Fermilab

Managed by Fermi Research Alliance, LLC for the U.S. Department of Energy Office of Science

Accelerator Physics Concepts

Jeffrey Eldred

for **Fermilab CERN Hadron Collider School**

8/26/2022

Learn more at US Particle Accelerator School (USPAS)

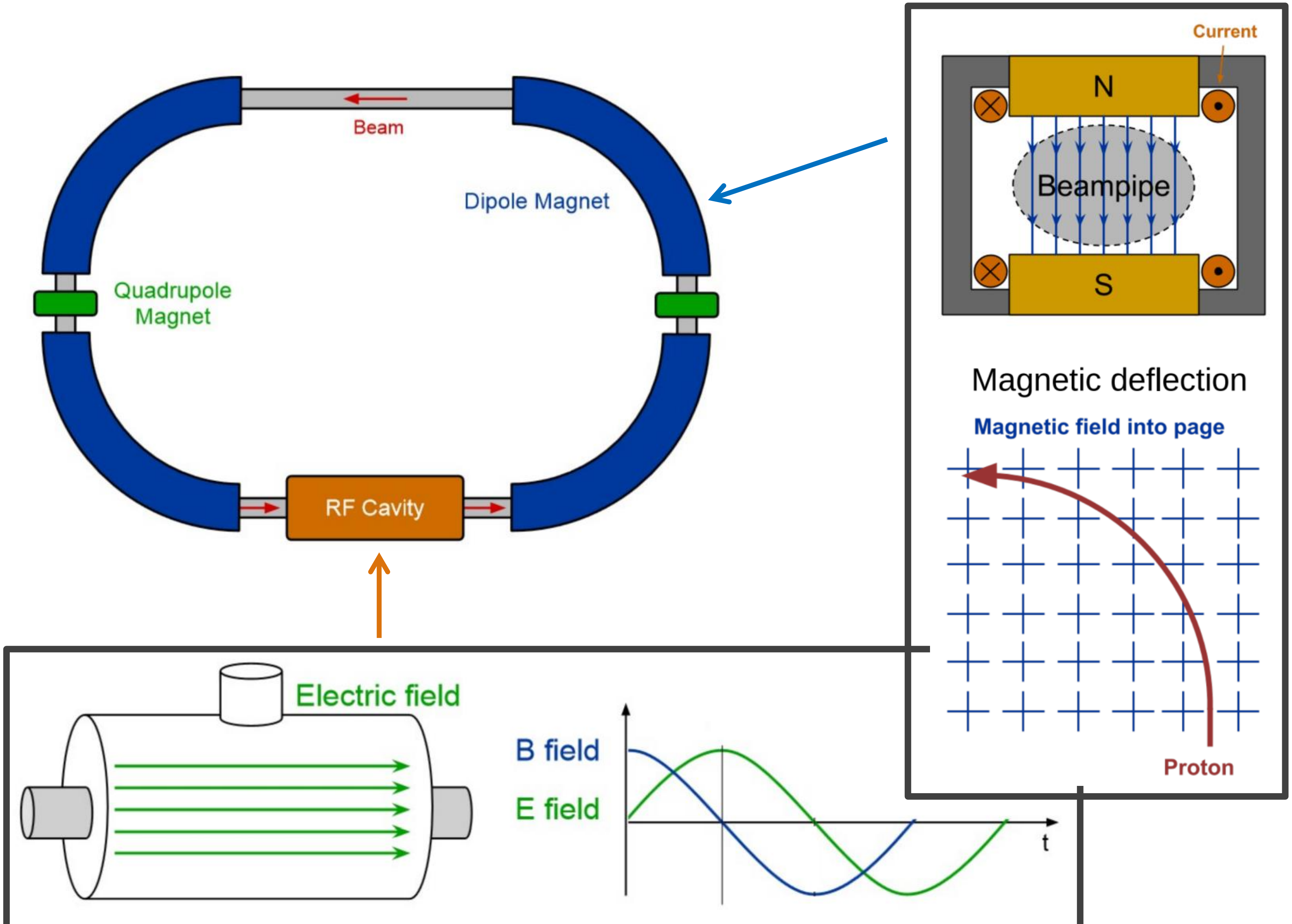
Free Recorded Classes:
- Eric Prebys' online course:
"Fundamentals of Accelerator Physics"
- Huang & I's online course:
"Mechanics & Electromagnetism for Accelerator Physics"

Textbook:
"An Introduction to the Physics of High Energy Accelerators"
Syphers and Edwards

Sign up for Live USPAS classes: website
- Two-week full-time sessions every June and January.
- Equivalent to graduate-level college-semester course!
- January 2023 session will be back to in-person (deadline Sept 15)

I took many USPAS classes as a graduate student, and now I regularly teach at USPAS.

Simplified Particle Accelerator



D. Barak, B. Harrison, A. Watts, Concepts Rookie Book,
special thanks **A. Watts**

Dipole Magnets for Bending

Maximum Dipole Field -> Maximum Proton Energy

Lorentz Force:

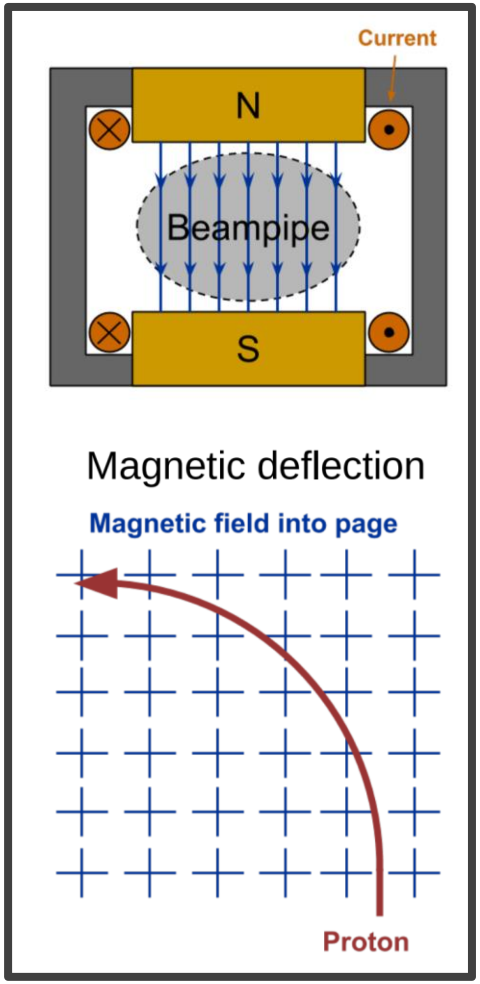
$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

Bending Radius in a constant dipole field:

$$B\rho = \frac{p}{Ze}$$

Dipole field → B; Bending radius → ρ; particle momentum → p; particle charge → Ze

Total bending for a ring is always 2π:



Maximum Proton Energy – LHC Example

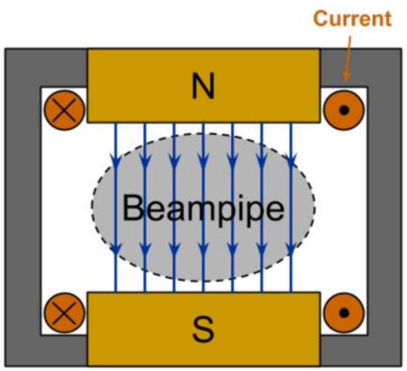
The LHC has **1232** dipoles, each **15m** long but effectively **14.3m** long.
- **18.5 km** of the **26.7 km** circumference is dipole magnet.
- We say “circumference” even though the LHC is a 1232-sided polygon.
- $1232*14.3/2\pi$ = **2800 m** magnetic bending radius ρ.

For a dipole field B of **7.7 T,** we can calculate the equivalent energy:
- 2800*7.7 / **3.3357** = 6,500 GeV/c proton
- **6.5 TeV** per beam
- **13 TeV** colliding energy

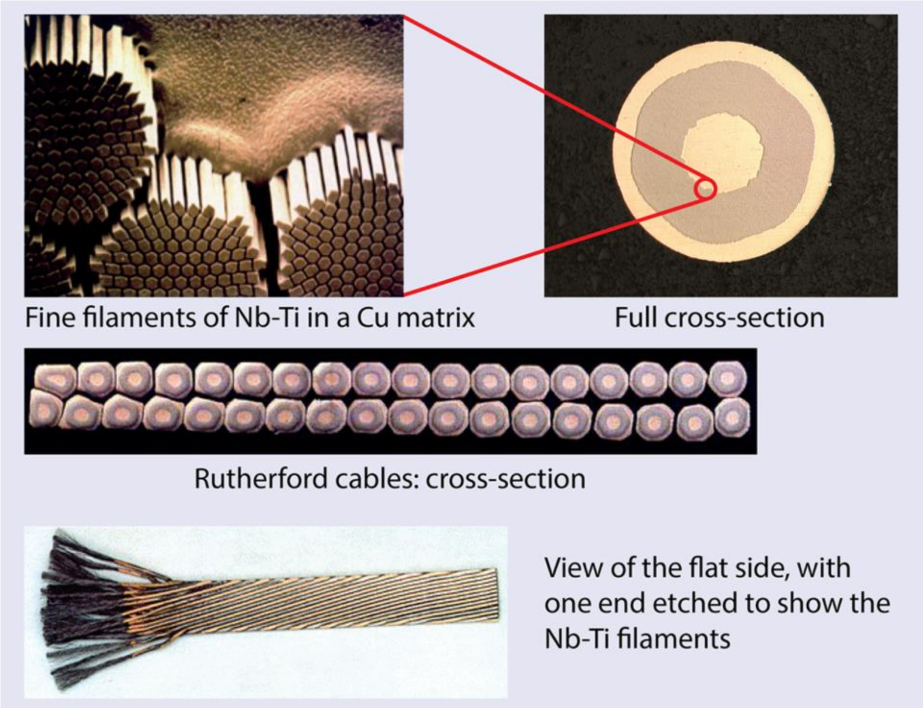
To go to higher energy:
- dig a longer tunnel
- and/or build a better dipole magnet:
    - superconductors have a critical temp, critical field, critical current.
    - manufacturing and reliability are part of dipole design as well.

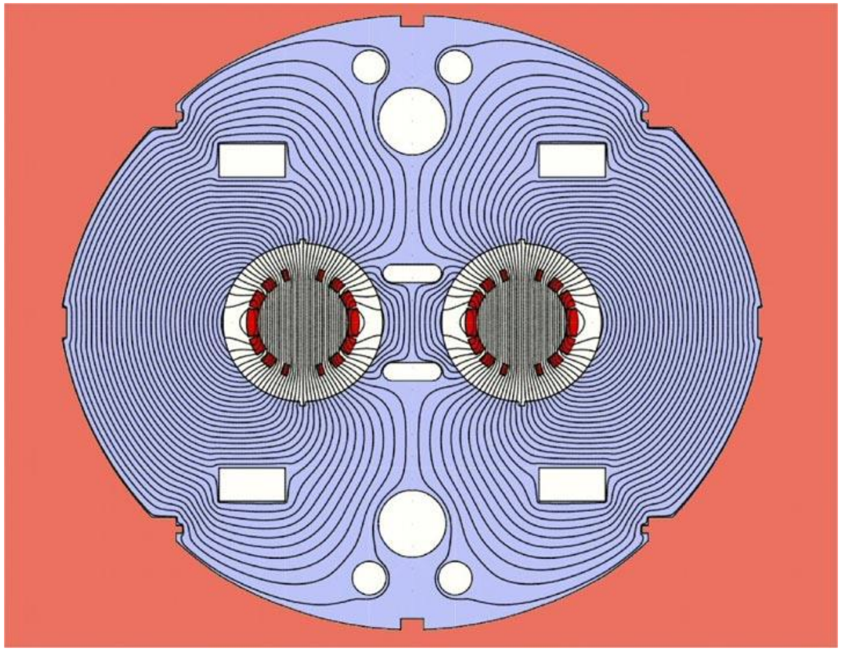
LHC Superconducting Dipoles

Normal Conducting Dipole



LHC Superconducting Dipole

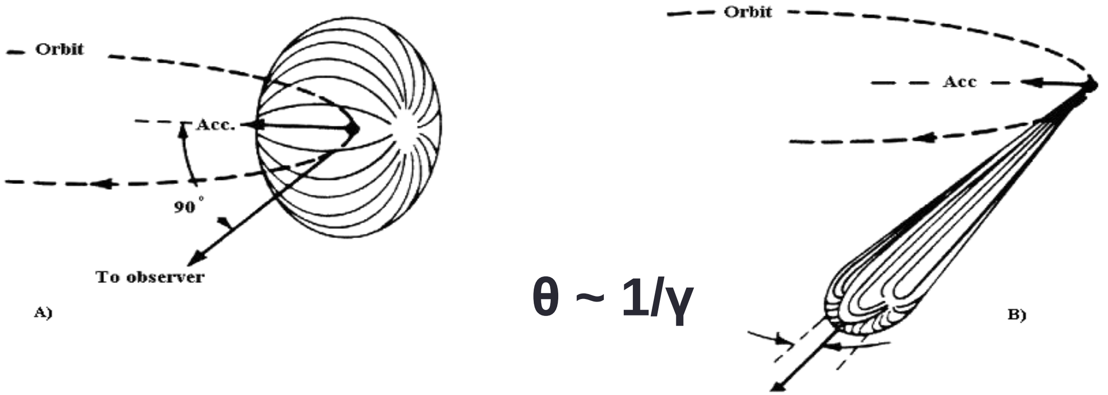




CERN Courier

Radiation Loss -> Maximum Electron Energy

Circular electron colliders are limited instead by radiation.
Any relativistic charged particle will give off synchrotron radiation.



