregory *et al.* (HPQCD), Phys.Rev. **D83**, 014506 (2011), arXiv:1010.3848 [hep-lat]

[11] D. Mohler and R. Woloshyn, Phys.Rev. **D84**, 054505 (2011), arXiv:1103.5506 [hep-lat]

[12] S. Dürr *et al.* (BMW), Phys. Lett. **B701**, 265 (2011), arXiv:1011.2403 [hep-lat]

[13] C. McNeile, C. Davies, E. Follana, K. Hornbostel, and G. Lepage (HPQCD), Phys.Rev. **D82**, 034512 (2010), arXiv:1004.4285 [hep-lat]
[14] J. Laiho and R. S. Van de Water, PoS **Lattice 2011**, 293 (2011), arXiv:1112.4861 [hep-lat]
[15] A. Bazavov *et al.* (MILC), PoS **LAT2009**, 079 (2009), arXiv:0910.3618 [hep-lat]
[16] T. Blum *et al.*, Phys.Rev. **D82**, 094508 (2010), arXiv:1006.1311 [hep-lat]
[17] C. Kelly (RBC and UKQCD), PoS **Lattice 2011**, 285 (2012), arXiv:1201.0706 [hep-lat]
[18] K. Nakamura *et al.* (Particle Data Group), J.Phys.G **G37**, 075021 (2010)
[19] K. Chetyrkin *et al.*, Phys.Rev. **D80**, 074010 (2009), arXiv:0907.2110 [hep-ph]
[20] I. Allison *et al.* (HPQCD), Phys.Rev. **D78**, 054513 (2008), arXiv:0805.2999 [hep-lat]
[21] C. Davies *et al.* (HPQCD), Phys.Rev. **D78**, 114507 (2008), arXiv:0807.1687 [hep-lat]
[22] S. Aoki *et al.* (PACS-CS), JHEP **0910**, 053 (2009), arXiv:0906.3906 [hep-lat]
[23] E. Shintani *et al.* (JLQCD), Phys.Rev. **D82**, 074505 (2010), arXiv:1002.0371 [hep-lat]
[24] B. Blossier *et al.*(2012), arXiv:1201.5770 [hep-ph]
[25] S. Bethke *et al.*(2011), arXiv:1110.0016 [hep-ph]
[26] I. F. Allison *et al.* (HPQCD, Fermilab Lattice, and UKQCD), Phys.Rev.Lett. **94**, 172001 (2005), arXiv:hep-lat/0411027 [hep-lat]
[27] A. Abulencia *et al.* (CDF), Phys.Rev.Lett. **96**, 082002 (2006), arXiv:hep-ex/0505076 [hep-ex]
[28] C. Aubin *et al.* (Fermilab Lattice, MILC, and HPQCD), Phys.Rev.Lett. **95**, 122002 (2005), arXiv:hep-lat/0506030 [hep-lat]
[29] M. Artuso *et al.* (CLEO), Phys.Rev.Lett. **95**, 251801 (2005), arXiv:hep-ex/0508057 [hep-ex]
[30] C. Aubin *et al.* (Fermilab Lattice, MILC, and HPQCD), Phys.Rev.Lett. **94**, 011601 (2005), arXiv:hep-ph/0408306 [hep-ph]
[31] K. Abe *et al.* (BELLE)(2005), arXiv:hep-ex/0510003 [hep-ex]
[32] J. Laiho, "Pion and kaon physics," Project X Physics Study (2012), `https://indico.fnal.gov/contributionDisplay.py?sessionId=9&contribId=55&confId=5276`
[33] S. Dürr *et al.* (BMW), Phys.Rev. **D81**, 054507 (2010), arXiv:1001.4692 [hep-lat]
[34] E. Follana, C. Davies, G. Lepage, and J. Shigemitsu (HPQCD), Phys.Rev.Lett. **100**, 062002 (2008), arXiv:0706.1726 [hep-lat]
[35] A. Bazavov *et al.* (MILC), PoS **LATTICE2010**, 074 (2010), arXiv:1012.0868 [hep-lat]
[36] Y. Aoki *et al.* (RBC and UKQCD), Phys.Rev. **D83**, 074508 (2011), arXiv:1011.0892 [hep-lat]
[37] V. Lubicz, F. Mescia, S. Simula, and C. Tarantino (ETM), Phys.Rev. **D80**, 111502 (2009), arXiv:0906.4728 [hep-lat]
[38] P. Boyle *et al.* (RBC and UKQCD), Eur.Phys.J. **C69**, 159 (2010), arXiv:1004.0886 [hep-lat]
[39] F. Farchioni *et al.* (ETM), PoS **LATTICE2010**, 128 (2010), arXiv:1012.0200 [hep-lat]
[40] A. Bazavov *et al.* (MILC), PoS **LATTICE2011**, 107 (2011), arXiv:1111.4314 [hep-lat]
[41] J. Laiho, E. Lunghi, and R. S. Van de Water, Phys. Rev. **D81**, 034503 (2010), updates at `http://www.latticeaverages.org`, arXiv:0910.2928 [hep-ph]
[42] G. Colangelo *et al.* (FLAG), Eur. Phys. J. **C71**, 1695 (2011), arXiv:1011.4408 [hep-lat]
[43] E. Gamiz *et al.* (HPQCD, UKQCD), Phys.Rev. **D73**, 114502 (2006), arXiv:hep-lat/0603023 [hep-lat]
[44] S. Dürr *et al.* (BMW), Phys.Lett. **B705**, 477 (2011), arXiv:1106.3230 [hep-lat]
[45] T. Bae *et al.* (SWME)(2011), arXiv:1111.5698 [hep-lat]
[46] J. Laiho, E. Lunghi, and R. Van de Water, PoS **Lattice 2011**, 018 (2012), updates at `http://www.latticeaverages.org`, arXiv:1204.0791 [hep-ph]
[47] J. Brod and M. Gorbahn, Phys.Rev.Lett. **108**, 121801 (2012), arXiv:1108.2036 [hep-ph]
[48] J. Brod, "Rare K Decays in the Standard Model," Project X Physics Study (2012), `https://indico.fnal.gov/getFile.py/access?contribId=23&sessionId=5&resId=0&materialId=slides&confId=5276`
[49] N. Christ, "Computing $K \to \pi\pi$ and $m_{K_L} - m_{K_s}$ from lattice QCD," Project X Physics Study (2012), `https://indico.fnal.gov/getFile.py/access?contribId=54&sessionId=9&resId=0&materialId=slides&confId=5276`
[50] A. J. Buras, G. Colangelo, G. Isidori, A. Romanino, and L. Silvestrini, Nucl. Phys. **B566**, 3 (2000), arXiv:hep-ph/9908371
[51] U. Haisch, "Some Thoughts About Rare Kaon Decays," Project X Physics Study (2012), `https://indico.fnal.gov/getFile.py/access?contribId=22&sessionId=5&resId=0&materialId=slides&confId=5276`

