PMNS and the number of additional neutrino flavors

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NuFact

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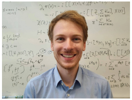
Based on

(i) *'Neutrino mixing, interval matrices and singular values'*,
Krzysztof Bielas, Wojciech Flieger, JG, Marek Gluza, PRD 98 (2018) 5, 053001





(ii) *'New limits on neutrino non-unitary mixings based on prescribed singular values'*,
W. Flieger, JG, Kamil Porwit, JHEP 03 (2020) 169

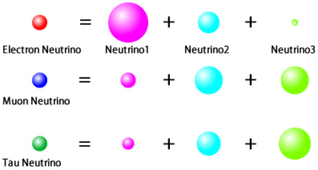
(iii) *'General neutrino mass spectrum and mixing properties in seesaw mechanisms'*,
W. Flieger, JG, Chin.Phys.C 45 (2021) 2, 023106

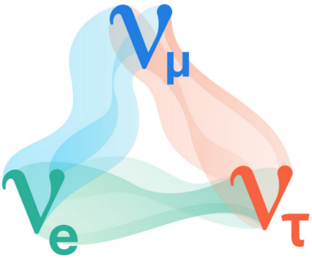
(iv) *'Geometry of the neutrino mixing space'*,
W. Flieger, JG, 2201.06036 [hep-ph], PRD in print

The Number 3 Stays with Us For Long: Neutrino Oscillations

Neutrino oscillations

$$\nu_\alpha^{(f)} = (U_{\mathrm{PMNS}})_{\alpha i}\nu_i^{(m)}$$





Mixing matrix

$$U_{\mathrm{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

source: http://www.hyper-k.org/en/index.html; https://neutrinos.fnal.gov

Setting

Experimental values of mixing parameters

$$\theta_{12} \in [31.61^\circ, 36.27^\circ], \quad \theta_{23} \in [41.1^\circ, 51.3^\circ],$$
$$\theta_{13} \in [8.22^\circ, 8.98^\circ], \quad \delta \in [144^\circ, 357^\circ]$$

Interval matrix build up from unitary matrices U_{PMNS} (3σ C.L.)

$$|U|_{int} = \begin{pmatrix} [0.797, 0.842] & [0.518, 0.585] & [0.143, 0.156] \\ [0.243, 0.490] & [0.473, 0.674] & [0.651, 0.772] \\ [0.295, 0.525] & [0.493, 0.688] & [0.618, 0.744] \end{pmatrix}$$

includes non-unitary matrices. $\delta \neq 0$: complex intervals, $\longrightarrow U_{int}$

- **Can we get from $|U|_{int}$ an additional information on existence and structure of hypothetical $N_\nu > 3$ states?**
- We explore U_{int} from matrix theory perspective.

Non-unitary Matrices and a Notion of Contractions

$$\|A\| \leq 1$$

Operator norm (spectral norm)

$$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sigma_{\max}(A)$$

Contractions as submatrices of the unitary matrix

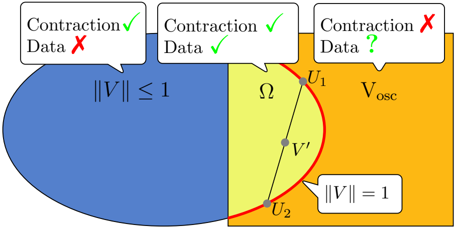
$$\text{If } UU^\dagger = 1 \Longrightarrow \left\| \begin{pmatrix} U_{3\times3} & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix} \right\| = 1 \Longrightarrow \|U_{3\times3}\| \leq 1.$$

PRD'2018.

Contractions allow us to determine the set of physically admissible mixing matrices $\Omega \subset U_{int}$

(I) U_{int} and the Physical Region of Mixing (Convex Hull of U_{PMNS})

$$\Omega := \mathrm{conv}(U_{\mathrm{PMNS}}) = \{\sum_{i=1}^{m} \alpha_i U_i \mid U_i \in U(3), \alpha_1, ..., \alpha_m \geq 0, \sum_{i=1}^{m} \alpha_i = 1,$$
$$\theta_{12}, \theta_{13}, \theta_{23} \text{ and } \delta \text{ given by experimental values}\}$$



We proved that the Carathéodory's number is **m** $\leq$ **4**, instead of 10(19) for CP ($\cancel{CP}$) cases, e.g., for the 3+1 scenario, two U_{PMNS} matrices are enough to span the corresponding subset of Ω region.

Fig. from PRD2018, $V_{osc} \equiv U_{int}$

(II) Physical Region Can Be Divided into Non-Overlapping Subregions !

`Not all` $U_{3\times 3}$ `entries known well (precision)` - see backup - `hard to avoid analysis based on Euler angles.`
Nonetheless, we can use the knowledge of Ω differently.

Ω is divided into four disjoint subsets by *singular values* (PRD2018)

Ω_1 : 3+**1** scenario: $\Sigma = \{\sigma_1 = 1.0, \sigma_2 = 1.0, \boldsymbol{\sigma_3 < 1.0}\}$,
Ω_2 : 3+**2** scenario: $\Sigma = \{\sigma_1 = 1.0, \boldsymbol{\sigma_2 < 1.0}, \boldsymbol{\sigma_3 < 1.0}\}$,
Ω_3 : 3+**3** scenario: $\Sigma = \{\boldsymbol{\sigma_1 < 1.0}, \boldsymbol{\sigma_2 < 1.0}, \boldsymbol{\sigma_3 < 1.0}\}$,
Ω_4 : PMNS scenario: $\Sigma = \{\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 1\}$.

$$\sigma_i(A) = \sqrt{\lambda_i(AA^\dagger)}$$

The connection between $\Sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $3 + N$ scenarios, with N additional νs, goes by the *dilation* procedure.