The magnitude of radiation is usually negligible in hadron machines, but is a dominant feature of electron machines.

$$P = \frac{q^2 c}{6\pi\epsilon_0} \frac{\beta^4 \gamma^4}{\rho^2}$$

Power radiated

Physical constants

Relativistic βγ factors

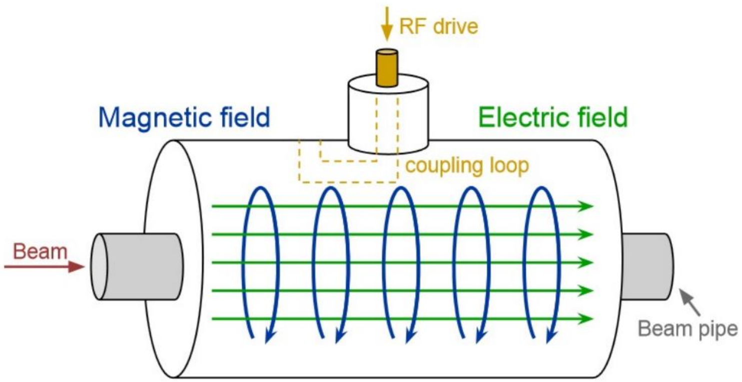
Bending radius

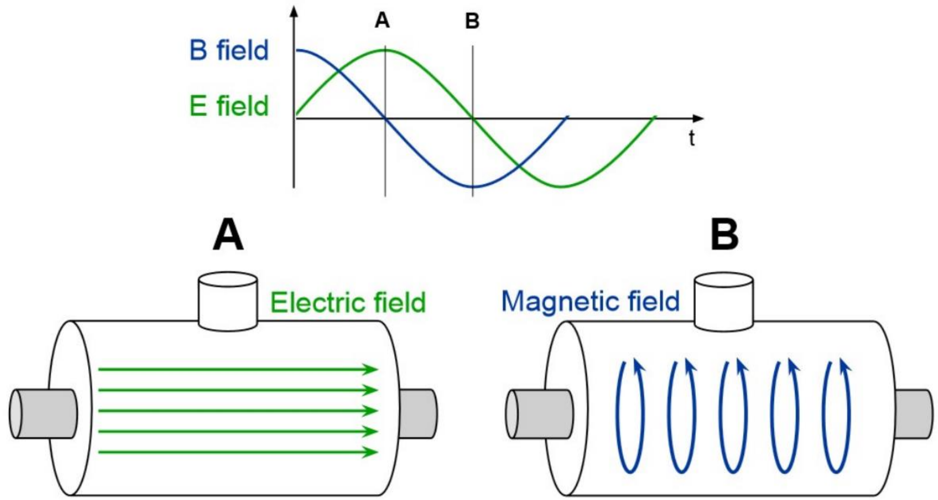
The maximum acceleration rate sets the maximum power loss.
For x2 the energy, the same power loss occurs at x4 the bending radius.

RF Cavities
for Acceleration
(and Synchronization)

RF Accelerating Cavity

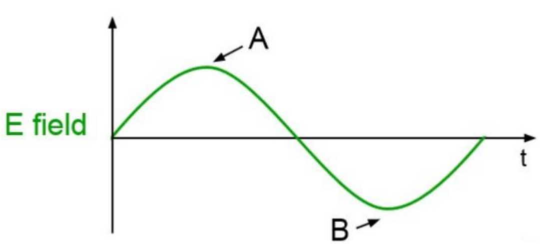
We use resonating radiofrequency (RF) cavities to efficiently trap an electromagnetic wave which accelerates the beam.

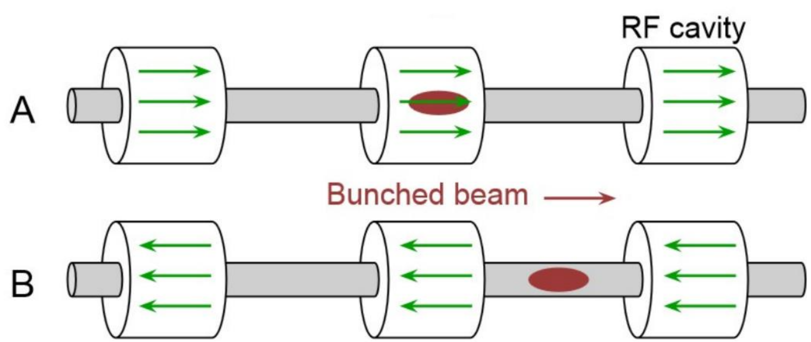




A. Watts, Concepts Rookie Book

The beam must arrive in synchronized bunches to be accelerated.





Fermilab

Change in Momentum

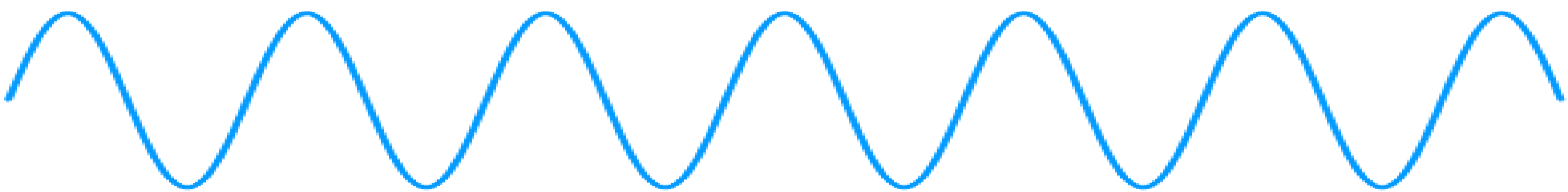
Fractional Momentum: $\delta \equiv \frac{p - p_0}{p_0}$

RF Acc. Per Pass: $\Delta E = qV \sin(\phi)$

Change Momentum per unit time:

$$\dot{\delta} = \frac{\dot{p}}{p_0} = \frac{\dot{E}}{\beta^2 E_0} = f_{rev} \frac{\Delta E}{\beta^2 E_0} = f_{rev} \frac{qV}{\beta^2 E_0} \sin(\phi)$$

Sinesoidal potential:



Phase-Slip Factor η

The arrival time of the particle depends on the momentum:

$$\delta \equiv \frac{p - p_0}{p_0} \qquad \frac{T - T_{rev}}{T_{rev}} \approx 0 + \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} \delta = \eta \delta$$

Higher momentum particles may arrive earlier or later than lower momentum particles:

$$\eta = \frac{1}{T_{rev}} \frac{\partial T}{\partial \delta} = \frac{1}{C} \frac{\partial C}{\partial \delta} - \frac{1}{\beta} \frac{\partial \beta}{\partial \delta} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}$$

Momentum compaction factor:

$$\text{where, } \alpha_c = \frac{1}{C} \frac{\partial C}{\partial \delta} = \frac{1}{C} \int_0^C \frac{D(s)}{\rho} ds$$

We can write the change in phase per unit time using the phase-slip factor:

$$\dot{\phi} = f_{rev} \Delta\phi = 2\pi f_{rev} \frac{\Delta T}{T_{rf}} = 2\pi f_{rev} h \frac{\Delta T}{T_{rev}} = 2\pi f_{rev} h \eta \delta$$

$$f_{rf} = h f_{rev}$$

Longitudinal Focusing

$$\dot{\phi} = 2\pi f_{rev} h \eta \delta, \ \ \dot{\delta} = f_{rev} \frac{qV}{\beta^2 E_0} \sin(\phi)$$

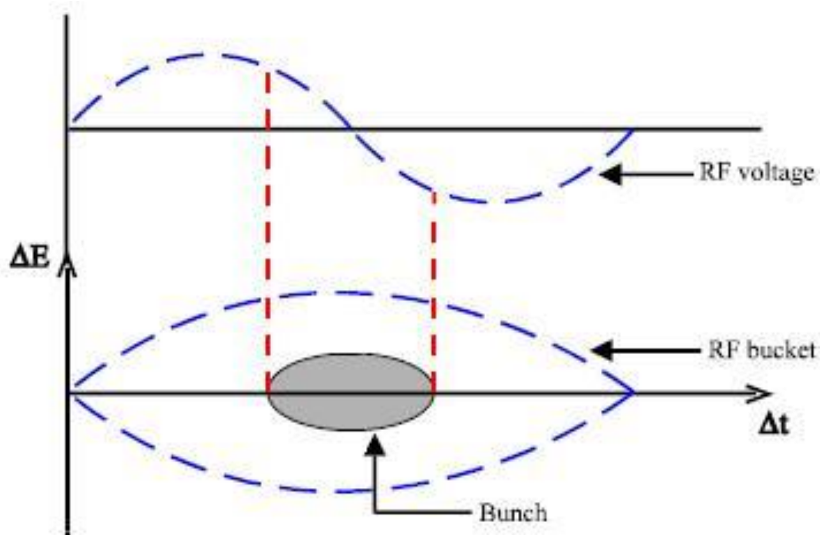
$$\ddot{\phi} = 2\pi f_{rev}^2 \frac{qV}{\beta^2 E_0} h\eta \sin(\phi)$$

$$\eta < 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi)$$

$$\eta > 0 : \ddot{\phi} = -\omega_s^2 \sin(\phi + \pi)$$

Synchrotron Tune

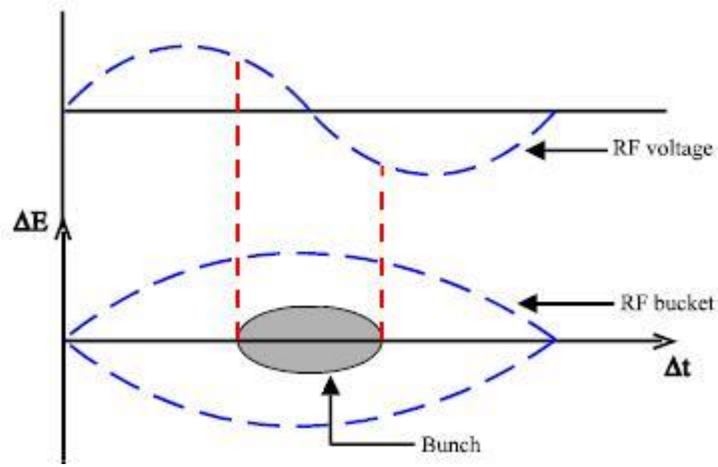
$$\omega_s = 2\pi f_{rev} \sqrt{\frac{qVh|\eta|}{2\pi\beta^2 E_0}}$$

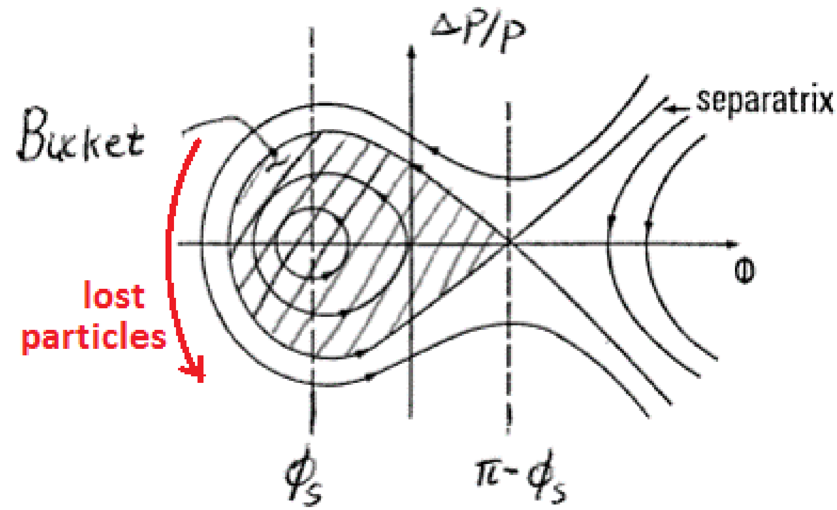


RF Acceleration

A fixed frequency beam longitudinally focuses the beam into a several beam “bunches” in individual RF “buckets”.

Particles in the bucket can be accelerated by adiabatically changing the RF frequency, the other particles are lost.





$$\dot{\delta} = f_{rev} V_\delta [\sin(\phi) - \sin(\phi_s)], \;\; \dot{\phi} = 2\pi f_{rev} h \eta \delta$$

Quadrupole Magets for Transverse Focusing

Luminosity

Luminosity is proportional to the number of particle interactions in colliding beams, which (to lowest order) is given by:

$$\mathcal{L} \approx \frac{N_1 N_2 f n_b}{4\pi\sigma_x\sigma_y}$$

Luminosity → $\mathcal{L}$

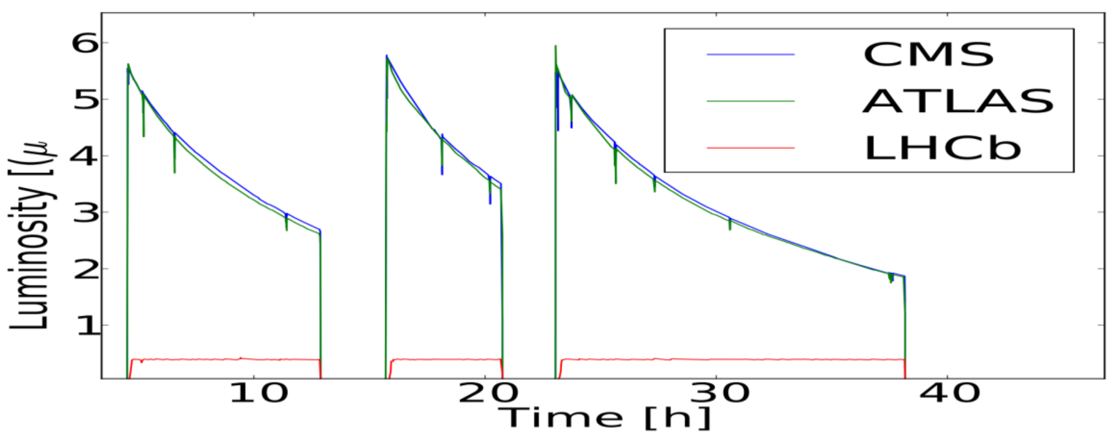
Particles per bunch → $N_1 N_2$

Rate of bunch crossing → $f n_b$

Horz, Vert beam sizes → $\sigma_x\sigma_y$

Luminosity benefits from achieving the highest possible particle density in the beams, transversely and longitudinally.

