Using Variational Methods and Hyperbolic Lattices to find the ground state of gapless systems

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Abstract

The ground state of a system is the foundation state, as all other states of a system are excitations from the ground state. Calculating different observables at the ground state can help us understand the behavior of the system at the ground and excited energy levels. In a quantum system, calculating the ground state is often a hard problem. We explore two approaches to find the ground state of a gapless system. We investigate the Maldacena duality or the AdS/CFT correspondence in our works and calculate the average energy of the ground state of the Conformal Formal Theory lying at the boundary of our 3, 7 hyperbolic space with Anti-de Sitter isometries, using inspiration from Monte Carlo Markov Chain Metropolis sampling algorithm. We also explore the Tensor Renormalization Group theory to compute the ground state of a large quantum lattice using quadratically lesser resources.

1 Introduction

1.1 Quantum Physics Problem

The problem we are interested in is the Anti-de Sitter/Conformal Field Theory correspondence [1], which essentially proposes a duality between a d-dimensional quantum mechanical system and a d+1 dimensional gravity theory. It was initially introduced by Maldacena in 1998 following the Black hole information loss paradox. The AdS/CFT theory proposes a correspondence between the 3-dimensional quantum mechanical theory describing the event horizon of a black hole and the 3+1-dimensional gravitational theory describing the singularity of the black hole. The theory defines the quantum mechanical theory to be invariant under conformal transformation, for example angle preserving transformations, and lie on the boundary of a negatively curved background d+1 gravitational theory.

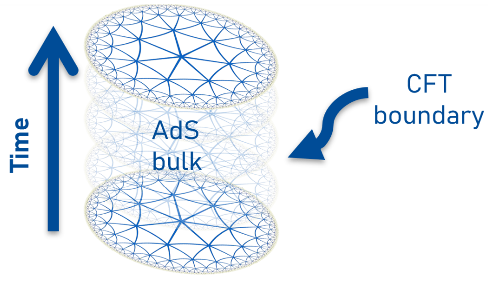

Figure 1: Three-dimensional AdS bulk represented as stacks of 14 hyperbolic lattice at different points in time. A lower dimensional CFT lies on the boundary

1.2 Analytical and Computational Approach

We took a two-step approach to investigating our problem. First, we translated our quantum physical problem into a quantum circuit using the structure of a hyperbolic lattice and second we simulated the quantum field theory on the boundary of our lattice to its ground state mimicking the quantum annealing process.The isommetries and the constant negative curvature background of a hyperbolic lattice [2] helps replicate the environment for the Maldacena duality. We then propose an algorithm to simulate the boundary of the hyperbolic lattice to the ground state using Metropolis inspired sampling method[3]. Simulating a large quantum lattice is not feasible with the current quantum technology and with the above-mentioned approach we were only able to simulate 2 layers of a hyperbolic lattice (consisting of 21 qubits). Another approach we are currently exploring is representing quantum systems as tensor networks. This approach gives us the opportunity to apply tensor renormalization on a large hyperbolic lattice to produce a small hyperbolic lattice representing the large system. The Tensor Renormalization Group theory[4] is a two-step process of coarse-graining and rescaling to represent large systems in terms of lesser degrees of freedom.

2 Quantum System

2.1 Model

The Ising model was introduced a century ago but is still a strong candidate for simple and effective representation of classical and quantum systems. The model proposes particles in a system to be in a down or up spin, $|0\rangle$ or $|1\rangle$ states respectively. A particle would

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thus be represented as:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1}$$

The hamiltonian of the Quantum Ising model is given as following:

$$H_{ising} = -\sum_{i=0}^{N_q-1}(\sigma_i^Z\sigma_{i+1}^Z) - \sum_{i=0}^{N_q}(\sigma_i^X) \tag{2}$$

The first summation in our hamiltonian describes all the neighbouring interactions measured in the Z basis while the second summation describes all the onsite interactions measured in the X basis.
A 2D classical Ising model looks like following:

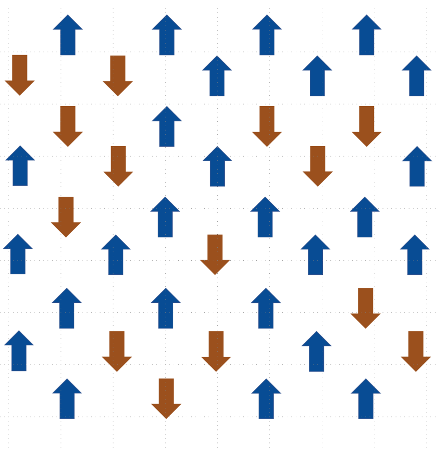

Figure 2: A 2D classical Ising model. The down and up spins correspond to scalar values of -1 and +1 in a classical model.