Unitary dilation

Unitary dilation: an operation that extends a matrix which is a contraction to a unitary matrix of an appropriate dimension

$$U_{\mathtt{int}} \xrightarrow{\text{dilation}} \begin{pmatrix} ||U_{int}|| \leq 1 & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix} \equiv U \rightarrow UU^{\dagger} = I$$

To find a unitary dilation of possible smallest dimension n: **CS decomposition**

$$U \equiv \begin{pmatrix} ||U_{int}|| \leq 1 & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array}\right) \begin{pmatrix} Q_1^{\dagger} & 0 \\ 0 & Q_2^{\dagger} \end{pmatrix}$$

$C \geq 0$ and $S \geq 0$ — diagonal matrices satisfying $C^2 + S^2 = I_{\mathbf{m}}$

$W_1, Q_1 \in M_{n-m \times n-m}$ and $W_2, Q_2 \in M_{m \times m}$ — unitary matrices.

C, S, W_1, Q_1 — fixed by the singular value decomposition of the $U_{\mathtt{int}}$.

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$C \geq 0$ and $S \geq 0$ — diagonal matrices satisfying $C^2 + S^2 = I_{\mathbf{m}}$

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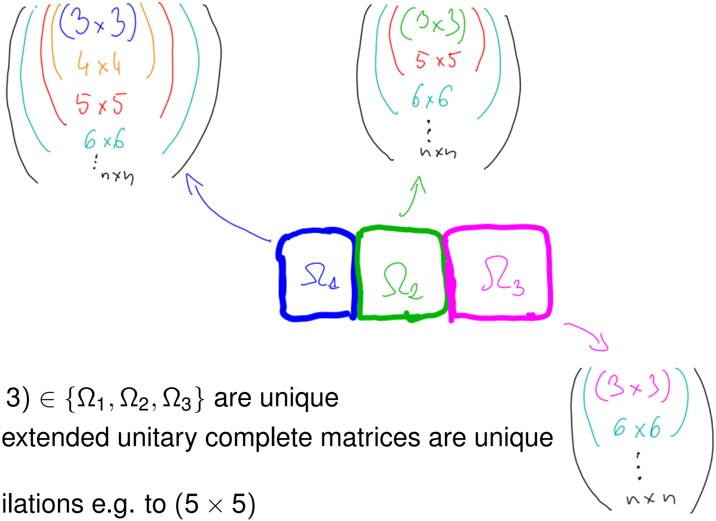
C, S, W_1, Q_1 — fixed by the singular value decomposition of the $U_{\mathtt{int}}$.

The dimension **m** of the defect space, is the minimal number of new neutrino species necessary to ensure unitarity.

Possible (Minimal) Extensions for Ω Subsets



1. $(3 \times 3) \in \{\Omega_1, \Omega_2, \Omega_3\}$ are unique
   $\longrightarrow$ extended unitary complete matrices are unique

2. Ω_2 dilations e.g. to (5×5)
   must be completed with Ω_1 dilations to (5×5), etc.

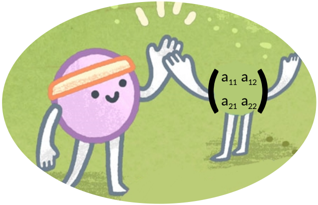
Matrix Theory and Neutrinos: Summary Before Analytical/Numerical Results

$$\mathbf{U}_{\mathrm{PMNS}} \longrightarrow \mathbf{U}_{\mathrm{int}}$$

Mixing & interval matrix

$$\sigma_i(A) = \sqrt{\lambda_i(A A^\dagger)}$$

Singular values



$$\begin{pmatrix} ||\mathbf{U}_{\mathrm{int}}|| \leq \mathbf{1} & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix}$$

Unitary dilation

$$||A|| \leq 1$$

Contractions

source (inside picture): https://www.symmetrymagazine.org

Questions, based on the knowledge of U_{int}

Q1 How much space do we have for the additional neutrinos and how quantify it within our approach?

Q2 Can we distinguish between $\Omega_1 - \Omega_3$ (3+n models) using U_{int}?

Q3 Can we estimate active-sterile mixing using singular values and U_{int}?

α-Parametrization and Prescribed Singular Values (backup slides)

α- parametrization of the deviation from unitarity in the neutrino sector:

$$\cancel{U}_{\mathtt{PMNS}} = (I - \alpha)W = TW,$$

where W is a unitary matrix and $T = I - \alpha$ is a lower triangular matrix [Zhi-Zhong Xing, 2008].

