Preparing for MUonE experiment: What can we learn from lattice and dispersive data?

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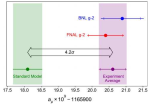
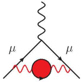
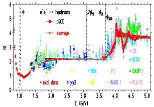
Outline

- Experiment vs Theory

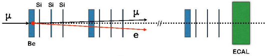


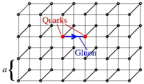
- Theory: HVP



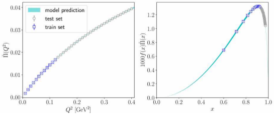


- HVP $\left\{ \begin{array}{l} \text{Timelike} \\ \text{Spacelike} \left\{ \begin{array}{l} \text{MUonE} \\ \text{Lattice} \end{array} \right. \end{array} \right.$

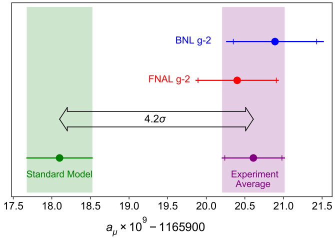




- Fitting procedure & its systematics



a_μ: Experiment vs Theory



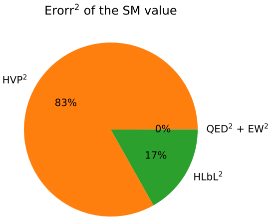
Exper.: Average of FNAL & BNL experiments

Theory: The Muon $g-2$ Theory Initiative

[arXiv:2104.03281]

Contribution	value $\times 10^{11}$
QED	$116\,584\,718.931(104)$
Electroweak	$153.6(1.0)$
HVP	$6845(40)$
HLbL	$92(18)$
Total SM value	$116\,591\,810(43)$
Difference: $a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	$251(59)$

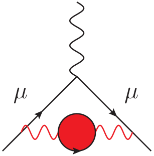
[arXiv:2006.04822, arXiv:2203.15810]

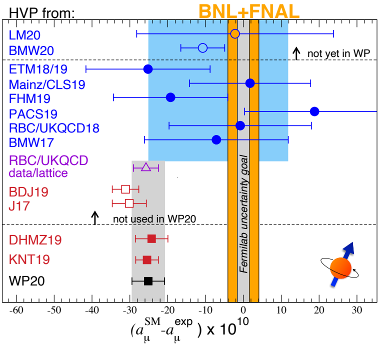


The SM value & HVP contribution

Progress to improve HVP precision:

- Current approaches:
  - Data-driven approach based on dispersive relations
  - Lattice QCD: calculations in progress
- New approaches: MUonE Experiment





red: data-driven results
blue: lattice-QCD calculations
blue band: lattice-QCD average (WP20)
black: (& gray band) the SM prediction (WP20)
filled: included in WP20
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[image: arXiv:2203.15810 (Snowmass 2021)]
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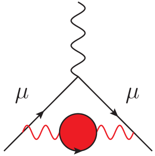

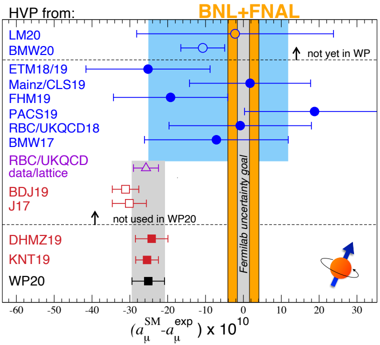

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New lattice calculations & some tension with data driven results

Data driven approach: dispersive methods

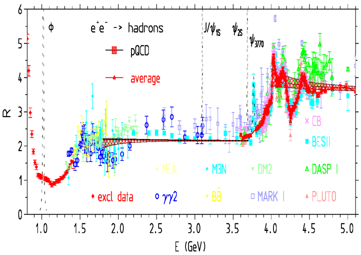
- Relation between the $\mathcal{R}e\,\Pi(Q^2)$ and $\mathcal{I}m\,\Pi(Q^2)$:

$$\boxed{\Pi(Q^2) - \Pi(0) = \frac{Q^2}{\pi} \int_0^\infty ds \frac{\mathcal{I}m\,\Pi(s)}{s(s - Q^2)}}$$