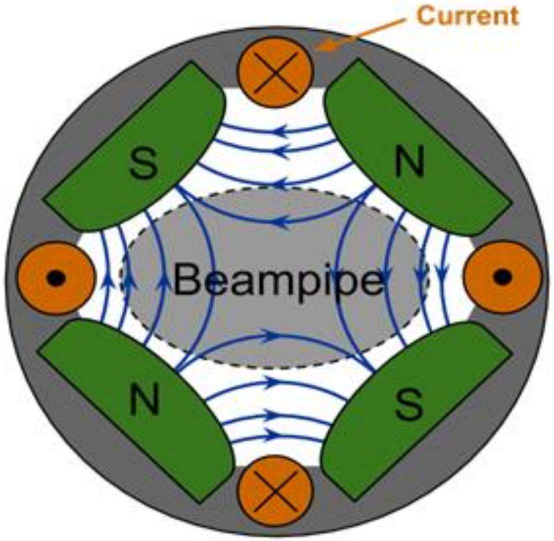
Luminosity leveling, for pile-up limits and beam-beam effects



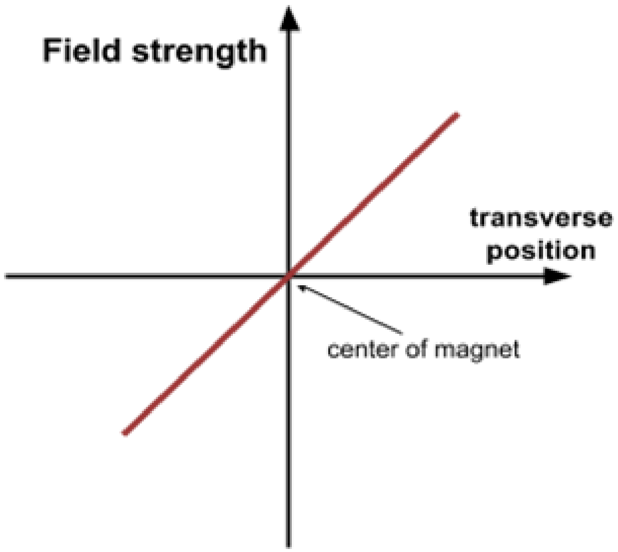
X. Buffat

Quadrupole Magnets for Transverse Focusing

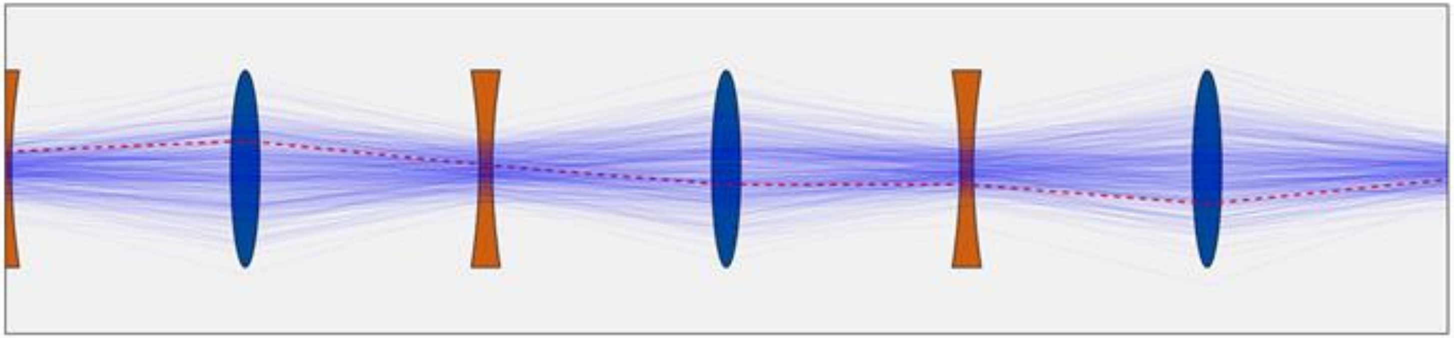
Quadrupole Magnet:



Linear Restoring Force:



Alternating Focusing Magnets:



D. Barak, B. Harrison, A. Watts, Concepts Rookie Book, special thanks A. Watts

Transverse “Betatron” Motion

Harmonic Oscillator

Hamiltonian: $$H = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2$$

Equations of motion:

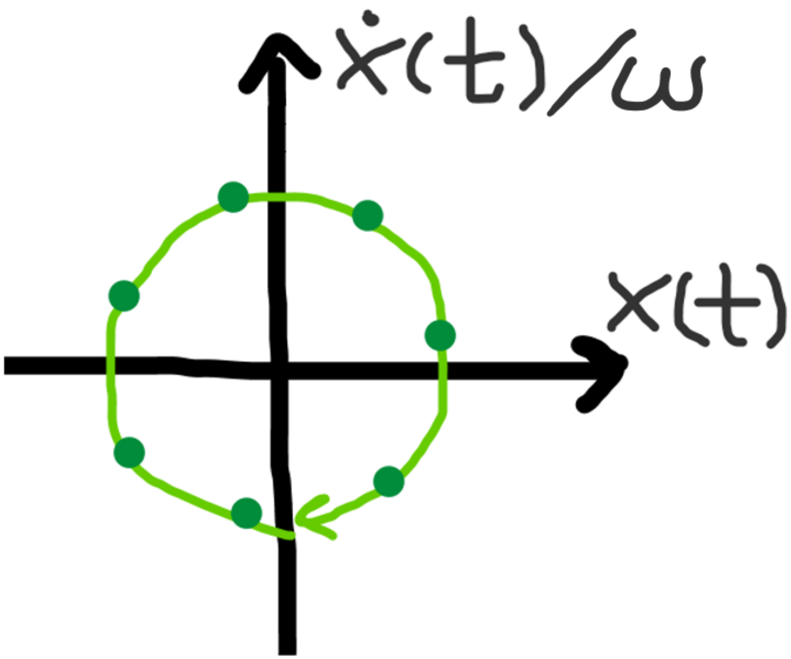
$$m\ddot{x} = -kx$$

$$\ddot{x} = -\omega^2 x$$

$$x(t) = A\cos(\omega t + \phi)$$

$$\dot{x}(t) = -\omega A\sin(\omega t + \phi)$$

Phase-space diagram:



Harmonic Oscillator

Hamiltonian: $H = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2$

Equations of motion, with action:

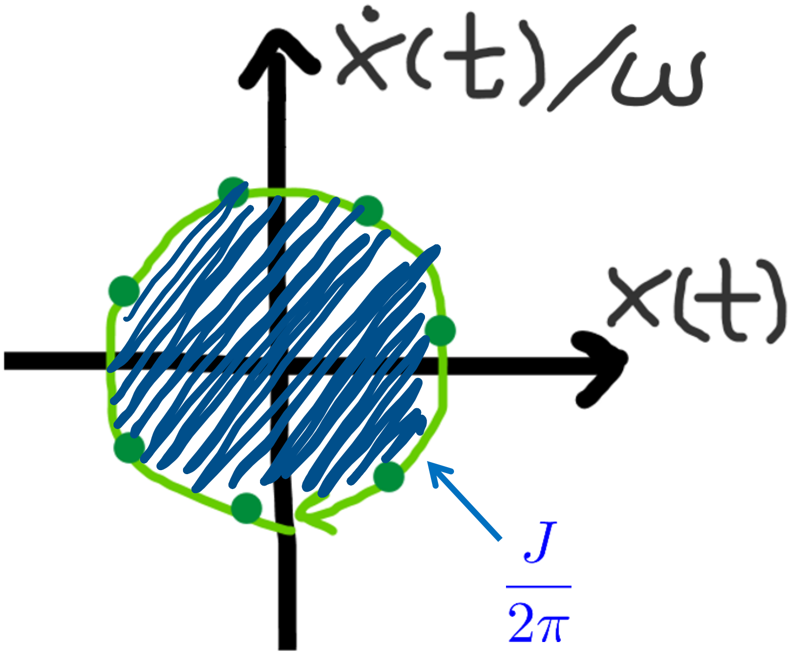
$$\ddot{x} = -\omega^2 x$$

$$x(t) = \sqrt{2J}\cos(\omega t + \phi)$$

$$\dot{x}(t) = -\sqrt{2J}\omega\sin(\omega t + \phi$$

$$x^2 + (\dot{x}/\omega)^2 = 2J$$

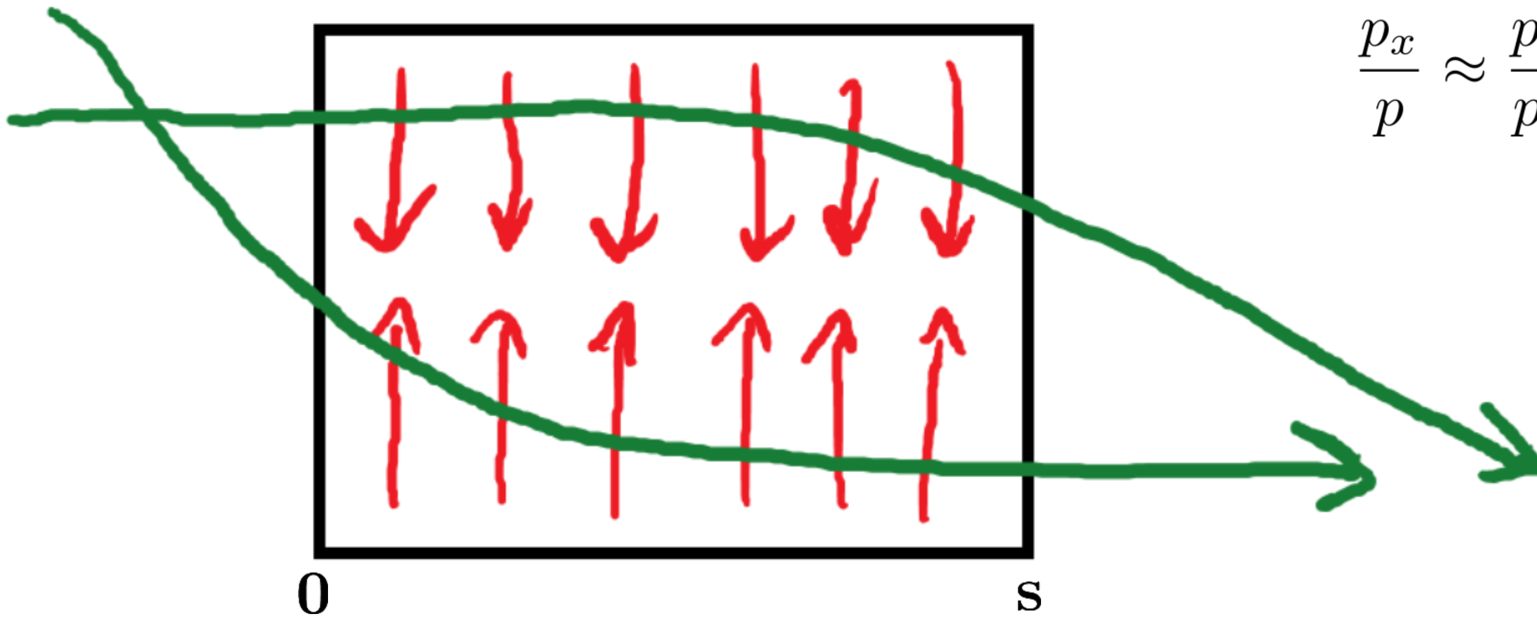
Phase-space diagram:



Fermilab

Linear Focusing

We can solve the linear Hill's equation: $x'' + K(s)x = 0$

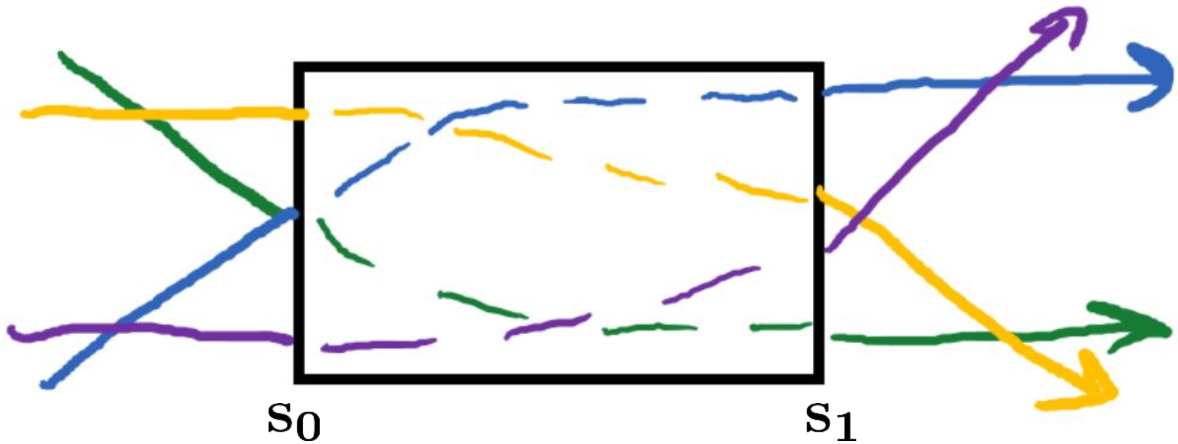


$$\frac{p_x}{p} \approx \frac{p_x}{p_0} \equiv x'$$

$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x_0' s, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \end{cases}$$

Transfer Matrices

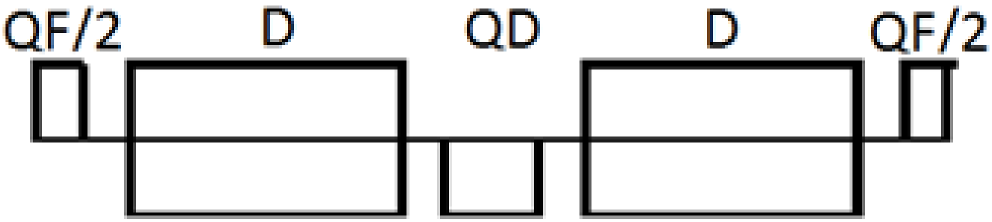
$$x(s) = \begin{cases} x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s), & K < 0 \\ x_0 + x_0' s, & K = 0 \\ x_0 \cos(\sqrt{K}s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}s), & K > 0 \end{cases}$$



The final position and slope is a linear combination of the initial position and initial slope. We can use matrices:

$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

Example: FODO Cell



$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

The transfer matrix for a sequence of elements can be obtained by multiplying the matrices for the components.