[52] T. Izubuchi, “New methods for HLbL,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=61&sessionId=9&resId=0&materialId=slides&confId=5276
[53] J. Yu, “Lattice calculation of the $K_L - K_S$ mass difference,” 30th International Symposium on Lattice Field Theory (2012), http://www.physics.adelaide.edu.au/cssm/lattice2012/talks/Meeting%20Room%203/Friday/1430-1610/Jianglei%20Yu/Jianglei%20Yu.pdf
[54] J. Brod, M. Gorbahn, and E. Stamou, Phys.Rev. **D83**, 034030 (2011), arXiv:1009.0947 [hep-ph]
[55] J. A. Bailey *et al.* (Fermilab Lattice and MILC), PoS **LATTICE2010**, 311 (2010), arXiv:1011.2166 [hep-lat]
[56] S.-W. Qiu *et al.* (Fermilab Lattice and MILC), PoS **Lattice 2011**, 289 (2011), arXiv:1111.0677 [hep-lat]
[57] G. Isidori, G. Martinelli, and P. Turchetti, Phys.Lett. **B633**, 75 (2006), arXiv:hep-lat/0506026 [hep-lat]
[58] C. Sachrajda, “Prospects for the computation of rare kaon-decay amplitudes,” New Horizons for Lattice Computations with Chiral Fermions (2012), https://indico.bnl.gov/getFile.py/access?contribId=30&resId=0&materialId=slides&confId=512
[59] B. Plaster, “High-Precision Measurements of g_A and g_V in Neutron and Nuclear β-Decay,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=64&sessionId=9&resId=0&materialId=slides&confId=5276
[60] H.-W. Lin and K. Orginos, Phys. Rev. **D79**, 034507 (2009), arXiv:0712.1214 [hep-lat]
[61] H.-W. Lin, T. Blum, S. Ohta, S. Sasaki, and T. Yamazaki, Phys. Rev. **D78**, 014505 (2008), arXiv:0802.0863 [hep-lat]
[62] T. Yamazaki *et al.* (RBC and UKQCD), Phys. Rev. Lett. **100**, 171602 (2008), arXiv:0801.4016 [hep-lat]
[63] J. D. Bratt *et al.* (LHPC), Phys. Rev. **D82**, 094502 (2010), arXiv:1001.3620 [hep-lat]
[64] D. Holmgren, “Future Computing Resources for Lattice Gauge Theory,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=63&sessionId=9&resId=0&materialId=slides&confId=5276
[65] T. Izubuchi, “New methods for HLbL,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=61&sessionId=9&resId=0&materialId=slides&confId=5276
[66] T. Bhattacharya *et al.*, Phys.Rev. **D85**, 054512 (2012), arXiv:1110.6448 [hep-ph]
[67] T. Izubuchi, “Proton Decay Matrix Elements from Lattice QCD,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=52&sessionId=9&resId=0&materialId=slides&confId=5276
[68] S. Cohen, “Probing TeV Physics through Neutron Decay Matrix Elements,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=52&sessionId=9&resId=0&materialId=slides&confId=5276
[69] S. Rao and R. Shrock, Phys.Lett. **B116**, 238 (1982)
[70] A. Martin and G. C. Stavenga, Phys.Rev. **D85**, 095010 (2012), arXiv:1110.2188 [hep-ph]
[71] G. Stavenga, “Skyrmion N-Nbar Suppression,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=139&sessionId=8&resId=0&materialId=slides&confId=5276
[72] M. Buchoff, “Lattice Calculations of Neutron-Antineutron Matrix Elements,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=140&sessionId=8&resId=0&materialId=slides&confId=5276
[73] E. Shintani, “Neutron EDM from Lattice QCD,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=49&sessionId=9&resId=0&materialId=slides&confId=5276
[74] E. Shintani, S. Aoki, and Y. Kuramashi, Phys. Rev. **D78**, 014503 (2008), arXiv:0803.0797 [hep-lat]
[75] E. Shintani *et al.*, Phys. Rev. **D72**, 014504 (2005), arXiv:hep-lat/0505022
[76] S. Aoki *et al.*(2008), arXiv:0808.1428 [hep-lat]
[77] T. Bhattacharya, “Neutron Electric Dipole Moment in the Standard Model and beyond from Lattice QCD,” Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=51&sessionId=9&resId=0&materialId=slides&confId=5276
[78] C. Aubin and T. Blum, Phys.Rev. **D75**, 114502 (2007), arXiv:hep-lat/0608011 [hep-lat]
[79] X. Feng, K. Jansen, M. Petschlies, and D. B. Renner, Phys.Rev.Lett. **107**, 081802 (2011), arXiv:1103.4818 [hep-lat]
[80] P. Boyle, L. Del Debbio, E. Kerrane, and J. Zanotti, Phys.Rev. **D85**, 074504 (2012), arXiv:1107.1497 [hep-lat]