2.2 Hyperbolic Lattice

A hyperbolic lattice is an interesting topological structure to explore as it can give rise to a holographic effect on the surface. We use a 3, 7 order tiling hyperbolic to represent our problem as a system. In such a structure, each point is connected to 7 more points forming 7 equilateral triangles around the point. The structure gains a constant negative curvature background as the angle made by 7 equilateral triangles exceeds 2pi which pushes the structure out of plane, giving it a holographic effect. Our hyperbolic structure is rotationally symmetric which makes the boundary conformal.
To reconstruct a hyperbolic structure, we can use the following equation to calculate the number of particles in each layer of our 3, 7 tiling, where n[1] = q, n[0] = 1, q = 7:

$$n[L] = (q-4) * n[L-1] - n[L-2] \tag{3}$$

Using the information about the number of particles in each layer of the hyperbolic space and the number of connections of each particle, we can trace out the connections of the particle in our space in form of a adjacency list.

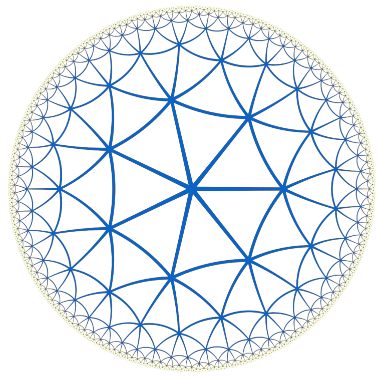

Figure 3: The 3, 7 tiling hyperbolic structure we are considering.

2.3 Ansatz

We prepare our ansatz with 14 stacks of hyperbolic lattice to introduce holography. Using the adjacency list, we apply two qubit unitary gates across the hyperbolic lattice to create entanglement in the system. We distribute the maximum amount of connections possible in a hyperbolic space in 7 stacks. Then we introduce single qubit unitary gates in between the two qubit gate stacks. Following is an example of how our ansatz would look like for one stack of the first layer of our hyperbolic lattice.

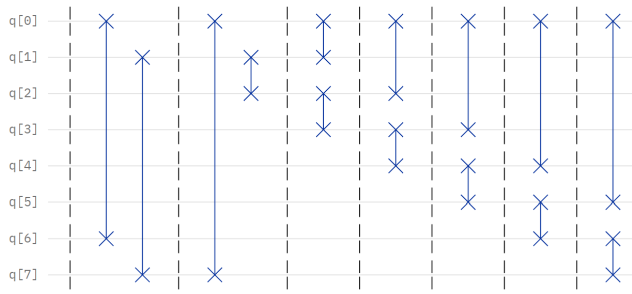

Figure 4: Our quantum circuit representing the first layer of the hyperbolic lattice. The dotted white lines represent the layers of one qubit gates acting on all qubits, separating the two qubit gates layers.

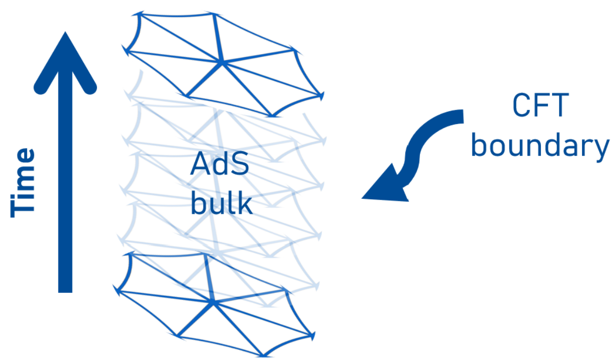

Figure 5: A pictorial representation of our ansatz. Our CFT of 7 qubits lies on the boundary while we have a 1 qubit bulk.

3 Metropolis-style random sampling

3.1 Algorithm

We propose an algorithm to mimic the process of annealing. Annealing is the process of getting to the ground state through random fluctuations through time given increasing beta. We use inspiration from the Monte Carlo Markov Chain Metropolis algorithm to randomly introduce deviation in the connectivity of two particles in the system and calculate the average energy on the boundary of our lattice. A random step is accepted straightaway if the average energy from the proposed change is lower than the initial average energy. If the proposed energy is greater than the initial average energy, we go through a likelihood analysis considering the temperature. We then select or reject based on the following Boolean which introduces a random characteristic to our method:

$$e^{-\beta(E_{prop}-E_{initial})} > random[0.0, 1.0] \tag{4}$$

We programmed our algorithm to apply this operation n times.