This is good for tracking particles, but how can we make sense of what is happening to the beam as a whole?

Solving Hill's Equation

Hill's Equation: $x'' + K(s)x = 0$

If we write: $x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)]$

$$x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\Big[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\Big]$$

This is a solution if we also require that:

$$\alpha_x(s) = -\frac{1}{2}\beta_x'(s)$$

$$\phi(s) = \int_0^L \frac{1}{\beta_x(s)}ds$$

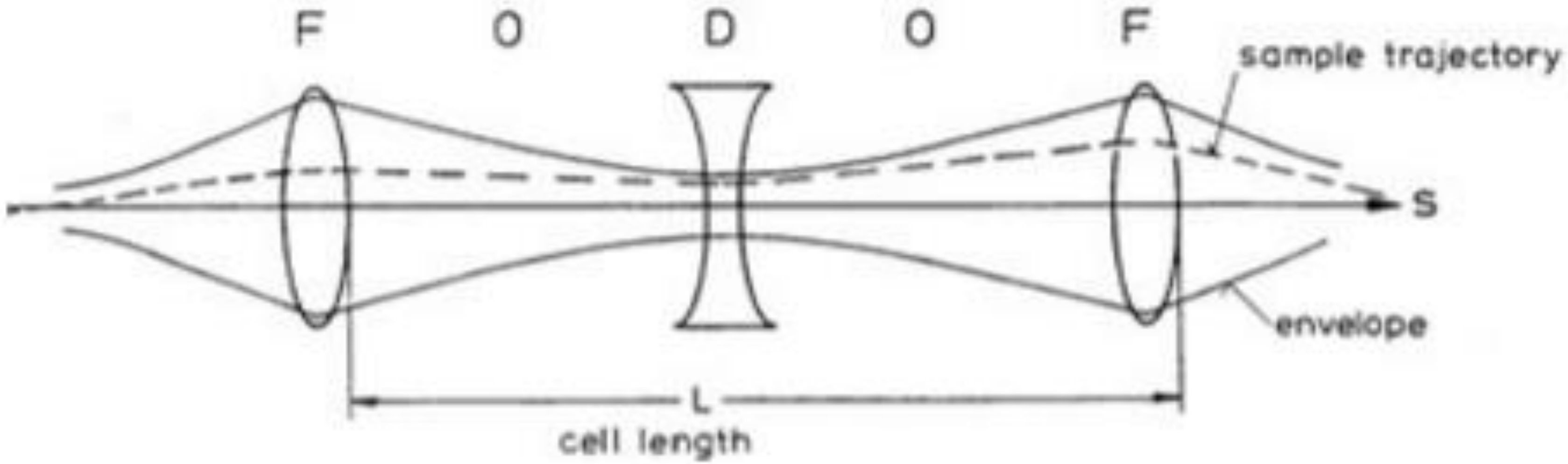
$$\alpha_x'(s) = K\beta_x(s) - \frac{1}{\beta_x(s)}(1+\alpha_x^2)$$

Amplitude & Beta function

$$\underline{x(s)} = \sqrt{2\underline{J_x}\,\underline{\beta_x(s)}}\cos[\underline{\phi_0} + \underline{\Delta\Phi_x(s)}]$$

J_x φ_0 specific to one particle, independent of accelerator location.

β_x $\Delta\Phi_x$ same all particles, depends on accelerator location.

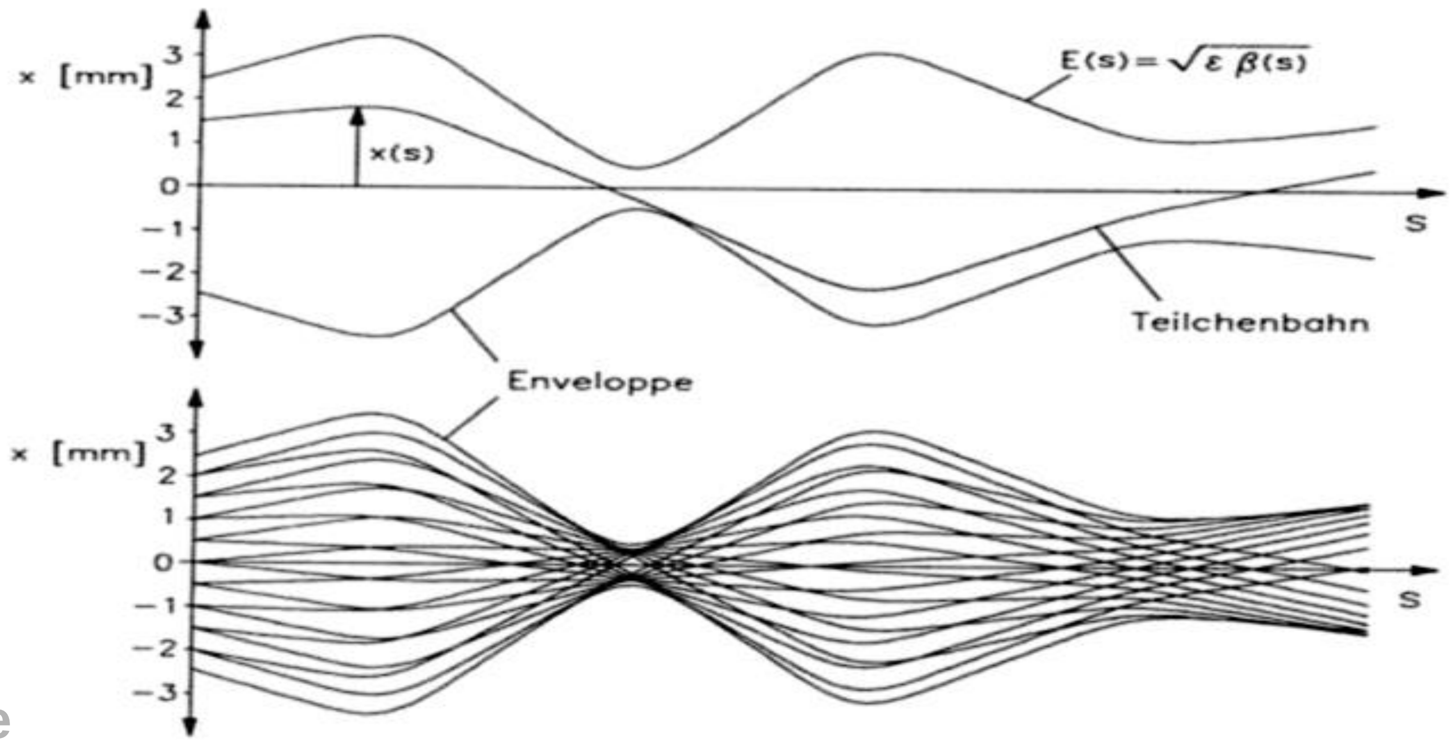


Bartolini

Amplitude & Beta function

$$x(s) = \sqrt{2J_x \beta_x(s)} \cos[\phi_0 + \Delta\Phi_x(s)]$$

RMS Beam size: $\sigma_x(s) = \sqrt{\epsilon_{rms}\beta_x(s)}$



Wille

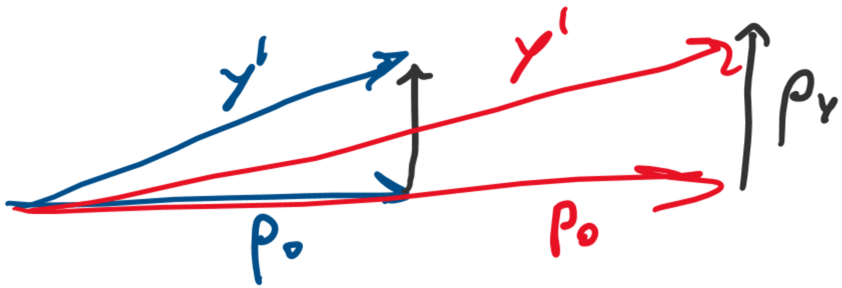
Fermilab

Adiabatic Damping

Completely separate from any time-dilation / length-contraction effects.

One coordinate for each of the three degrees of freedom is relative to the reference momentum p_0.

$$\frac{p - p_0}{p_0} = \delta \qquad \frac{p_x}{p} \approx \frac{p_x}{p_0} \equiv x' \qquad \frac{p_y}{p} \approx \frac{p_y}{p_0} \equiv y'$$



As the beam accelerates, p_0 scales as βγ:

Geometric **Normalized**

$$x \propto \sqrt{\epsilon_{g,x}} \qquad \delta \propto \sqrt{\epsilon_{g,L}}$$

$$x' \propto \sqrt{\epsilon_{g,x}} \qquad \Delta t \propto \sqrt{\epsilon_{g,L}}$$

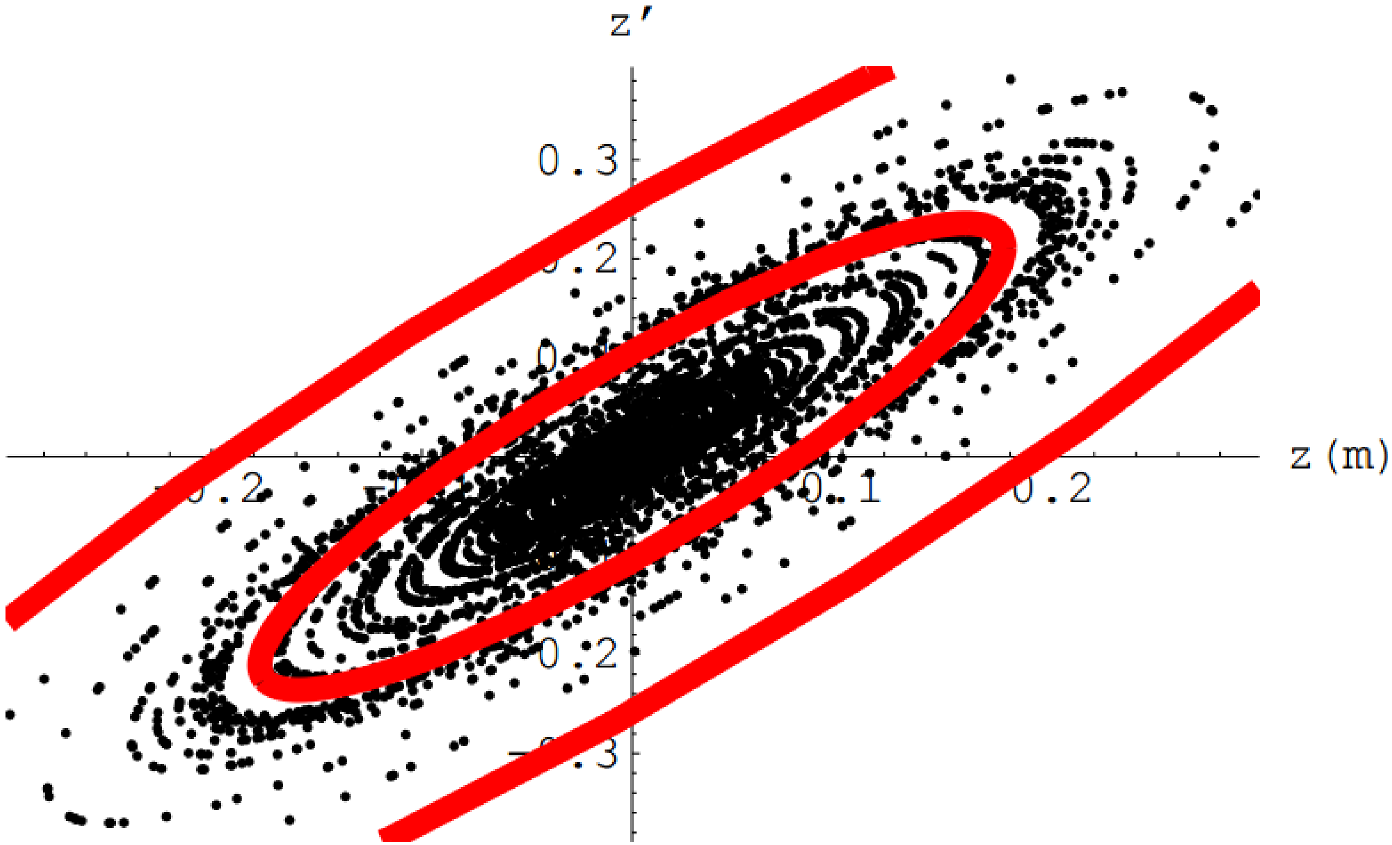
$$\epsilon_g = \frac{\epsilon_N}{\beta\gamma}$$