The method of analysis:
The construction of lower triangular matrices with prescribed singular values and eigenvalues [C-K. Li and R. Mathias, 2004]

$$\begin{aligned} \prod_{i=1}^{k} |\lambda_i| &\leq \prod_{i=1}^{k} \sigma_i \\ \Sigma &= \{\sigma_1, \sigma_2, \sigma_3\} \to \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \subset T = \begin{pmatrix} T_{11} & 0 & 0 \\ T_{21} & T_{22} & 0 \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \end{aligned}$$

Data

The limits for the T matrix (95% CL):

$$T = \begin{pmatrix} T_{11} & 0 & 0 \\ T_{21} & T_{22} & 0 \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

Entry	(I): $m >$ EW	(II): $\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$	(III): $\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$
$T_{11} = 1 - \alpha_{11}$	$0.99870 \div 1$	$0.976 \div 1$	$0.990 \div 1$
$T_{22} = 1 - \alpha_{22}$	$0.99978 \div 1$	$0.978 \div 1$	$0.986 \div 1$
$T_{33} = 1 - \alpha_{33}$	$0.99720 \div 1$	$0.900 \div 1$	$0.900 \div 1$
$T_{21} = \lvert\alpha_{21}\rvert$	$0.0 \div 0.00068$	$0.0 \div 0.025$	$0.0 \div 0.017$
$T_{31} = \lvert\alpha_{31}\rvert$	$0.0 \div 0.00270$	$0.0 \div 0.069$	$0, 0 \div 0.045$
$T_{32} = \lvert\alpha_{32}\rvert$	$0.0 \div 0.00120$	$0.0 \div 0.012$	$0.0 \div 0.053$

[M. Blennow et al., 2017]

Q1. Amount of Space for n Neutrinos: Singular Values

		σ_1	σ_2	σ_3
3+1	$m >$ EW			0.9968
	$\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$			0.900
	$\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$			0.889
3+2	$m >$ EW		0.9987	0.9986
	$\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$		0.976	0.975
	$\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$		0.986	0.985
3+3	$m >$ EW	0.9998	0.9996	0.9996
	$\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$	0.979	0.977	0.9773
	$\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$	0.991	0.989	0.989

Error: 0.00003 (follows from Weyl's inequality, see backup slides)

(I): Fidelity of Results: Comparison with a Quark Sector

Wolfenstein parametrization ($s_{12} = \lambda, \quad s_{23} = A\lambda^2, \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$)

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(\rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

Experimental values ($\bar{\rho} = \rho(1 - \lambda^2/2)$ and $\bar{\eta} = \eta(1 - \lambda^2/2)$)

$$\lambda = 0.22506 \pm 0.00050, \quad A = 0.811 \pm 0.026, \bar{\rho} = 0.124^{+0.019}_{-0.018}, \quad \bar{\eta} = 0.356 \pm 0.011,$$

$$\begin{aligned} \sigma_1 &\in [0.99997, 1.00101], \\ \sigma_2 &\in [0.99965, 1.00037], \\ \sigma_3 &\in [0.99890, 1.00002]. \end{aligned}$$

Distribution of contractions

All matrices within V_{CKM} are contractions with 2 permil accuracy

6% of $\|\mathrm{V_{CKM}}\| = 1.002,$

94% of $\|\mathrm{V_{CKM}}\| = 1.001$

$$0.961 \leq \|\mathrm{U_{int}}\| \leq 1.178$$

Q1. Amount of Space for n Neutrinos: Geometry and Volume

The Ω region is a subset of the unit ball of the spectral norm

$$\mathcal{B}(n) = \{A \in \mathbb{C}^{n\times n} : \|A\| \leq 1\}.$$

This fact allows us to give another characterization of the Ω region as the intersection of the $\mathcal{B}(3)$ with the interval matrix U_{int} i.e.

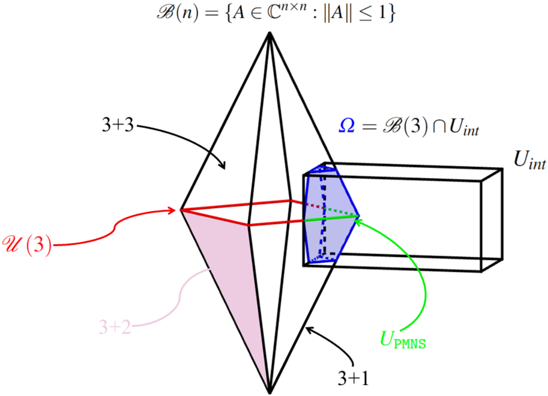
$$\Omega = \mathcal{B}(3) \cap U_{int}. \tag{1}$$

Faces of the Ω region:

$$\mathcal{F} = \{U \begin{pmatrix} I_r & 0 \\ 0 & A \end{pmatrix} V : A \in \mathcal{B}(n-r)\}.$$

Subsets $\Omega_1, \ldots, \Omega_4$ of the region Ω are (relative) interiors of faces $\mathcal{F}$ of the unit ball $\mathcal{B}(3)$ under parameters restriction for U and V to the experimental data and admissible eigenvalues of A.

From a Human Point of View



The extreme points of the $\mathcal{B}(n)$ are unitary matrices.
Edges correspond to $3 + 1$.
Sides to $3 + 2$.
Interiors to $3 + 3$.