- Imaginary part of $\Pi(s)$ is related to the experimental total cross-section in e+e- annihilation:

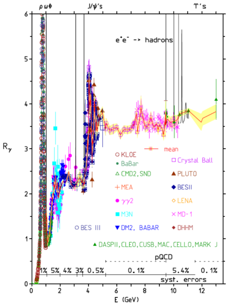
$$\boxed{\mathcal{I}m\,\Pi(s) = \frac{\alpha}{3} R(s)}$$

- Important contributions : $\rho, \omega, \phi, J/\psi$
- **O(1000)** channels
- Model calculations had to be used for some channels
- [Keshavarzi, Nomura, Teubner, Phys.Rev. D97 (2018) no.11]



Time-like vs space-like evaluation of a_μ^{HVP}

Time-like



[Credit: F. Jegerlehner]

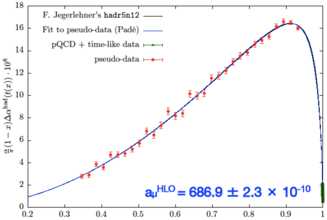
[Lautrup, Peterman de Rafael '72]

[T. Blum '03]

Space-like

Statistical reach of MUonE on a_μ^{HLO}

(2 years of data taking at 1.3 x10⁷μ/s)



A **0.3%** stat error can be achieved on a_μ^{HLO} in 2 years of data taking with ~10^7 μ/s (4×10^{14}μ total)

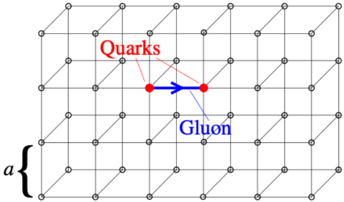
[Credit: G. Venanzoni, G. Abbiendi]

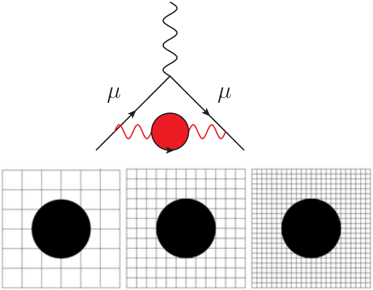
$$a_\mu^{\text{HVP}} = \frac{\alpha}{\pi^2}\int_0^\infty \frac{ds}{s}\,K(s)\,\text{Im}\Pi_{\text{had}}(s) \qquad \rightarrow \qquad \frac{\alpha}{\pi}\int_{-\infty}^0 \frac{dt}{t\beta}\left(\frac{\beta-1}{\beta+1}\right)^2 \hat{\Pi}_{\text{had}}(t)$$

- Π_{had} is the hadronic part of the photon vacuum polarization
- $\text{Im}\Pi_{\text{had}}(s)$ is related to the experimental total cross-section in e^+e^- annihilation

Space-like evaluation of a_μ^{HVP}: Lattice QCD

(1) Formulate QCD on a (finite-size) lattice in Euclidean time
(2) Generate ensembles of field configuration with MC simulations
(3) Compute correlation function of fields as a function of time/momentum
(4) Average over configurations
(5) Extrapolate to continuum, infinite volume, and physical quark masses



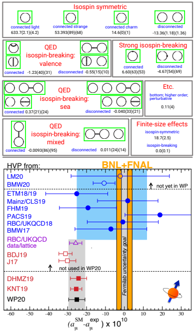


Progress in lattice-QCD value of $a_\mu^{\text{HVP, LO}}$

- In WP20, lattice results ($<$ Mar/2020) were averaged; uncertainty 2.6%
- BMW20 reported first lattice result with sub-percent uncertainty:
  - reduced tension with experiment: $\sim 1.5\sigma$,
  - some tension with the R-ratio method (WP20); $\sim 2.1\sigma$

	value $\times 10^{10}$	error %
$a_\mu^{\text{HVP, LO}}$ (R-ratio, WP20)	$693.1(4.0)$	0.6%
$a_\mu^{\text{HVP, LO}}$ (lattice, WP20)	$711.6(18.4)$	2.6%
$a_\mu^{\text{HVP, LO}}$ (lattice, BMW20)	$707.5(5.5)$	0.8%