ϵ_N conserved

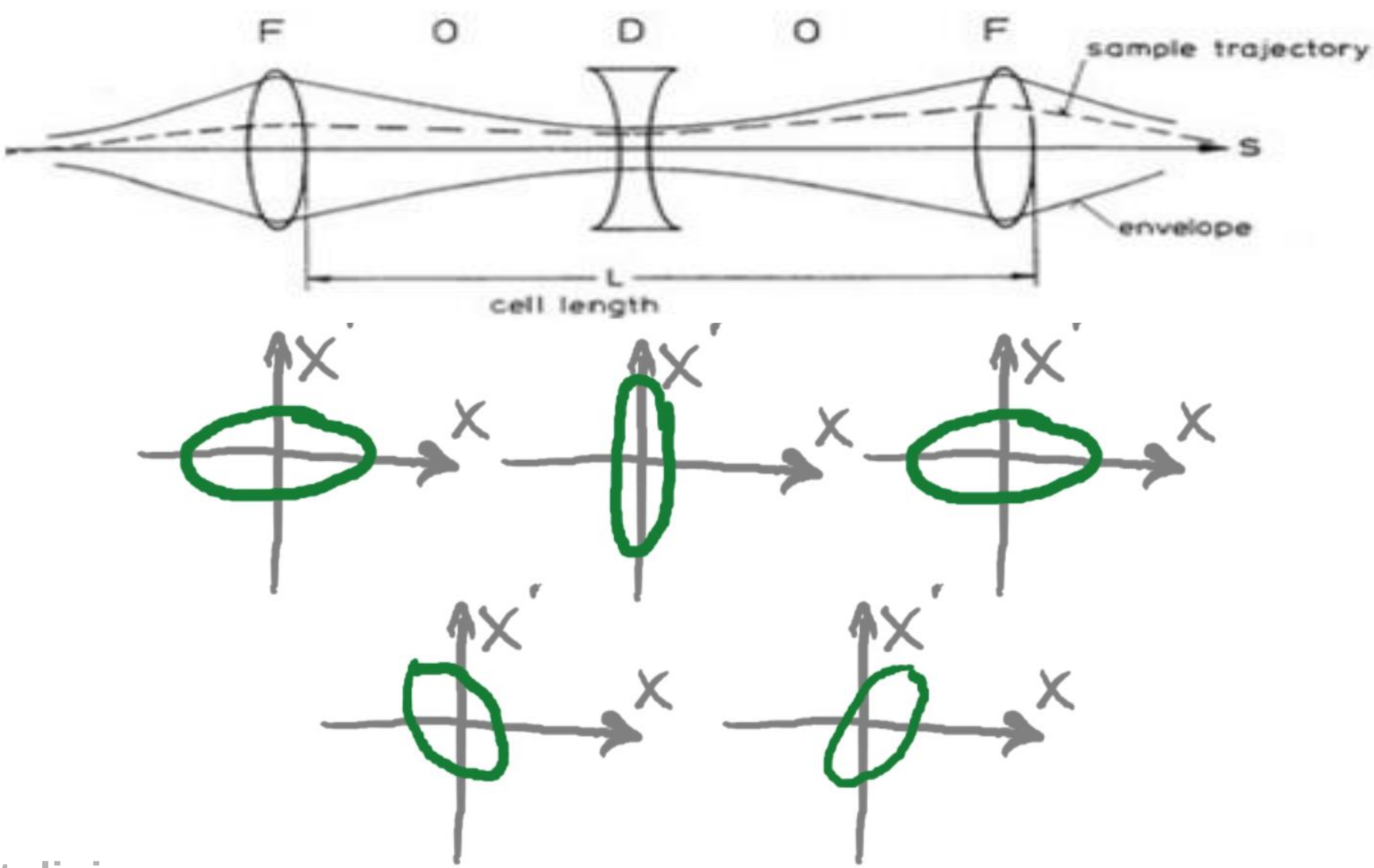
Fermilab

Transverse Phase-space

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\left[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\right]$$



Betatron Motion



Bartolini

Fermilab

Normalized Coordinates

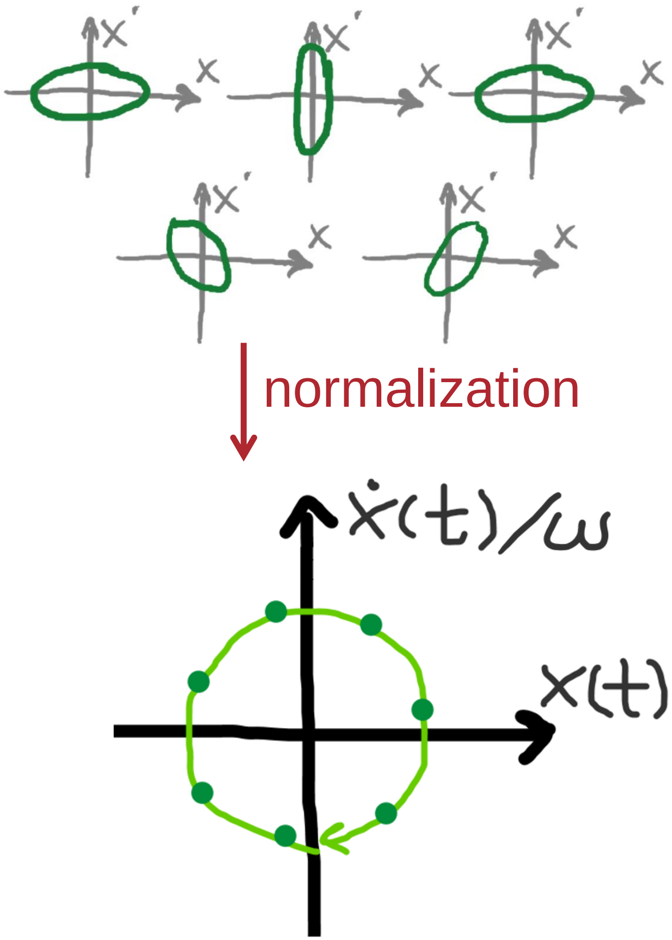
$$x(s) = \sqrt{2J_x \beta_x(s)} \cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}} \left[\sin[\phi_x(s)] + \alpha_x(s) \cos[\phi_x(s)] \right]$$

We can "normalize" these coordinates by a scale-skew transformation:

$$\begin{pmatrix} X \\ P_x \end{pmatrix} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix}^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$X = \frac{1}{\sqrt{\beta_x}} x = \sqrt{2J_x} \cos[\phi_x(s)]$$

$$P_x = \frac{\alpha_x}{\sqrt{\beta_x}} x + \sqrt{\beta_x} x' = -\sqrt{2J_x} \sin[\phi_x(s)]$$



Betatron Oscillation

Using these continuous forms of motion:

$$x(s) = \sqrt{2J_x\beta_x(s)}\cos[\phi_x(s)] \qquad x'(s) = \sqrt{\frac{2J_x}{\beta_x(s)}}\Big[\sin[\phi_x(s)] + \alpha_x(s)\cos[\phi_x(s)]\Big]$$

Relating this to the matrices, see any general transfer matrix can be parameterized and decomposed:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1\sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\Phi & \sin\Delta\Phi \\ -\sin\Delta\Phi & \cos\Delta\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix}^{-1}$$

An inverse transformation, a rotation, and transformation.

Courant-Snyder (TWISS) Parameters

The transfer matrix for a general transfer matrix:

$$M(s_2|s_1) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\Phi + \alpha_1 \sin\Delta\Phi) & \sqrt{\beta_1\beta_2}\sin\Delta\Phi \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\Phi + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\Phi & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\Phi - \alpha_2\sin\Delta\Phi) \end{pmatrix}$$

Transfer matrix for an entire ring, impose $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$:

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\alpha_x = -\frac{\beta_x'}{2}$$

$$\gamma_x = \frac{1+\alpha_x^2}{\beta_x}$$

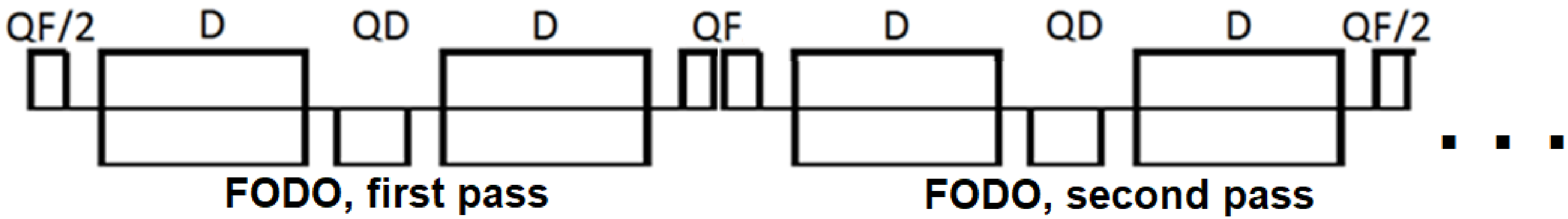
These α, β are known as Courant-Snyder or TWISS parameters. We can think of them either as parameterization of the transfer matrix or as functions which solve the Hill's equation.

Courant-Snyder (TWISS) Parameters

The transfer matrix for an entire ring

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

For example, we can calculate TWISS for a repeating FODO ring:

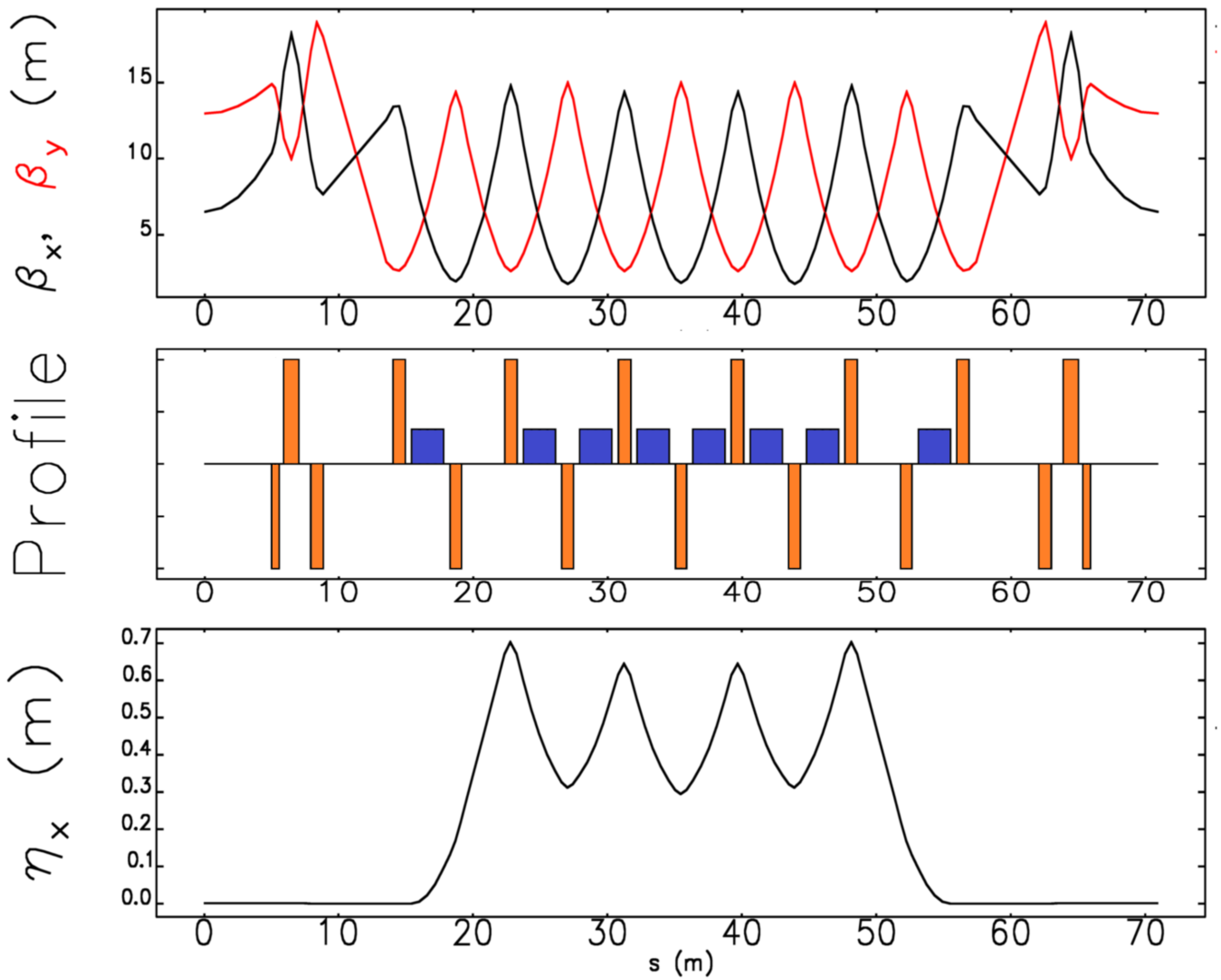


$$M(s_0) = \begin{pmatrix} 1 - \frac{L^2}{2f} & 2L\left(1 + \frac{L}{2f}\right) \\ -\frac{L}{2f^2}\left(1 - \frac{L}{2f}\right) & 1 - \frac{L^2}{2f} \end{pmatrix}$$

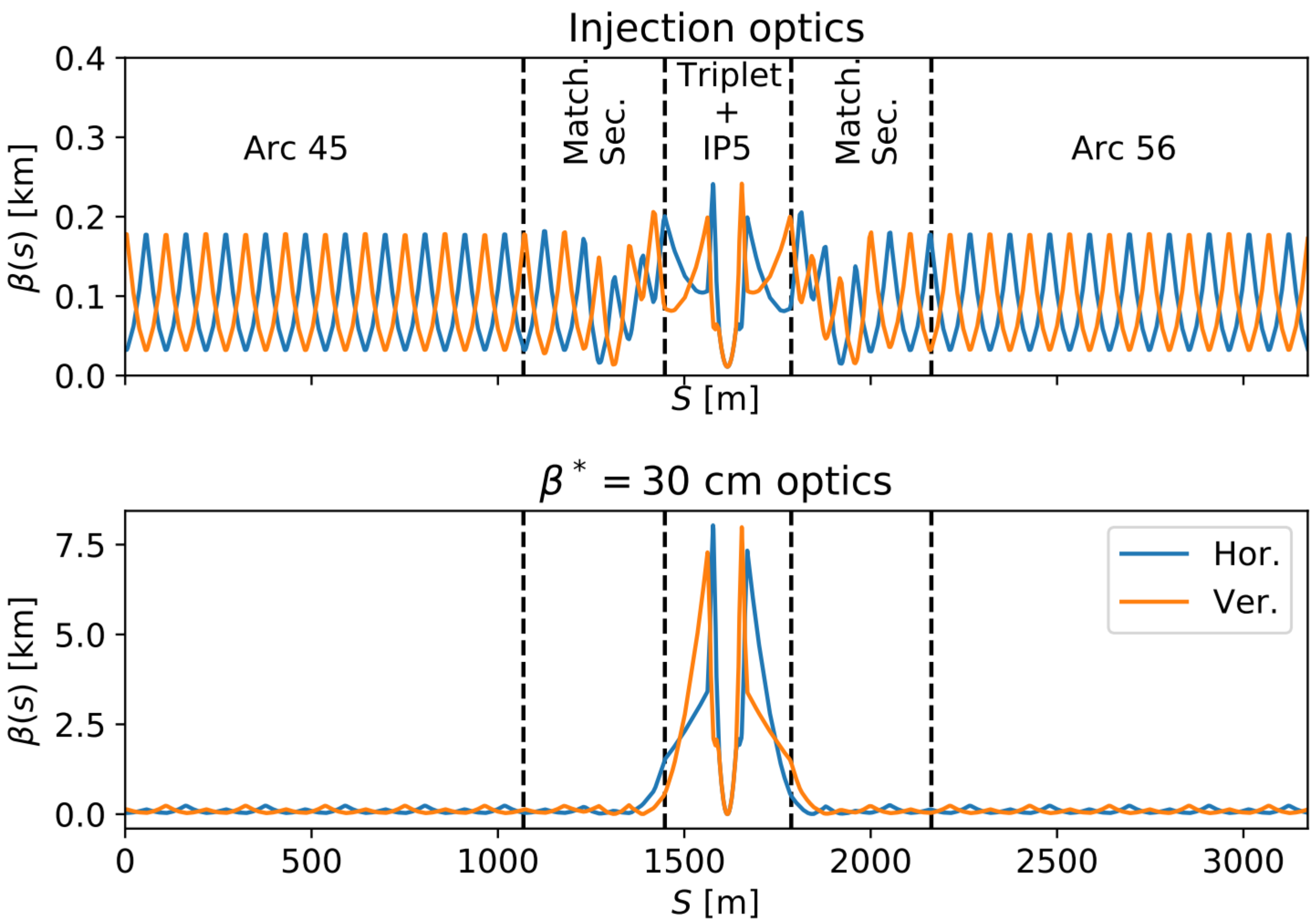
$$\cos\Phi = \frac{1}{2}\mathrm{Tr}(M) = 1 - L^2/2f$$