(II): Physical Volume Vanishes with Descending Errors in U_{int}

CP-conserving	CP-violating
Total volumes	
$\mathcal{SO}(3) \subset \mathcal{O}(3) \subset \tilde{\mathcal{B}}(3)$	$\mathcal{SU}(3) \subset \mathcal{U}(3) \subset \mathcal{B}(3)$
Experimentally restricted volumes	
$O_{\mathtt{PMNS}} \subset \tilde{\Omega} = \tilde{\mathcal{B}}(3) \cap O_{int}$	$U_{\mathtt{PMNS}} \subset \Omega = \mathcal{B}(3) \cap U_{int}$

E.g.:

$$vol(\mathcal{B}(n)) = \frac{1}{(2\pi)^n n!} vol(\mathcal{U}(n))^2 \int_0^1 \prod_{k=1}^{n} \sigma_k \prod_{i<j} |\sigma_j^2 - \sigma_i^2|^2 \prod_{k=1}^{n} d\sigma_k .$$

$$vol(\Omega) = \frac{1}{(2\pi)^3 3!} vol(PMNS)^2 \int_{\sigma_{min}}^1 \prod_{k=1}^{n} \sigma_k \prod_{i<j} |\sigma_j^2 - \sigma_i^2|^2 \prod_{k=1}^{n} d\sigma_k .$$

Measurement fidelity:

$$\mathcal{D} \equiv 1 - \frac{I_\Omega}{I_{\mathcal{B}(3)}}$$

$$\begin{aligned}
\mathcal{D}_{\cancel{CP}} &= 1 - 2.19 \times 10^{-4} = 0.999781, \\
\tilde{\mathcal{D}}_{CP} &= 1 - 1.8 \times 10^{-5} = 0.999982, \\
\mathcal{D}_{\mathtt{CKM}} &= 1 - 1.077 \times 10^{-26}.
\end{aligned}$$

Q2: Can we distinguish between $\Omega_1 - \Omega_3$ (3+n models)?

$$\cancel{U}_{\mathtt{PMNS}} = (I - \alpha)W = TW,$$

Data are global, for 3+n

Entry	$(\mathtt{I})$: $m >$ EW	$(\mathtt{II})$: $\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$	$(\mathtt{III})$: $\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$
$T_{11} = 1 - \alpha_{11}$	$0.99870 \div 1$	$0.976 \div 1$	$0.990 \div 1$
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$T_{31} = \lvert\alpha_{31}\rvert$	$0.0 \div 0.00270$	$0.0 \div 0.069$	$0, 0 \div 0.045$
$T_{32} = \lvert\alpha_{32}\rvert$	$0.0 \div 0.00120$	$0.0 \div 0.012$	$0.0 \div 0.053$

$$\Sigma = \{\sigma_1, \sigma_2, \sigma_3\} \to \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \subset T = \begin{pmatrix} T_{11} & 0 & 0 \\ T_{21} & T_{22} & 0 \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

The 3+1 scenario ($\sigma_3 < 1$): Results

	$(\mathrm{I}): m > \mathrm{EW}$	$(\mathrm{II}): \Delta m^2 \gtrsim 100\ \mathrm{eV}^2$	$(\mathrm{III}): \Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$
(1, 1)	0.99885 ÷ 0.99999	0.97641 ÷ 0.99996	0.99020 ÷ 0.99999
Exp:	0.99870 ÷ 1	0.976 ÷ 1	0.990 ÷ 1
(2, 2)	0.99980 ÷ 0.99999	**0.99331** ÷ 0.99999	0.98646 ÷ 0.99999
Exp:	0.99978 ÷ 1	**0.978** ÷ 1	0.986 ÷ 1
(3, 3)	0.99721 ÷ 0.99996	0.90040 ÷ 0.99985	0.90015 ÷ 0.99958
Exp:	0.99720 ÷ 1	0.900 ÷ 1	0.900 ÷ 1
(2, 1)	0.00001 ÷ 0.00062	0.00031 ÷**0.02214**	0.00014 ÷ 0.01615
Exp:	0.0 ÷ 0.00068	0.0 ÷ **0.025**	0.0 ÷ 0.017
(3, 1)	0.00002 ÷ 0.00266	0.00048 ÷ 0.06892	0.00012 ÷ 0.04500
Exp:	0.0 ÷ 0.00270	0.0 ÷ 0.069	0.0 ÷ 0.045
(3, 2)	0.00008 ÷ 0.00113	0.00052 − 0.01196	0.00024 ÷ 0.05281
Exp:	0.0 ÷ 0.00120	0.0 ÷ 0.012	0.0 ÷ 0.053

Q2: Can we distinguish between $\Omega_1 - \Omega_3$ (3+n models)?

- 3+1 is different. So far no distinction between 3+2 and 3+3 scenarios is possible (results overlap with T).
- Note non-zero lower bounds (in blue), the biggest differences are in red.

Q3: Can We Estimate Active-Sterile Mixing Using Singular Values and U_{int}?

Ω_1 : 3+1 scenario: $\Sigma = \{\sigma_1 = 1.0, \sigma_2 = 1.0, \sigma_3 < 1.0\}$

$$\begin{pmatrix} \cancel{U}_{\mathtt{PMNS}} & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ \hline 0 & 0 & s & c \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}.$$

We are interested in the estimation of the light-heavy mixing sector which is given by

$$U_{lh} = W_1 S_{12} Q_2^\dagger,$$

where $W_1 \in \mathbb{C}^{3\times 3}$ is unitary, $S_{12} = (0, 0, -s)^T$ and $Q_2 = e^{i\theta}, \theta \in (0, 2\pi]$.
Taking into account exact values of the W_1 we can estimate the light-heavy mixing by the analytical formula

$$|U_{i4}| = |w_{i3}| \cdot |\sqrt{1 - \sigma_3^2}|, \quad i = e, \mu, \tau.$$

Answer to Q3: Estimation of the "light-heavy" mixing: Results for 3+1

Estimation of the "light-heavy" mixing via CS decomposition

- (I): $m > \mathrm{EW}$.