[WP20: arXiv:2006.04822, BMW20: arXiv:2002.12347]



- Cross-checks with similar precision are crucial to scrutinize high-precision lattice methodology

Space-like evaluation of a_μ^{HVP}: Lattice QCD

Dominant sources of error:

1) Determination of signal at small Q^2
   - integrand peaked at Q^2 about $m_\mu^2/4$

$$\left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2)$$

   - These very low momenta cannot be directly accessed on current lattices ($L \approx 10$ fm required)
   - Analytic functions (like Padé) in combination of the method of time moments have been suggested & used to describe $\hat{\Pi}(Q^2)$ over small values of Q^2; increase of statistical error at higher moments
   - In alternative, coordinate-space representation the problem with small Q^2 shows itself as exponential growth of the relative statistical error at large time

2) Continuum extrapolation, scale setting errors, finite volume effects, disconnected diagrams, $\cdots$

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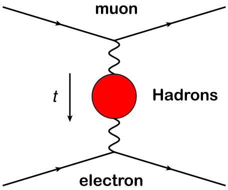
   - In alternative, coordinate-space representation the problem with small Q^2 shows itself as exponential growth of the relative statistical error at large time
   - A hybrid method (with MUonE & PT) has been proposed to circumvent this problem
2) Continuum extrapolation, scale setting errors, finite volume effects, disconnected diagrams, $\cdots$

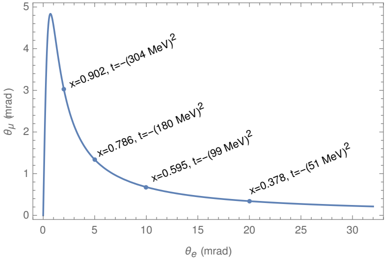
Space-like evaluation of a_μ^{HVP}: MUonE

- HVP contributes to the running of QED fine structure coupling

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \hat{\Pi}(Q^2)}$$

- Comparing experimental data & perturbative calculations yields HVP through its contribution to $\alpha(Q^2)$
- MUonE extracts $\Delta\alpha_{\text{had}}(Q^2)$ from the shape of the differential $\mu - e$ scattering cross section by a template fit method





[arXiv:2004-13663]

Space-like evaluation of a_μ^{HVP}: hybrid method

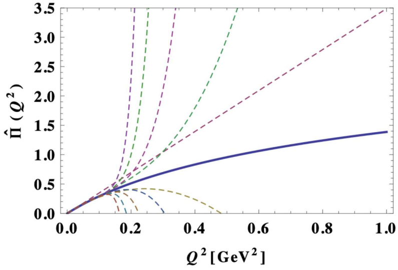
Divide & Conquer:

$$
\begin{aligned}
a_\mu^{\mathsf{HVP}} &= I_0 + I_1 + I_2 \\
I_0 &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^{0.14} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2) \\
I_1 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{0.14}^{Q^2_{\mathsf{max}}} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2) \\
I_2 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q^2_{\mathsf{max}}}^{\infty} dQ^2 f(Q^2)\hat{\Pi}_{\mathsf{had}}(Q^2)
\end{aligned}
$$

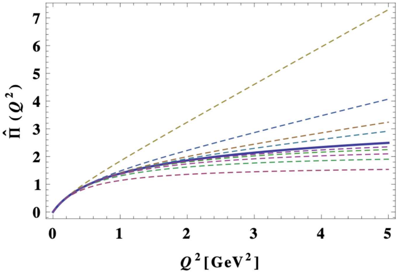
- I_0: contains $\sim 84\%$ of the $a_\mu^{\mathsf{had,\ LO}}$ & can be calculated precisely with the MUonE experiment
- I_1: use lattice QCD (or R-ratio)
- I_2: use perturbation theory

HVP, Stieltjes functions, Padé approximants

- Pade approximants for low Q^2 regions $\hat{\Pi}(Q^2)$ in lattice QCD [Aubin, Blum, Golterman, Peris (2012); Golterman, Maltman, Peris (2013)]
  - was introduced to deal with low signal at small Q^2
  - solely based on known mathematical properties of HVP (can be systematically improved if data with higher precision become available)
  - investigated using mock data from dispersive τ-based $I = 1$ data: $\hat{\Pi}^{I=1}(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} ds \frac{\rho^{I=1}(s)}{s(s+Q^2)}$