$$\beta = \frac{(2L)(1 + L/2f)}{\sin\Phi} \qquad \alpha = 0$$

Computed Calculation of TWISS Plots



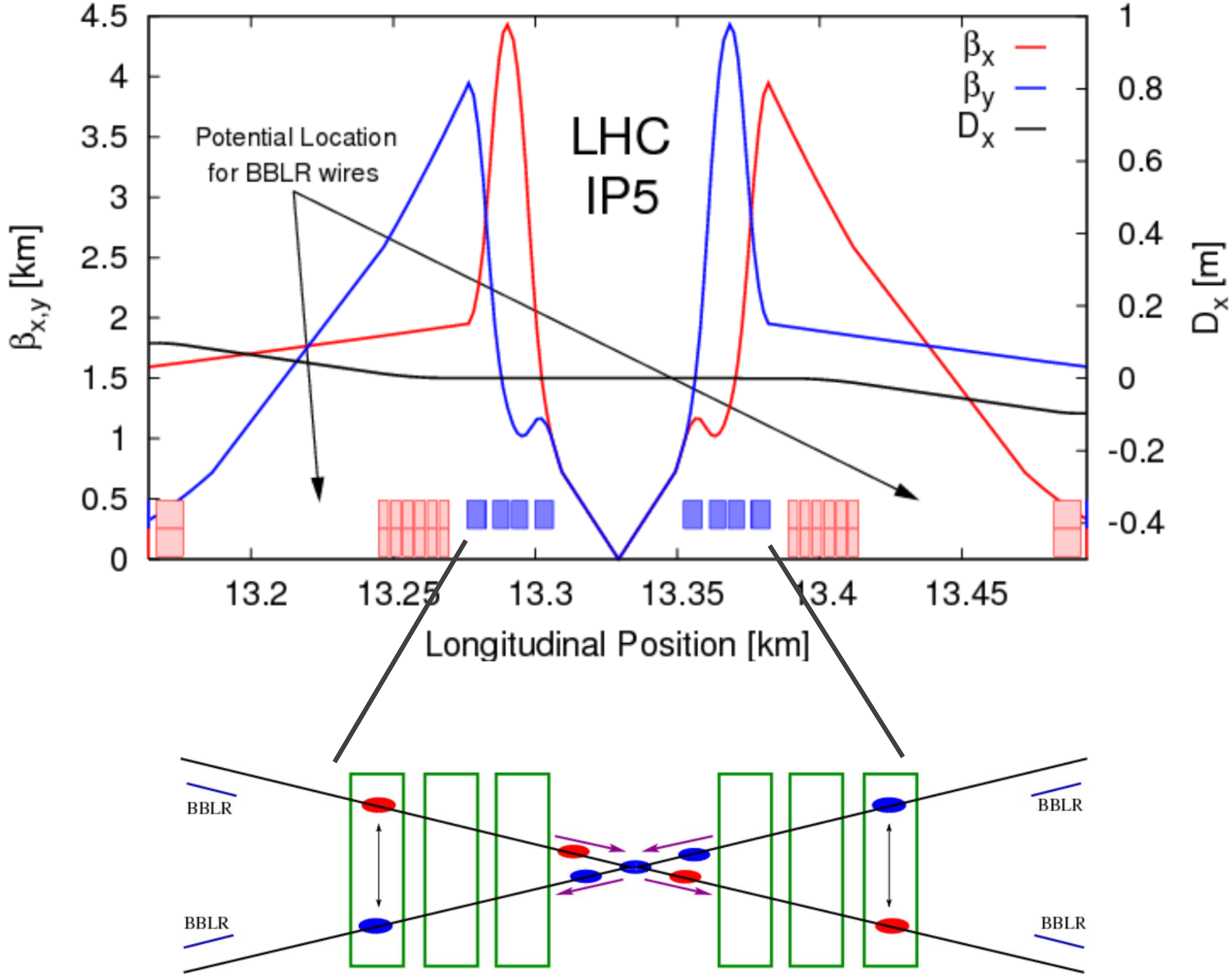
LHC IP5 TWISS Plot



R Calaga, Long-range beam-beam experiments in the relativistic heavy ion collider

Fermilab

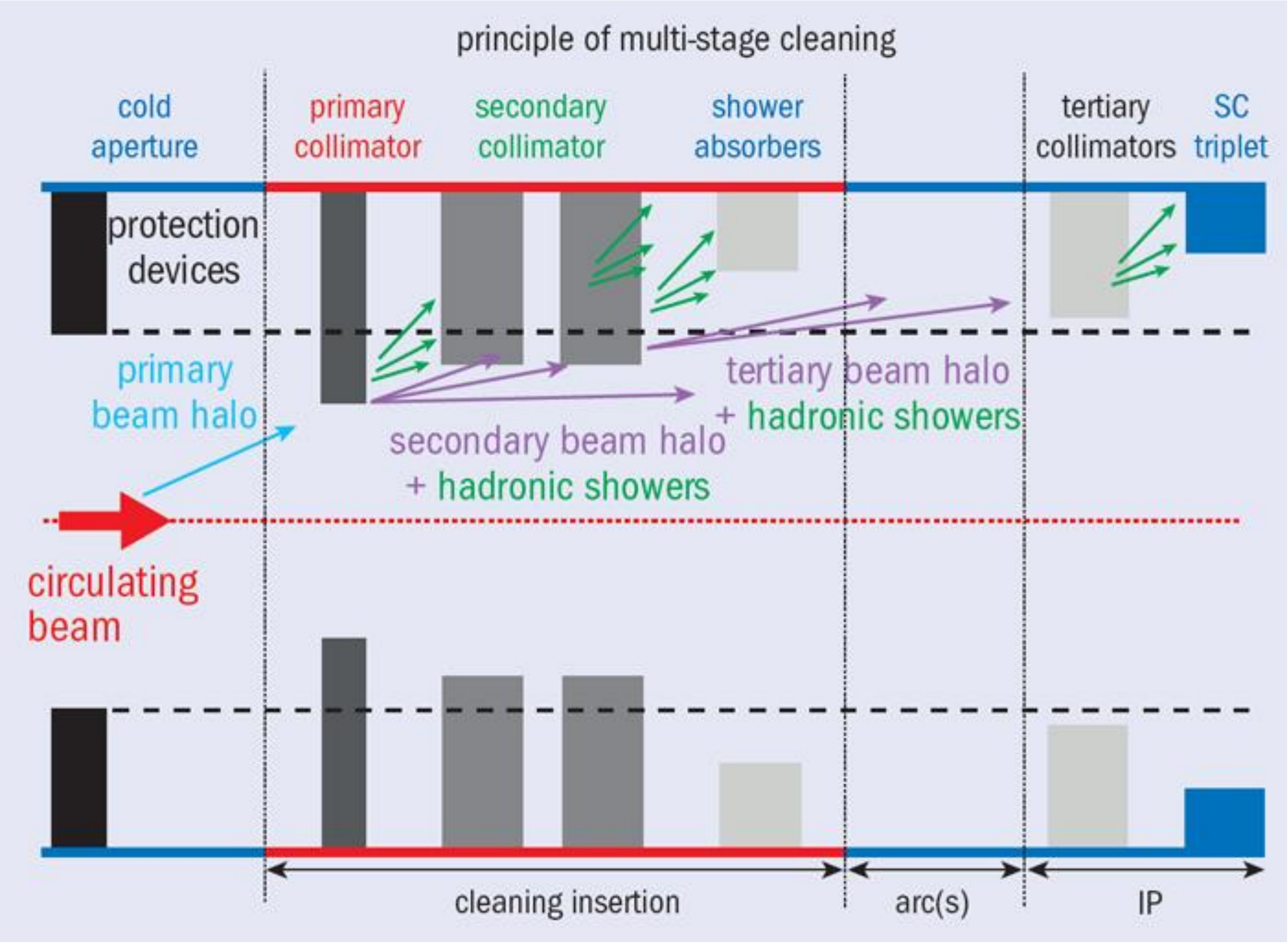
LHC IP5 TWISS Plot (cont.)



Jaime Maria Coello, Dissertation

Collimators and Phase-Advance

Machine Protection through Collimation



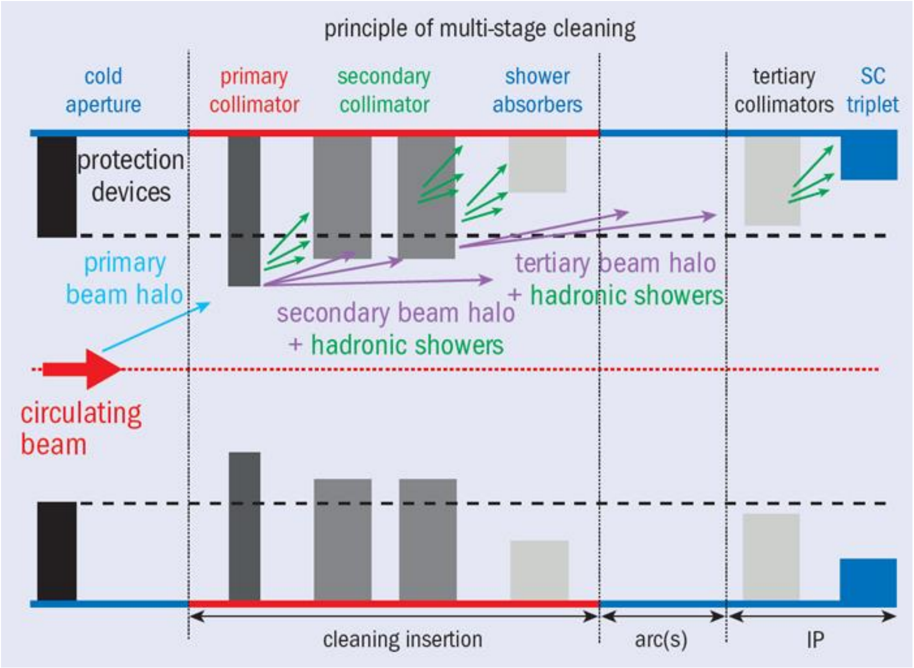
CERN Courier

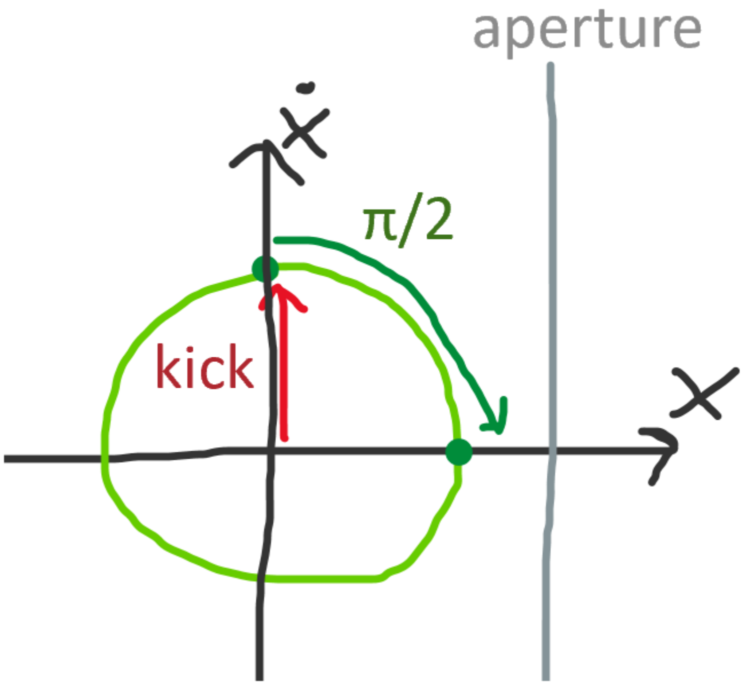
Betatron Phase Advance

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x)$$

$$x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x \cos(\phi_x)]$$

$$\phi_x(s_2) - \phi_x(s_1) = \Delta\phi_x = \int_{s_1}^{s_2} \frac{1}{\beta_x(s)} ds'$$