  $$Ours : |U_{e4}| \in [0, 0.021] , \quad |U_{\mu 4}| \in [0.00013, 0.021] , \quad |U_{\tau 4}| \in [0.0115, 0.075] .$$
  $$Others : |U_{e4}| \leq 0.041 , \quad |U_{\mu 4}| \leq 0.030 , \quad |U_{\tau 4}| \leq 0.087$$ [J. de Blas, 2013]

- (II): $\Delta m^2 \gtrsim 100\ \mathrm{eV}^2$.

  $$Ours : |U_{e4}| \in [0, 0.082] , \quad |U_{\mu 4}| \in [0.00052, 0.099] , \quad |U_{\tau 4}| \in [0.0365, 0.44] .$$

- (III): $\Delta m^2 \sim 0.1 - 1\ \mathrm{eV}^2$.

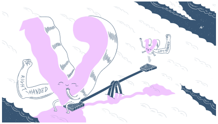
  $$Ours : |U_{e4}| \in [0, 0.130] , \quad |U_{\mu 4}| \in [0.00052, 0.167] , \quad |U_{\tau 4}| \in [0.0365, 0.436] .$$
  $$Others : |U_{e4}| \in [0.114, 0.167] , \quad |U_{\mu 4}| \in [0.0911, 0.148] , \quad |U_{\tau 4}| \leq 0.361 .$$
  [C. Giunti et al., 2017] [M. Dantler et al., 2018]

$\longrightarrow$ In some cases we improved (blue), in some not (red).

How Light and Heavy Masses (Eigenvalues) Influence Active-Sterile Mixings (Eigenvectors)?

Seesaw (SS) mass matrix

$$M_{SS} = \begin{pmatrix} 0 & M_D \\ M_D^T & M_R \end{pmatrix}, |M_D| \ll |M_R|$$



https://www.symmetrymagazine.org

When 3 light νs? SS-I, II, III, ESS, ISS, LSS - we can rearange to the same structure,
W. Flieger, JG, Chin.Phys.C 45 (2021) 2, 023106

$$|M_D| \ll \lambda(M_R),\ \lambda(M_{SS}) \simeq \lambda(M_R) \pm |M_D|$$

A relation between light and heavy masses and their mixings

$$\| \sin \Theta(V_{Light}, V'_{heavy}) \| \leq \frac{1}{\delta} \| M_{SS} - M_R \| = \frac{1}{\delta} \| M_D \|,\ \delta = min(M_{N_i}) - max(m_{\nu_j})$$

P. Denton et al, Bull.Am.Math.Soc. 59 (2022) 1

$$|v_{i,j}|^2 \prod_{k=1;k\neq i}^{n} (\lambda_i(A) - \lambda_k(A)) = \prod_{k=1}^{n-1} (\lambda_i(A) - \lambda_k(M_j)) \ .$$

To be (3ν) or not to be (3ν)

We investigated U_{int} within matrix theory defining admissible region Ω and analyzing Ω geometrical structure, and based on that we discussed:

Q1 How much space do we have for the additional neutrinos and how quantify it within our approach?
$\Longrightarrow$ **A lot.**
We can judge on that using U_{int} (singular values of U_{int} and volumes of admissible CP, $\cancel{CP}$ mixing matrices).

Q2 Can we distinguish between $\Omega_1 - \Omega_3$ (3+N models) using U_{int}?
$\longrightarrow$ **Difficult.**
3+1 is different. So far no distinction between 3+2 and 3+3 scenarios is possible (results overlap with complete T-matrix data in both cases).

Q3 Can we estimate active-sterile mixing using singular values and U_{int}?
$\longrightarrow$ **Yes.**
We estimated it in the 3+1 scenario using dilation and CS decomposition, getting for some Δm^2 mass scenarios better constraints than from other analysis (slide 22).

To be (3ν) or not to be (3ν)

- Better precision of future ν-experiments can open a way to distinguish between $3+n$ mixings using the approach based on prescribed singular values.
- Matrix theory can help to find relations between eigenvectors, eigenvalues of light and heavy mass spectrum $\longrightarrow$ mass matrix modeling.

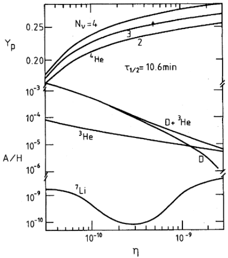
Thank you for your attention, and the Organizers for the invitation.

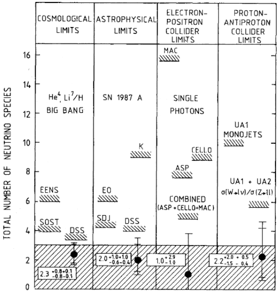
Backup slides

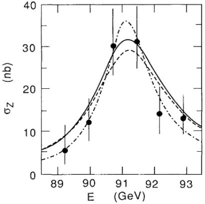
N_{eff}: (Good) Things Come in 3s?