$\longrightarrow$



Taylor expansions for $N = 1, \cdots, 10$

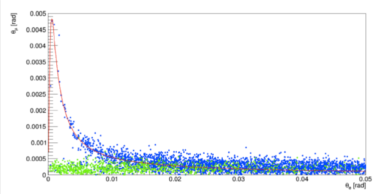
$[N, N-1]$ & $[N, N]$ Padé's: $[1, 1] \to [5, 5]$

[Credit: L. Lellouch]

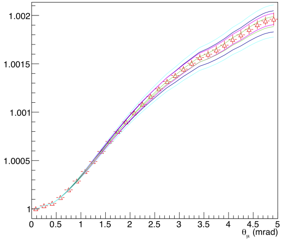
Padé approximants for MUonE (?)

- The extraction of the hadronic contribution to μ-e scattering is carried out by a template fit method
- Proposed template fit inspired by contribution of lepton-pairs the space-like photon vacuum polarization look very good on test data

$$\Delta\alpha_{\mathrm{had}}(t) = k\left\{ -\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right) \frac{2}{\sqrt{1-\frac{4M}{t}}} \log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right| \right\}$$



[arXiv:2201.13177, arXiv:2102.11111]



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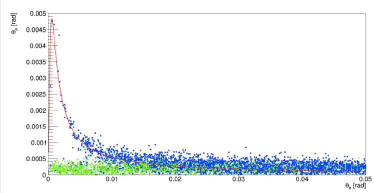

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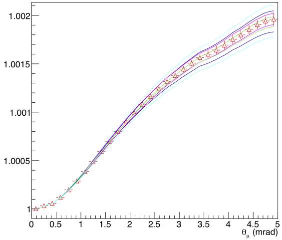
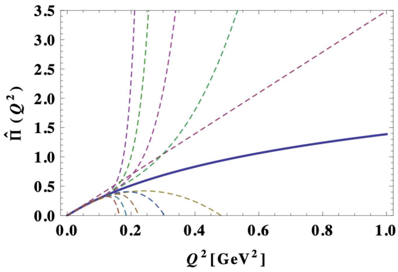


- The above template fit can be potentially problematic with highly precise data
- A natural alternative is to use Padé based template fits

- Writing the HVP in terms of a Stieltjes function warrants an existence of the converging sequence of order $[N-1, N]$ and $[N, N]$ Padé approximants (PAs), defined as

$$\Delta\alpha_{\text{had}}(Q^2) = c_0 + Q^2\left(a_0 + \sum_{i=1}^{N}\frac{a_i}{b_i + Q^2}\right),$$

where $Q^2 = -t$ and $a_0 = 0$ in $[N-1, N]$ PAs



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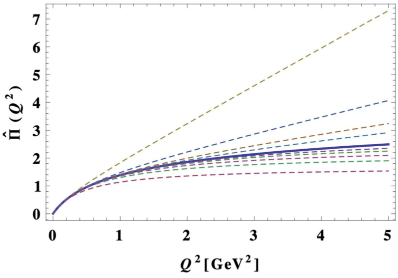
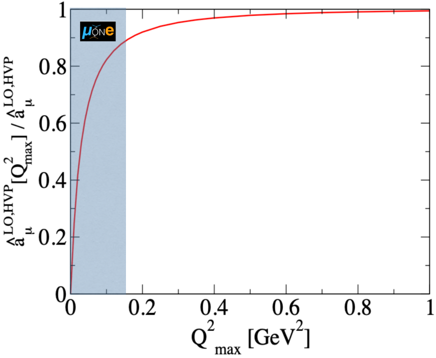


$[N, N{-}1]$ & $[N, N]$ Padé's: $[1, 1] \to [5, 5]$

[Credit: L. Lellouch]