Nonlinearities and Beam Resoances

Betatron Phase Advance

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x)$$

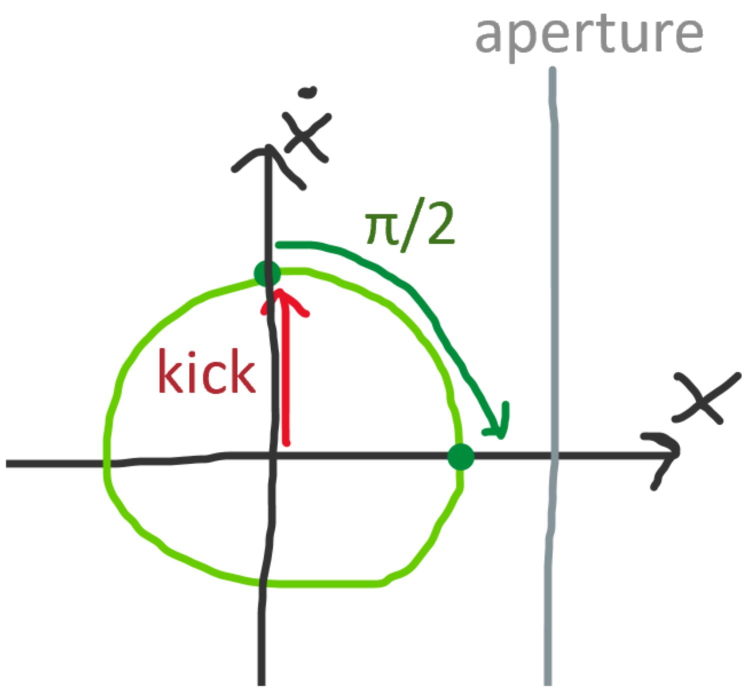
$$x' = -\sqrt{\frac{2J_x}{\beta_x}}[\sin(\phi_x) + \alpha_x\cos(\phi_x)]$$

$$\phi_x(s_2) - \phi_x(s_1) = \Delta\phi_x = \int_{s_1}^{s_2}\frac{1}{\beta_x(s)}ds'$$

Betatron Tune:

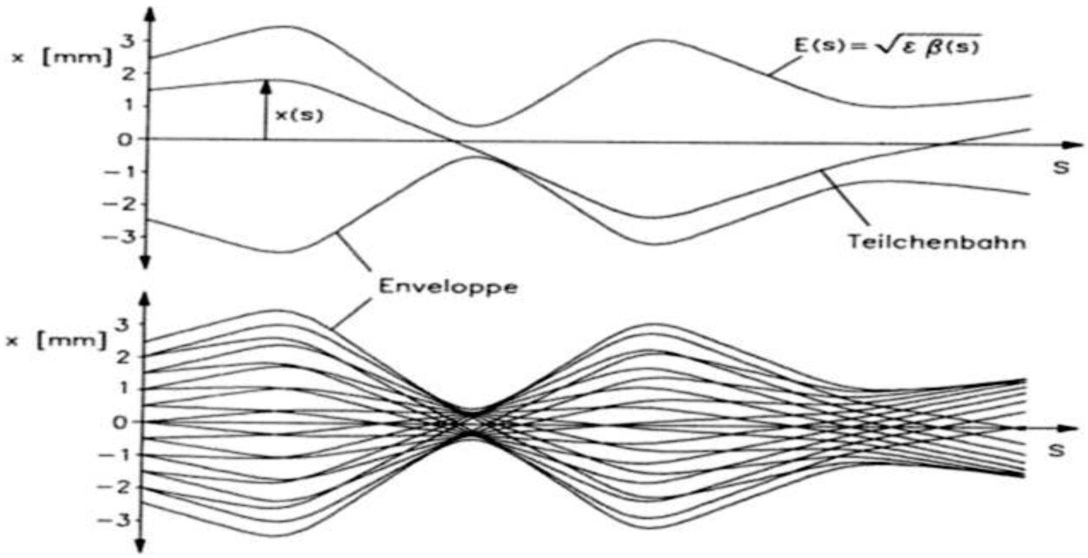
$$\nu = \frac{1}{2\pi}\int_{s_0}^{s_0+C}\frac{1}{\beta_x(s)}ds'$$

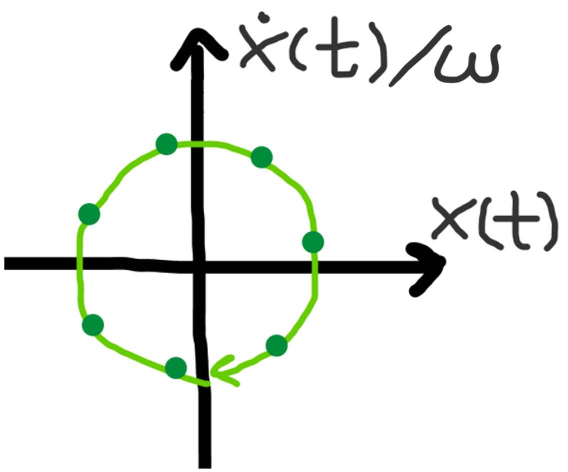
$$\phi_x(t + NT_{rev}) = \phi_x(t) + 2\pi N\nu$$



Discrete Sampling

Depending on the ratio between betatron frequency the revolution frequency, the phase of oscillation with each passage of the beam may fall under regular patterns.



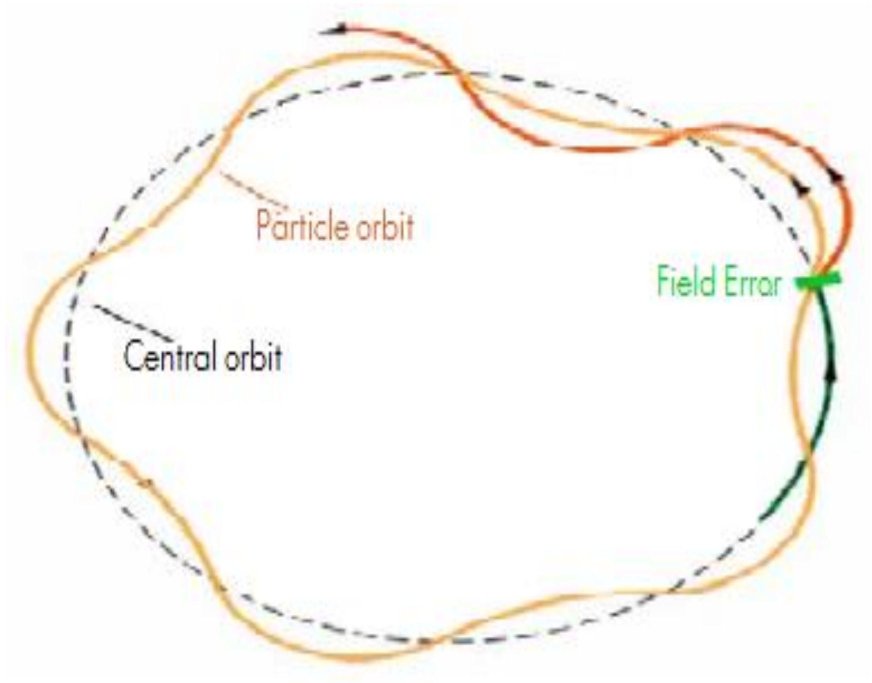


Betatron Tune: $$\nu = \frac{1}{2\pi}\int_{s_0}^{s_0+C} \frac{1}{\beta_x(s)} ds'$$

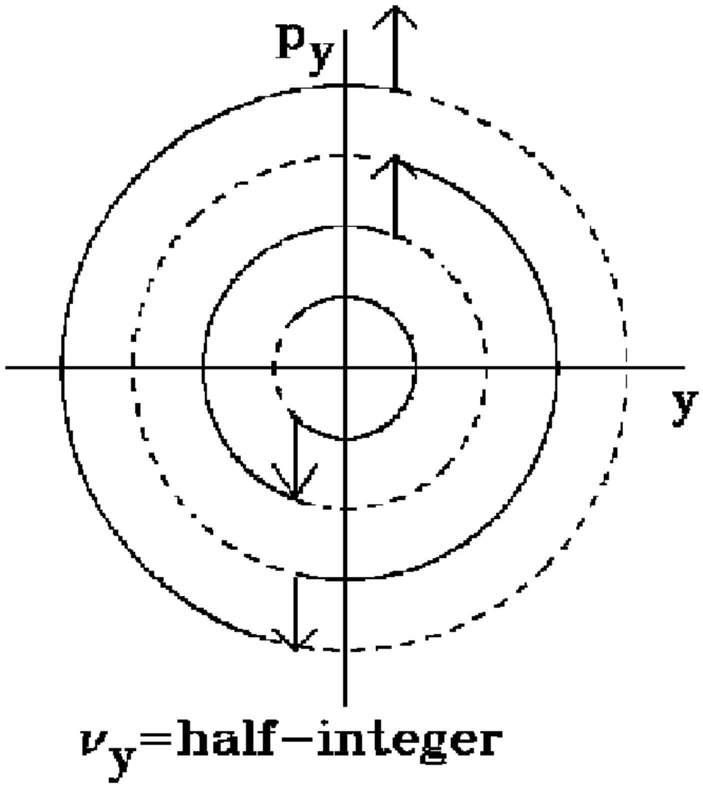
Betatron Tune Resonance

Perturbation will accumulate if tune is a fraction corresponding to the symmetry of the applied fields.

Dipole-Integer Resonace:

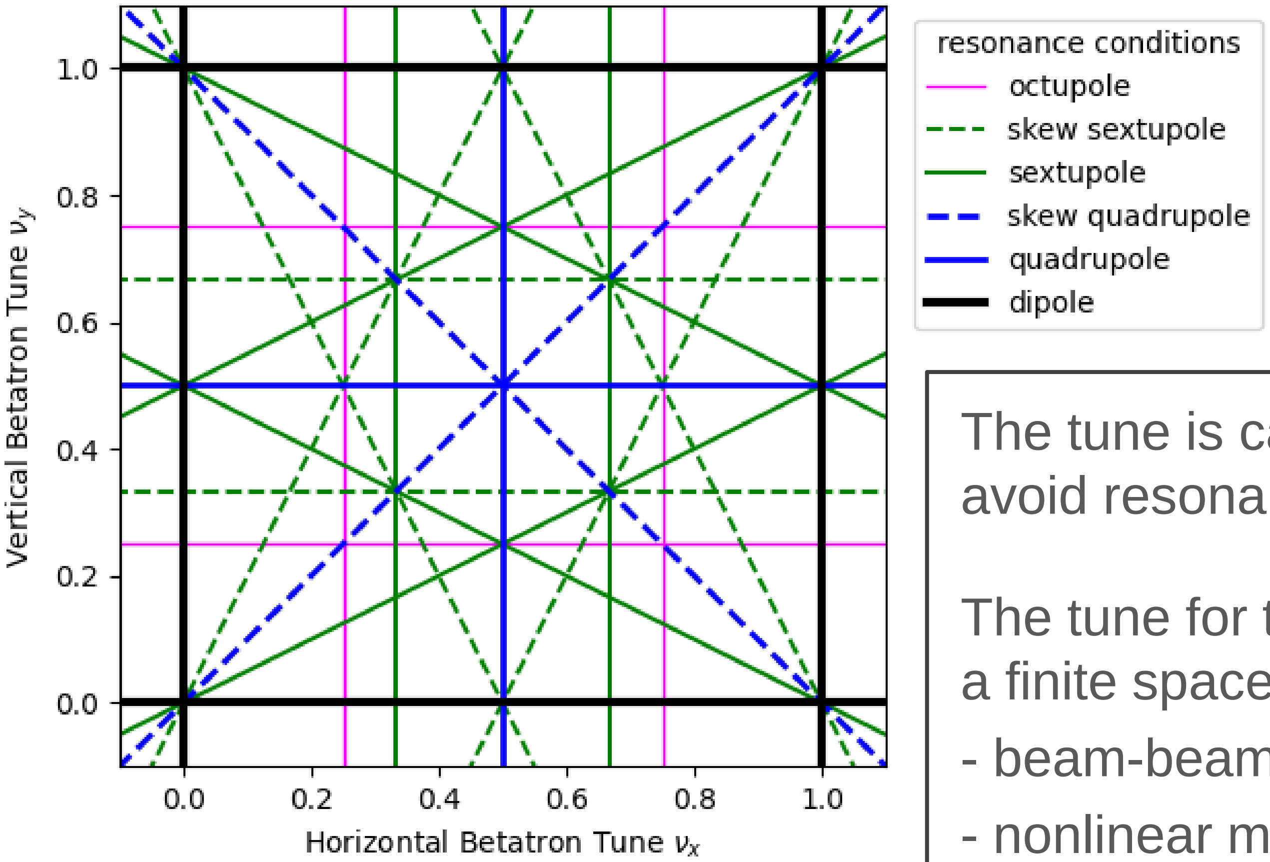


Quadrupole Resonace:



SY Lee

Tune Diagrams (by magnet-type)

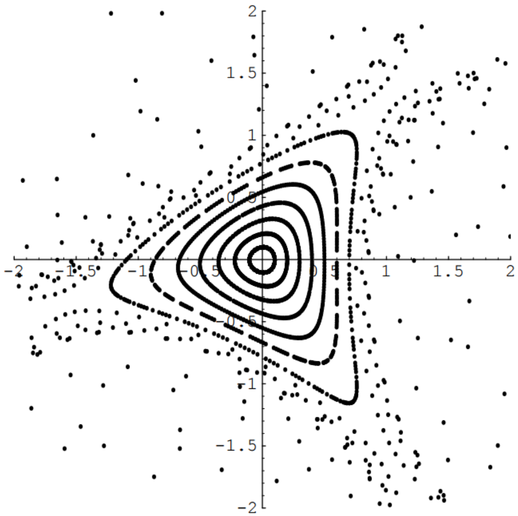


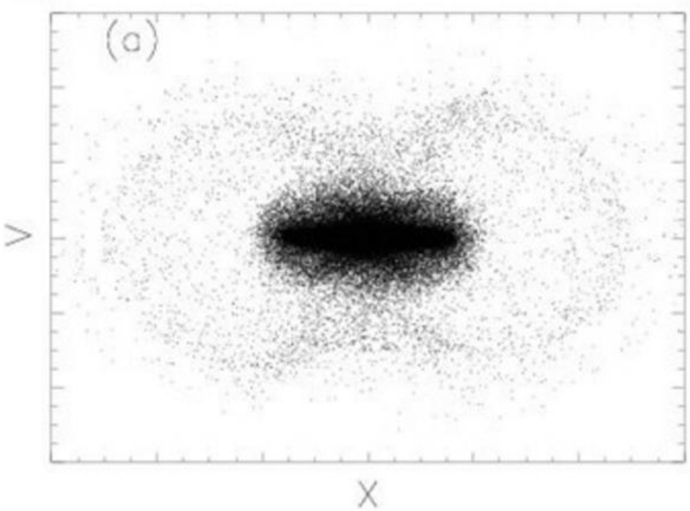
The tune is carefully picked to avoid resonances.