The Number of Neutrino Species,

D. Denegri, B. Sadoulet, M. Spiro, Rev.Mod.Phys. 62 (1990) 1







SLC, $N_\nu = 3.8 \pm 1.4$

106 Z events

1989:

Initial measurements of Z-boson resonance parameters in e^+e annihilation, SLC Colaboration

Phys. Rev. Lett. 63, 724

$N_{\rm eff}$: LEP and Now

ALEPH, OPAL, L3, DELPHI, MARKII (SLC): $N_\nu = 3.12 \pm 0.19$
CERN, 13.10.1989, Video ($\sim$12,000 Z decays)
[LEP, 2006] ($\sim$17 mln Z decays)

$$N_\nu = 2.9840 \pm 0.0082$$

Update: [P. Janot and S. Jadach, 2019](only 1σ off from N=3)

$$N_\nu = 2.9963 \pm 0.0074$$

Theorem: [C. Jarlskog, 1990]
In the Standard Model with n left-handed lepton doublets and $N - n$ right-handed neutrinos, the effective number of neutrinos, N_ν, defined by

$$\Gamma(Z \to \nu' s) \equiv N_\nu \Gamma_0,$$

where Γ_0 is the standard width for one massless neutrino, satisfies

$$N_\nu \leq n.$$

Cosmology: $N_{eff} = 3.044$. J. Froustey, C. Pitrou, M. Volpe, JCAP 12 (2020) 015, J. Bennett, G. Buldgen, M. Drewes, Y. Wong, JCAP 03 (2020) 003, JCAP 03 (2021) A01

Determination of U_{PMNS} entries

$$U_{PMNS} = \begin{pmatrix} U_{e1} & U_{e2} & \mathbf{U_{e3}} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

Appearence/Disappearence, SBL/LBL experiments sensitive to different U_{PMNS} entries or their combinations.

- E.g., $\mathbf{U_{e3}}$ - Daya Bay ($\bar{\nu}_e$ disappearence).
- Least knowledge about the τ entries.

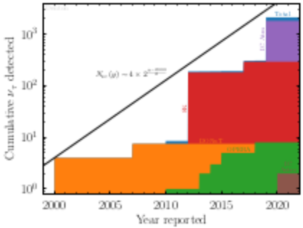
Determination of U_{PMNS} entries

Unitarity violation: tau row

Leptons: tau row is the weakest

1. Existing global analyses use OPERA and SNO
2. More data from atmospheric ν_τ appearance!

PBD 2109.14576



PBD, et al. 2203.05591 (whitepaper)

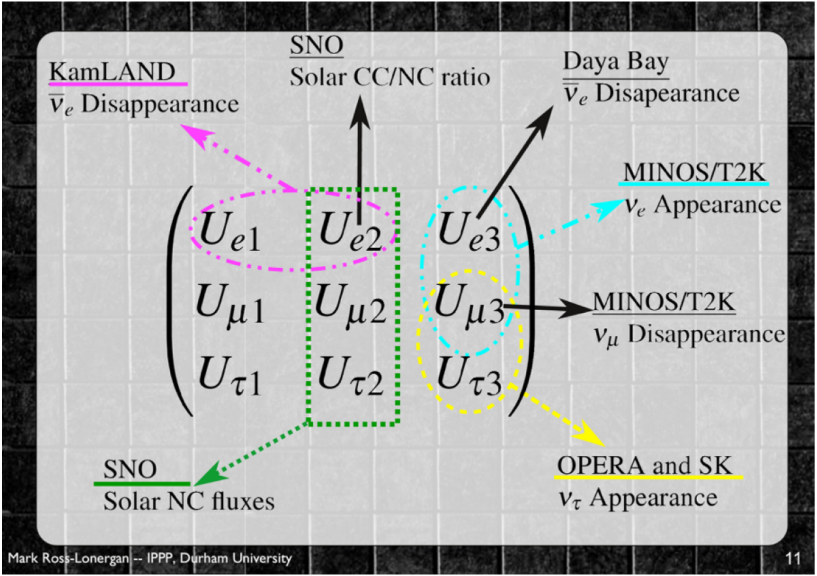
Also astrophysical ν_τ appearance; weak but distinct!

PBD, J. Gehrlein 2109.14575

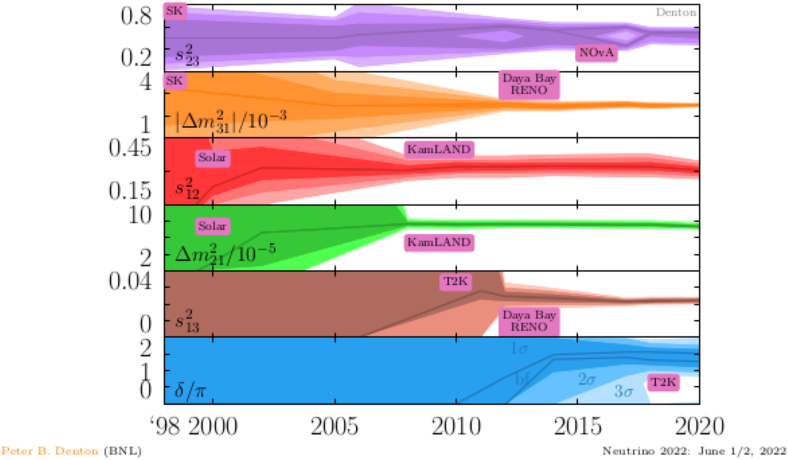
Works because τ is in direct region

Tau neutrino data set doubles every two years!