3.2 Results

After constructing our ansatz representing the quantum system consisting of the first layer of our 3, 7 hyperbolic lattice, we ran our anneahiling algorithm on it. We used IBM's qasm simulator to simulate our circuit of 8 qubits and got the following result:

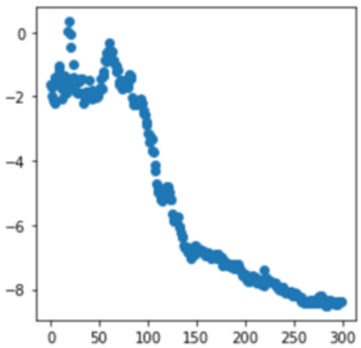

Figure 6: Energy distribution of the boundary as its approaching its ground state. The final energy we observed was -8.4 which is close to the analytical eigenvalue of the hamiltonian in a 7 qubit system of -8.9.

3.3 Need for another approach

For more reliable results of our system we would need to analyse a large hyperbolic lattice. As we can notice from equation 3, the number of particles increase as we go more towards the boundary. Currently, we do not have a quantum system capable of simulating a system consisting of 3 layers. This encourages us to look at the approach of representing as Tensor Networks and applying Tensor Renormalization to reduce the amount of particles in the system.

4 Tensor Renormalization Group

4.1 Ising to Tensors

A quantum Ising system can also be represented as a tensor network. One property of a tensor network is that if we are able to do an operation of a small set of tensors, we can generalize it across the tensor network easily. A 2x2 Ising system can be represented by the following 4-index tensor:

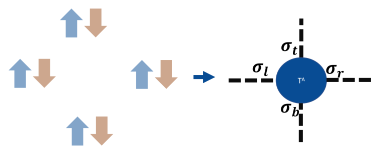

Figure 7: A 4 spin model to a 4-index tensor

The tensor can be formulated through the following equation where beta = 1/T:

$$T^A = \sum_{r,t,l,b=0}^{r,t,l,b=1} e^{-\beta(\sigma_t\sigma_r+\sigma_r\sigma_b+\sigma_b\sigma_l+\sigma_l\sigma_t)} \tag{5}$$

In a Tensor network, the same tensor is reproduced across the network.

4.2 Singular Value Decomposition

SVD is a linear algebric tool to factorize a four-index tensor into two two-index tensors. We use it to separate corners of each square in the tensor network.

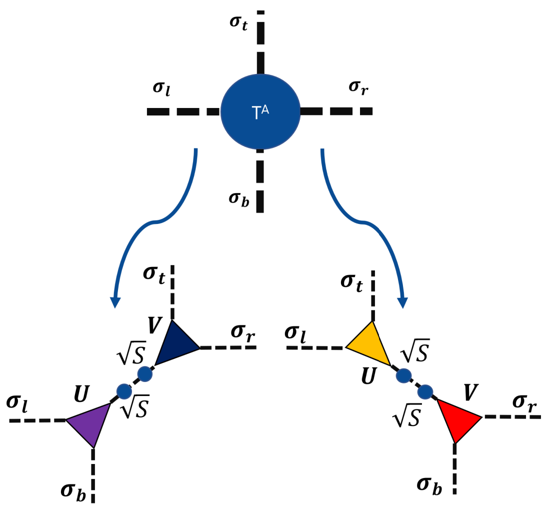

Figure 8: A tensor can be reduced to two-index tensors of (U_{lb} and V_{tr}) or (U_{lt} and V_{br})
which are equivalent to each other.

We can write this transformation mathematically as follows:

$$T^A = U_{I\alpha}\sqrt{S_{\alpha\beta}}V^{\dagger}_{\beta J} \tag{6}$$

4.3 Tensor Renormalization Group

Renormalization is the process of changing ones scaling perspective from a micro level to a macro level. The Tensor Renormalization Group is the process of rescaling and coarse-graining a large quantum lattice in a small quantum lattice. This technique enables us to renormalize a large quantum lattice into producing lesser number of degrees of freedom. Finding the ground state of the smaller system would be computationally easier. The algorithm consists of steps of linear algebric operations across the tensor network. The two steps can be shown as follows:

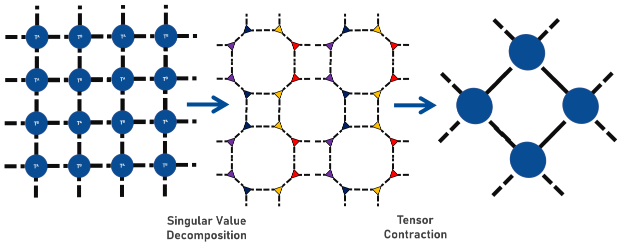

Figure 9: The two-step process of rescaling and coarse-graining

5 Conclusions and Future Research

The tensor renormalization can be applied to reduce a large hyperbolic lattice to a smaller lattice. A triad-network tensor can be used to reduce the number of qubits by 3. This approach would help renormalizing a large bulk in a bulk involving lesser computational power.

Another field I am actively researching in is if machine learning through neural networks has similarities with the tensor renormalization group.

References

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