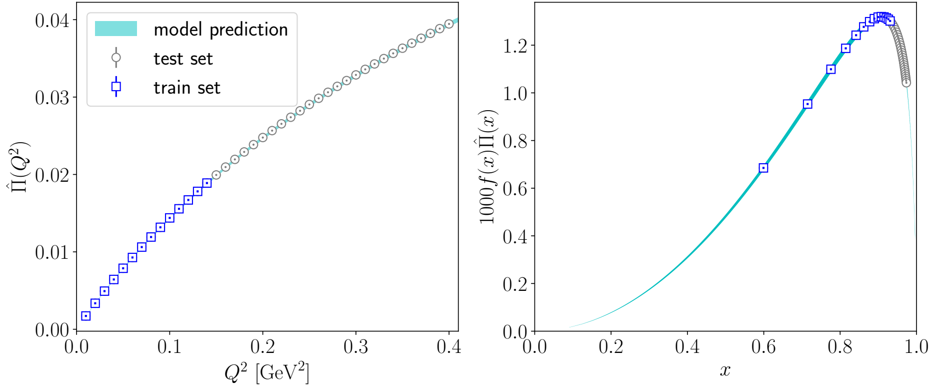
Goals of employing Padé approximants for MUonE

- Quantitatively examine the systematics of MUonE fits in the planed region of the hybrid method $Q^2 < 0.14$ GeV^2



- Investigate extrapolation of MUonE fits to higher values of Q^2
- In the test study that follows we use the mock data from dispersive τ-based $I = 1$ data (curtesy of Kim Maltman)

Padé-based fits to τ-based $I = 1$ model; Padé order [2, 2]



Mock data: 40 highly correlated data (curtesy of Kim Maltman);
~ 30 of eigenvalues consistent with zero

Challenge: extremely small eigenvalue lead to ill-conditioned covariance matrices;
a challenge for least square fits

Treatment: employ modified SVD (svdcut$=10^{-10}$ in the above example)

Message: better template functions are needed with very precise covariance matrices

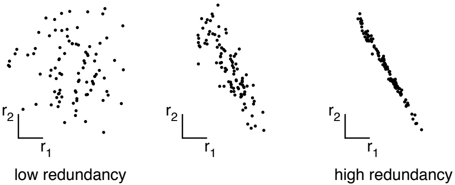
Summary & Outlook

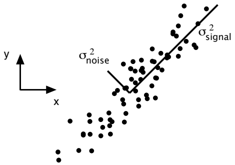
- Current status of $g-2$
- HVP contributions dominating uncertainties in the muon $g-2$
- New lattice calculations and independent space-like calculation by MUonE
- Hybrid method (MUonE $+$ lattice $+$ PT)
- τ-based phenomenological model:
  informs the choice of the fit function in low-Q^2 region

Back-up Slides

Principal Component Analysis (PCA)

- PCA is an analysis that used for dimensional reduction
  - identifies the *principal components* in data that show as much variation in data as possible
  - projects data onto only the first few principal axes/directions to obtain lower-dimensional data with most variation





[https://arxiv.org/pdf/1404.1100.pdf]

- Principal axes are eigenvectors of the data's covariance/correlation matrix.
- Principal components are the projection of data on the principal axes

Principal Component Analysis using SVD

- Say C is the covariance matrix of our data
- SVD or eigenvalue decomposition of C (an $n \times n$ matrix)

$$C = \sum_{i=1}^{n} \lambda_i \, |i\rangle\langle i|, \quad (\lambda_i > \lambda_{i+1})$$

- Modified SVD inflates the smallest eigenvalues of C and replaces the inverse of C with

$$\tilde{C}^{-1} = c_0 \sum_{i=1}^{n-k} \frac{1}{\lambda_i} \, |i\rangle\langle i| + \frac{c_1}{\lambda_{\text{cut}}} \sum_{\text{rest}} |i\rangle\langle i|$$

PCA & SVD for the Method of Least Square

- The method of least squares (LS) is a standard approach in data fitting that minimizes the sum of squared residuals

$$\chi^2 = r^T C^{-1} r, \qquad r = y - f(x)$$

- Highly correlated data lead to ill-conditioned covariance matrices, which makes it challenging to perform the method of least squares
- Singular values of a correlation matrix related to the MUonE project