Peter B. Denton (BNL) 2109.14575 & 2109.14576 Neutrino 2022: June 1/2, 2022 32/34


SNO
Solar CC/NC ratio
KamLAND
$\overline{\nu}_e$ Disappearance
Daya Bay
$\overline{\nu}_e$ Disapearance
MINOS/T2K
ν_e Appearance
$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$
MINOS/T2K
ν_μ Disappearance
SNO
Solar NC fluxes
OPERA and SK
ν_τ Appearance
Mark Ross-Lonergan -- IPPP, Durham University
11

Determination of U_{PMNS} entries



Singular values

Singular values σ_i of a given matrix A are positive square roots of the eigenvalues λ_i of the matrix $AA^\dagger$

$$\sigma_i(A) = \sqrt{\lambda_i(AA^\dagger)}$$

Properties:

- generalization of eigenvalues
- always non-negative
- stable under perturbations

Unitary matrices

$$UU^\dagger = I = diag(1, 1, ..., 1) \Longrightarrow \text{all singular values equal to 1}$$

Unitarity and Contraction: a Toy/Naive Example

For U_{PMNS} holds

$$\sum_{\alpha} P_{\alpha\beta} = 1,$$

However, for a nonunitary U this relation is not fulfilled. $\Theta_2 = \Theta_1 + \epsilon$

$$U_{toy} = \begin{pmatrix} \cos\Theta_1 & \sin\Theta_1 \\ -\sin\Theta_2 & \cos\Theta_2 \end{pmatrix}$$

In this case we get, $\Delta_{ij} \propto (m_i^2 - m_j^2)\frac{L}{E}$

$$P_{ee} + P_{e\mu} = 1 + 4\epsilon \sin^2\Delta_{21} \sin\Theta_1 \cos\Theta_1 \cos 2\Theta_1 + \mathcal{O}(\epsilon^2)$$

$$P_{\mu e} + P_{\mu\mu} = 1 - 4\epsilon \sin^2\Delta_{21} \sin\Theta_1 \cos\Theta_1 \cos 2\Theta_1 + \mathcal{O}(\epsilon^2)$$

Calculating the contraction, we get a unique answer about the non-physical features of matrices:

$$\|U_{toy}\| \geq 1.$$

Unitary dilation: an example

As an illustration let us take two U_{PMNS} matrices

$$U_1 : \ \theta_{12} = 31.38^\circ, \theta_{23} = 38.4^\circ, \theta_{13} = 7.99^\circ,$$
$$U_2 : \ \theta_{12} = 35.99^\circ, \theta_{23} = 52.8^\circ, \theta_{13} = 8.90^\circ,$$

and let us construct a contraction as

$$V = \frac{1}{2}U_1 + \frac{1}{2}U_2,$$

The set of singular values

$$\sigma_1(V) = 1, \ \boldsymbol{\sigma_2(V) = 0.991}, \ \boldsymbol{\sigma_3(V) = 0.991}$$

for which we get the following unitary dilation

$$U = \left(\begin{array}{ccc|cc} 0.822411 & 0.548133 & 0.146854 & 0.0169583 & -0.0368511 \\ -0.468394 & 0.520442 & 0.70103 & -0.133845 & 0.0197681 \\ 0.311417 & -0.643236 & 0.686702 & 0.0250273 & 0.130689 \\ \hline -0.0524981 & 0.122242 & -0.0336064 & 0.599485 & 0.788536 \\ -0.0671638 & 0.00403263 & 0.119588 & 0.788536 & -0.599485 \end{array}\right)$$

Amount of space for n neutrinos: Analysis

- Construction of matrices with prescribed singular values, e.g., in 3+1 scenario we take $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 < 1$, together with the requirement on the elements to stay within experimental limits.
- Go with the "free" singular values as low as possible, e.g., in the 3+1 scenario we take σ_3 the smallest possible.

Distinction of the 3+1 scenario: Analysis

$$\sigma_1 = \sigma_2 = 1.$$

- In each massive scenario 10^8 matrices are produced, starting from σ_3 as large as possible and lowering it systematically to the smallest obtained value (previous slide).
- For each value of σ_3 the smallest and the largest values of produced matrix elements are taken.
- Repeating the procedure over possible σ_3 values, the allowed ranges of the 3×3 matrix elements are determined.

Narrowing mixing spreads for individual sing. val.

- Generation of matrices with a prescribed set of singular values and with elements within experimental ranges.
- From the set of these matrices take the smallest and the largest value of each element.

E.g.: $\Delta m^2 \gtrsim 100\ \mathrm{eV}^2, \Sigma = \{1, 1, 0.900\}$:

$$|A_{0.900}| = \begin{pmatrix} 0.999623 \div 0.999999\ (1.5\%) & 0 & 0 \\ 0.000002 \div 0.000753\ (3\%) & 0.999623 \div 0.999999\ (2\%) & 0 \\ 0.000606 \div 0.011919\ (16\%) & 0.000606 \div 0.011923\ (94\%) & 0.900002 \div 0.900678\ (1\%) \end{pmatrix}$$

Values in the brackets represent the percentage of the current experimental bounds.
For the other massive cases these values do not exceed 15%.

