We define the *opacity* as the line integral through the density of rock along some line of sight through the pyramid

$$\varrho = \int \rho\left(x, y, z\right) dl.$$

There will be some lower muon energy threshold that allows traversal of a specified amount of opacity. This is given by $E_{min}\left(\varrho\right)$ and it can be calculated/tabulated from the known stopping power equation for muons. From plots in Lesparre et al, it appears to be linear on a log-log plot so I think we can assume, for now,

$$E_{min}\left(\varrho\right) \approx a\varrho,$$

where $a \sim 0.6$.

The measured intensity of muons detected will be given by integrating the known spectrum $\Phi\left(E\right)$ above this threshold energy.

$$I\left(\varrho\right) = \int\limits_{E_{min}\left(\varrho\right)}^{\infty} \Phi\left(E\right) dE.$$

Let's assume that we have a prior model for the pyramid and thus a model for the opacity ϱ along any specified line. We are seeking to detect voids or other reductions in density that cause an excess of muons to be detected because the opacity is lower than modeled by $\Delta\varrho$. So we are looking for

$$\Delta I = I\left(\varrho - \Delta\varrho\right) - I\left(\varrho\right) = \int\limits_{E_{min}\left(\varrho-\Delta\varrho\right)}^{\infty} \Phi\left(E\right) dE - \int\limits_{E_{min}\left(\varrho\right)}^{\infty} \Phi\left(E\right) dE = \int\limits_{E_{min}\left(\varrho-\Delta\varrho\right)}^{E_{min}\left(\varrho\right)} \Phi\left(E\right) dE.$$

Now the spectrum goes approximately as

$$\Phi\left(E\right) = cE^{-b},$$

where $b \sim 2.7$. We will calculate $c$later.

So we have

$$\Delta I = c \int\limits_{a\left(\varrho-\Delta\varrho\right)}^{a\varrho} E^{-b} dE.$$

So

$$\Delta I = c\left[\frac{1}{1-b}E^{1-b}\right]_{a\left(\varrho-\Delta\varrho\right)}^{a\varrho} = c\left(\frac{1}{1-b}\right)\left[a^{(1-b)}\varrho^{(1-b)} - a^{(1-b)}\left(\varrho-\Delta\varrho\right)^{(1-b)}\right].$$

This gives

$$\Delta I = c\left(\frac{a^{(1-b)}}{1-b}\right)\left[\varrho^{(1-b)} - \left(\varrho-\Delta\varrho\right)^{(1-b)}\right],$$

or

$$\Delta I = c\left(\frac{a^{(1-b)}}{1-b}\right)\varrho^{(1-b)}\left[1 - \left(1 - \frac{\Delta\varrho}{\varrho}\right)^{(1-b)}\right].$$

If $\Delta\varrho \ll \varrho$, then we can use the binomial expansion

$$\Delta I \approx c\left(\frac{a^{(1-b)}}{1-b}\right)\varrho^{(1-b)}\left[1 - \left(1 - (1-b)\frac{\Delta\varrho}{\varrho}\right)\right].$$

A little algebra gives

$$\Delta I = ca^{(1-b)}\varrho^{(1-b)}\left[\frac{\Delta\varrho}{\varrho}\right],$$

which reduces to

$$\Delta I = ca^{(1-b)}\left[\frac{\Delta\varrho}{\varrho^b}\right].$$

This makes sense to me. We get a bigger difference in number of muons if the number of muons (governed by c) is larger and if the opacity difference $\Delta\varrho$ is larger, and we get a smaller difference in number of detected muons if the background opacity ϱ is larger because fewer are getting trhough in the first place.

For this difference ΔI to be detectable against the background fluctuations of a similar measurement through opacity ϱ without such a void, we want it to be larger than $n\sqrt{I}$, where I is the number of detected muons through opacity ϱ without the void and $\sqrt{I}$ its standard deviation since it will be Poisson distributed, and n is the degree of significance ("n sigmas of significance").

We have

$$I(\varrho) = c\int_{a\varrho}^{\infty} E^{-b}dE = c\left[\frac{1}{1-b}E^{1-b}\right]_{a\varrho}^{\infty} = -\frac{c}{1-b}(a\varrho)^{1-b} = \frac{c}{b-1}(a\varrho)^{1-b}.$$

We need

$$\Delta I > n\sqrt{I},$$

so

$$ca^{(1-b)}\left[\frac{\Delta\varrho}{\varrho^b}\right] > n\sqrt{\frac{c}{b-1}(a\varrho)^{1-b}}.$$

We can then detect opacity differences

$$\Delta\varrho > \frac{n\varrho^b}{ca^{(1-b)}}\sqrt{\frac{c}{b-1}(a\varrho)^{1-b}}$$

This becomes

$$\Delta\varrho > \frac{n}{\sqrt{c}\sqrt{b-1}}\frac{\sqrt{a^{1-b}}}{a^{(1-b)}}\sqrt{\varrho^{1-b}}\varrho^b$$

and then

$$\Delta\varrho > \frac{n}{\sqrt{c}\sqrt{b-1}}\frac{1}{\sqrt{a^{(1-b)}}}\sqrt{\varrho^{1+b}}.$$

As noted before, we have $a = 0.6$ and $b = 2.7$. What about c? This is the spectral constant in

$$\Phi(E) = cE^{-b}$$

and is given by

$$c = A\Omega Tk,$$

where A is the bin area in cm^2 , Ω is the bin solid angle in steradians, and T is the measurement time in seconds. k is the spectral normalization constant in units of muons/ cm^2 Sr s. In the Daya-Bay spectrum, k=0.14 muons/ cm^2 Sr s. Note that we are ignoring the zenith angle dependence here.

We find the following:

1. For transmission through 100 m of rock we have ~1.3 muons/cm^2/sr/day.

2. For transmission through 100 m of rock and detection in a bin of area (48 cm)^2 and angular span 0.0004 Sr, we expect, after a month of viewing, to detect ~36 muons total. Not very many.

3. For a given void, many bins will view it throughout the entire multi-year viewing process. So I think we can change effective view time to ~1000 days and effective bin size to (96 cm)^2. I find that we could be sensitive to about 1 m void in 100 m of background.