[81] M. Della Morte, B. Jager, A. Juttner, and H. Wittig, JHEP **1203**, 055 (2012), arXiv:1112.2894 [hep-lat]
[82] C. Aubin, "Hadronic vacuum polarization contribution to muon $g-2$ using staggered fermions," Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=56&sessionId=9&resId=0&materialId=slides&confId=5276
[83] C. Aubin, T. Blum, M. Golterman, and S. Peris(2012), arXiv:1205.3695 [hep-lat]
[84] T. Blum, "The muon anomalous magnetic moment," 30th International Symposium on Lattice Field Theory (2012), http://www.physics.adelaide.edu.au/cssm/lattice2012/talks/Hall%20A/Friday/1640-1830/Tom%20Blum/blum.pdf
[85] D. Renner, "QCD vacuum-polarization corrections," Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=57&sessionId=9&resId=0&materialId=slides&confId=5276
[86] M. Hayakawa, T. Blum, T. Izubuchi, and N. Yamada, PoS **LAT2005**, 353 (2006), arXiv:hep-lat/0509016 [hep-lat]
[87] T. Blum, "Hadronic light-by-light contribution to the muon anomalous magnetic moment," Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=58&sessionId=9&resId=1&materialId=slides&confId=5276
[88] S. Cohen, "Neutral-Pion Two-Photon Decays from Lattice QCD," Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=60&sessionId=9&resId=1&materialId=slides&confId=5276
[89] X. Feng *et al.* (JLQCD)(2012), arXiv:1206.1375 [hep-lat]
[90] S. Peris, "The anomaly triangle and muon $g-2$," Project X Physics Study (2012), https://indico.fnal.gov/getFile.py/access?contribId=59&sessionId=9&resId=0&materialId=slides&confId=5276