Matrix norm

A matrix norm is a function $\|\cdot\|$ from the set of all complex (real matrices) into $\mathbb{R}$ that satisfies the following properties

$$\begin{aligned}
&\|A\| \geq 0 \text{ and } \|\mathrm{A}\| = 0 \iff \mathrm{A} = 0, \\
&\|\alpha A\| = |\alpha| \|A\|, \alpha \in C, \\
&\|A + B\| \leq \|A\| + \|B\|, \\
&\|AB\| \leq \|A\| \|B\|
\end{aligned}$$

Examples of matrix norms

- spectral norm: $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$
- Frobenius norm: $\|A\|_F = \sqrt{Tr(A^\dagger A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\sum_{i=1}^n \sigma_i^2}$
- maximum absolute column sum norm: $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_\infty = \max_j \sum_i |a_{ij}|$
- maximum absolute row sum norm: $\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$

Weyl's inequality for singular values

Let A and B be a $m \times n$ matrices and let $q = \min\{m, n\}$. Then

$$\sigma_j(A + B) \leq \sigma_i(A) + \sigma_{j-i+1}(B) \textit{ for } i \leq j$$

Error Estimation

Let us assume that the V matrix which realizes some *BSM* scenario includes an error matrix E which is of the form $V+E$. Using Weyl inequalities for decreasingly ordered pairs of singular values of V and $V+E$, the following relation takes place

$$|\sigma_i(V+E)-\sigma_i(V)| \leq ||E||.$$

A precision for elements of the A in the $m > \mathrm{EW}$ is 10^{-5}. In our analysis we keep the same precision for all massive cases. This does not contradict experimental results since we still work within experimentally established intervals. Thus, all entries of Error matrix can be taken as $E_{ij} \approx 0.00001$. Therefore, uncertainty of the calculated singular values is bounded by $||E|| = 0.00003$.

Algorithm

The following steps lead to a contraction settled by U_{PMNS} and then to its unitary dilation of a minimal dimension

1) Select a finite number of unitary matrices U_i, $i = 1, 2, ...m$, within experimentally allowed range of parameters θ_{13}, θ_{23} and δ.
2) Construct a contraction U_{11} as a convex combination of selected matrices U_i

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_1, ..., \alpha_m \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

3) Find singular value decomposition of V, i.e.

$$V = W_1 \Sigma Q_1^\dagger$$

where W_1, Q_1 are unitary, Σ is diagonal, and determine number η of singular values strictly less than 1.
4) Use CS decomposition

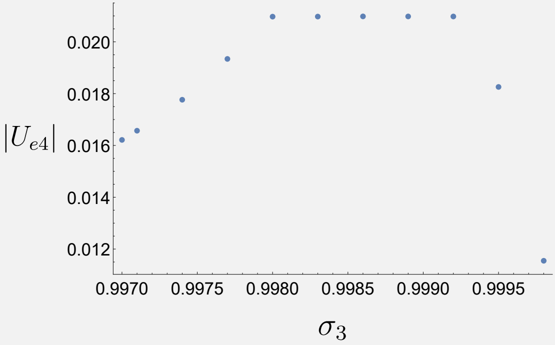
$$U = \begin{pmatrix} U_{\mathrm{PMNS}} & U_{lh} \\ U_{hl} & U_{hh} \end{pmatrix} =$$

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array} \right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

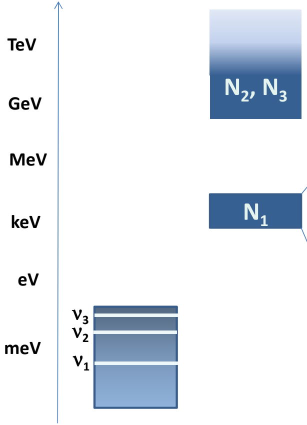
to find the unitary dilation $U \in \mathbb{M}_{(3+\eta)\times(3+\eta)}$ of contraction U_{11}.

Results on slide 21 have been obtained by taking exact maximal values of w_{e3}, $w_{\mu 3}$ and $w_{\tau 3}$, which follow from the singular value decomposition.

$$|U_{i4}| = |w_{i3}| \cdot |\sqrt{1-\sigma_3^2}|, \quad i = e, \mu, \tau.$$

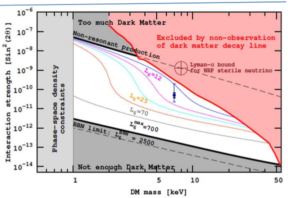


FCC-ee, Tera-Z option



can generate Baryon Asymmetry of Universe
if $m_{N2,N3}$ > 140 MeV

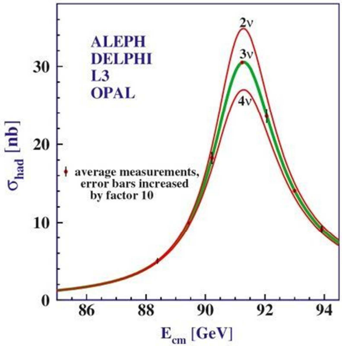
constrained:
mass: 1-50 keV
mixing :
10^{-7} to 10^{-13}
decay time:
$\tau_{N1} > \tau_{Universe}$



$N_1 \rightarrow \nu\,\gamma$ *may* have been seen:
arxiv:1402:2301 and arxiv:1402.4119

Blondel et al 1411.5230

$$n_\nu \equiv \left(\frac{\Gamma_{inv}}{\Gamma_{lept}}\right)^{meas} \Big/ \left(\frac{\Gamma_{\nu\bar{\nu}}}{\Gamma_{lept}}\right)^{SM}$$



$$N_\nu = 2.9840 \pm 0.0082 \quad ALEPH,\ 2005$$