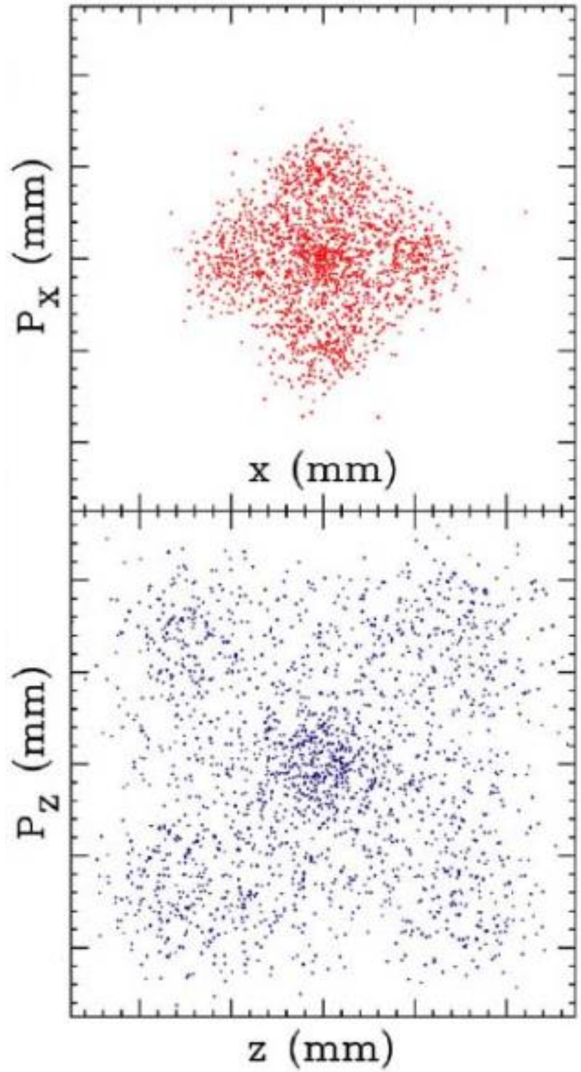
The tune for the beam occupies a finite space:

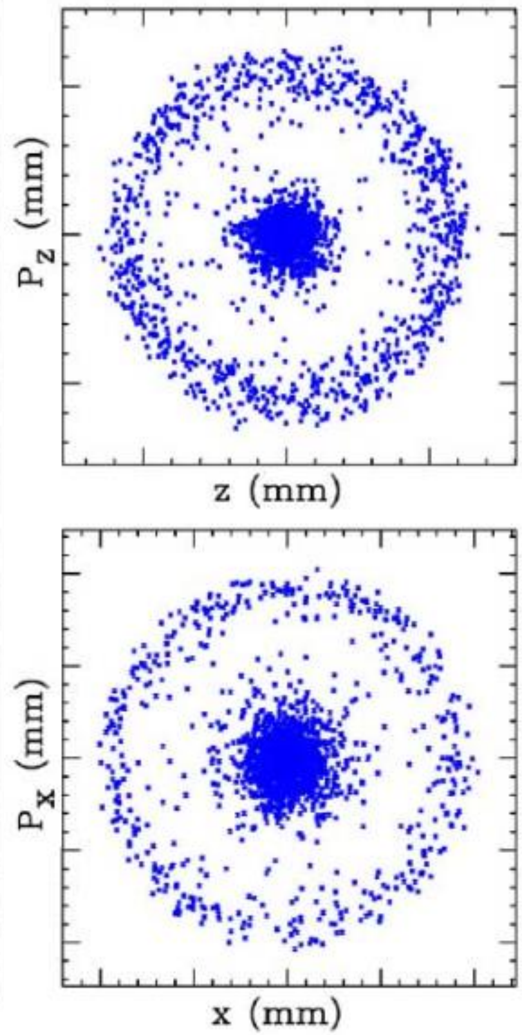
- beam-beam effects.
- nonlinear magnets.
- chromaticity.

Phase-space Distortions

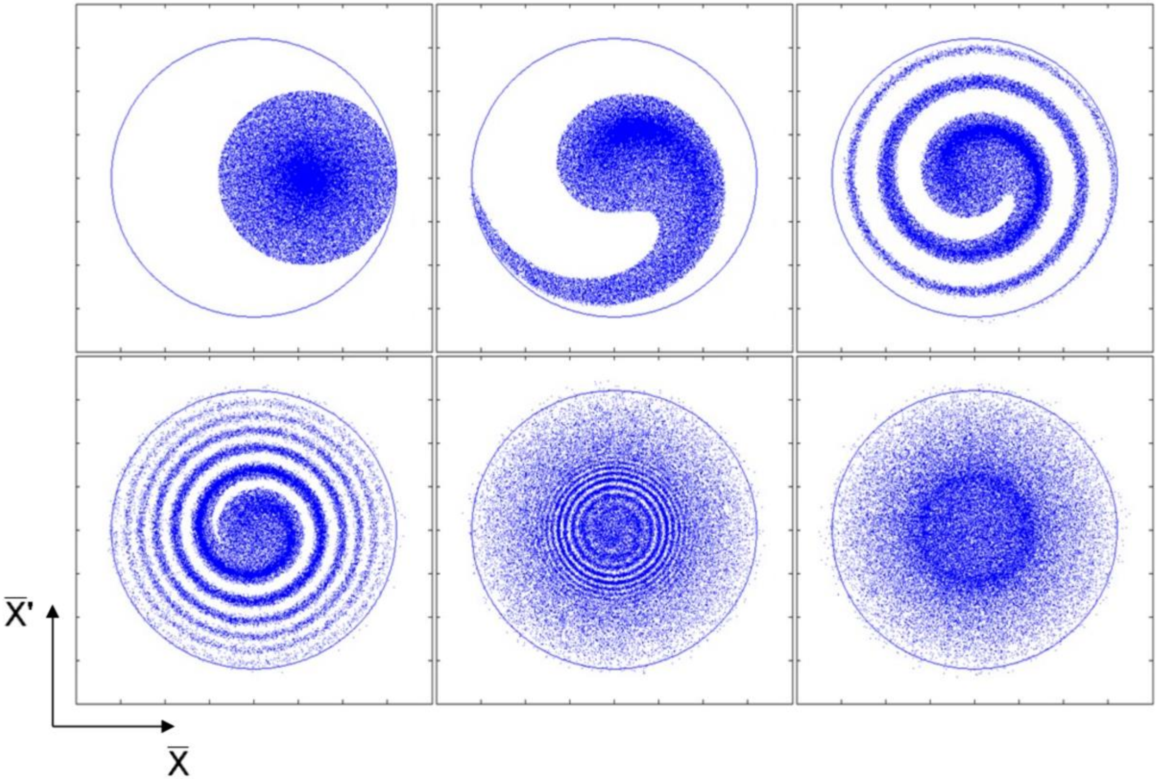








Nonlinear Decoherence



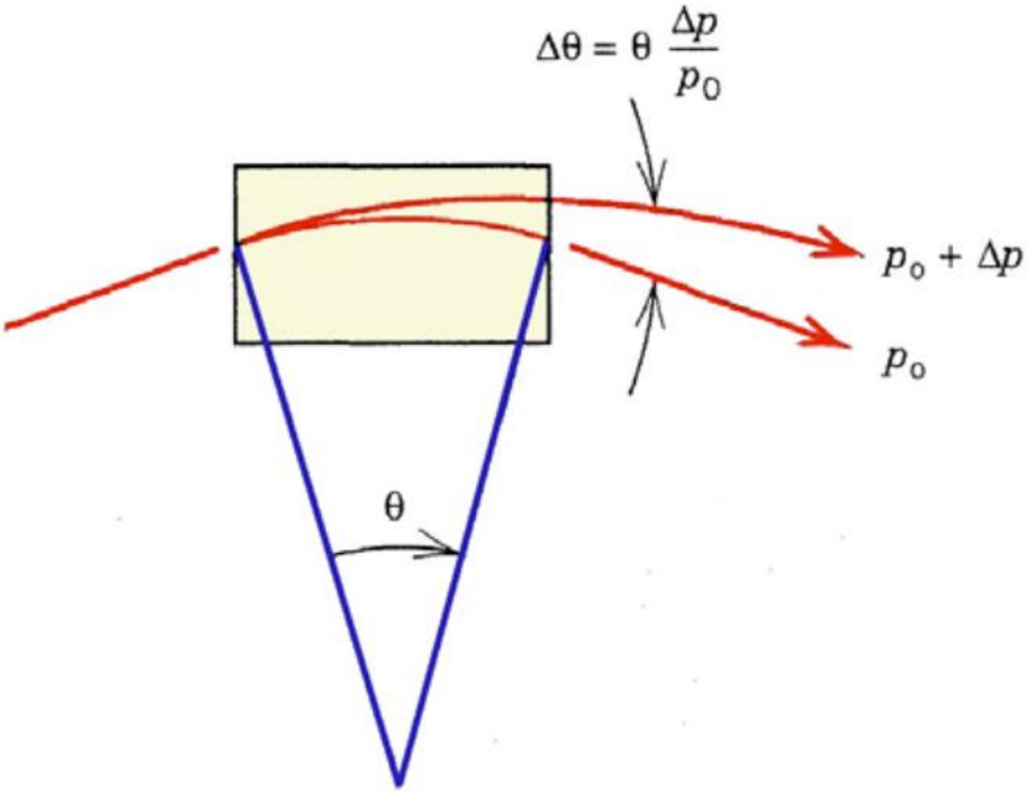
Injection errors, instabilities, and sudden lattice changes may cause a phase-space mismatch.

Tune spread causes the beam to fill-out along the phase-space contours.

Rather than normalized emittance being conserved, in practical terms its more accurate to say that it doesn't decrease.

Off-Momentum Particles

Dispersion



$$\delta \equiv \frac{p - p_0}{p_0}$$

Dispersion:

$$D'' + K_x(s)D = \frac{1}{\rho}$$

$$x(s) = \sqrt{2\beta_x J_x}\cos(\phi_x) + D\delta$$

Spot Size:

$$\sigma_{x,rms}^2 = \beta_x \epsilon_{rms} + D^2\delta^2$$

Barletta

Chromaticity

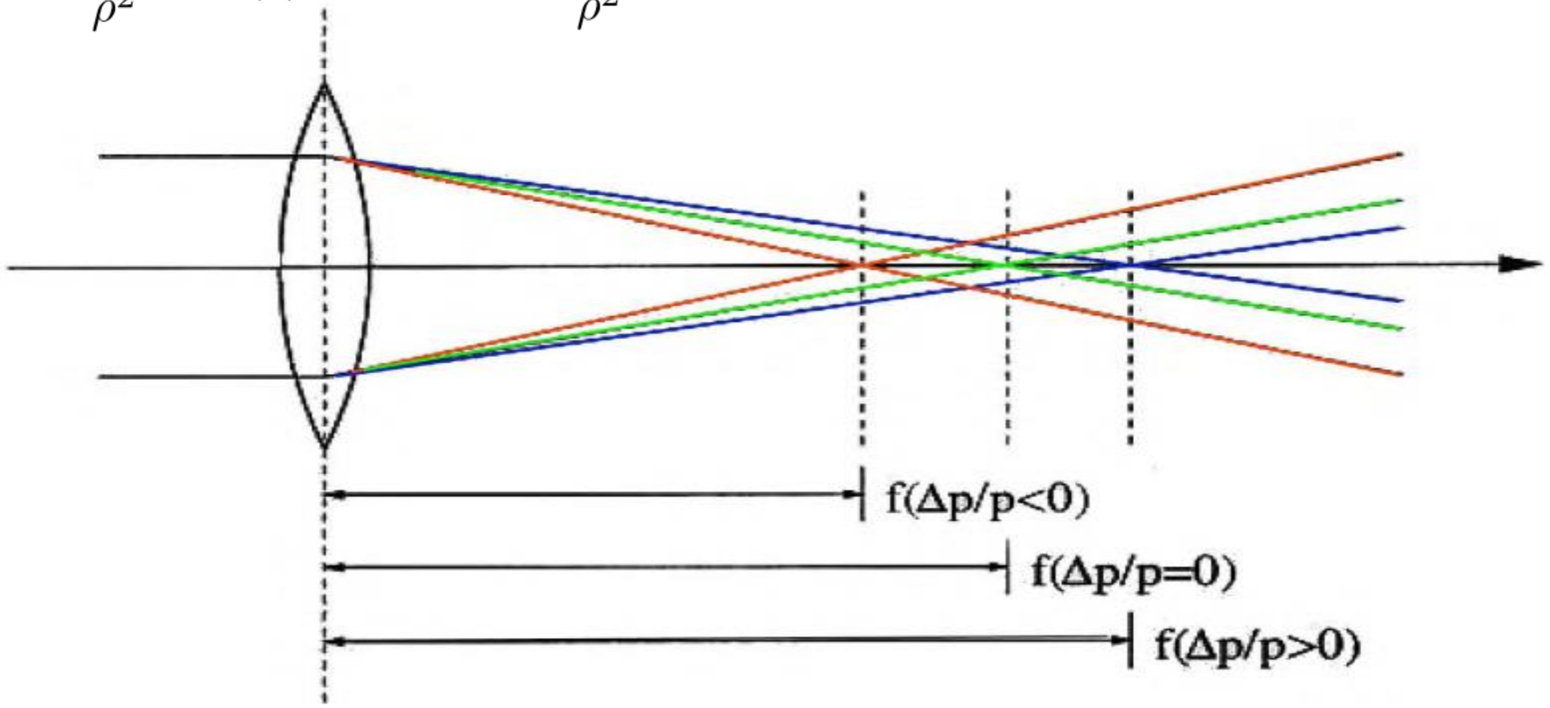
Change in tune with momentum:

$$x_\beta'' + (K_x + \Delta K_x \delta) x_\beta = 0$$

$$K_x = \frac{1}{\rho^2} - K(s) \qquad \Delta K_x = -\frac{2}{\rho^2} + K(s)$$

Chromaticity:

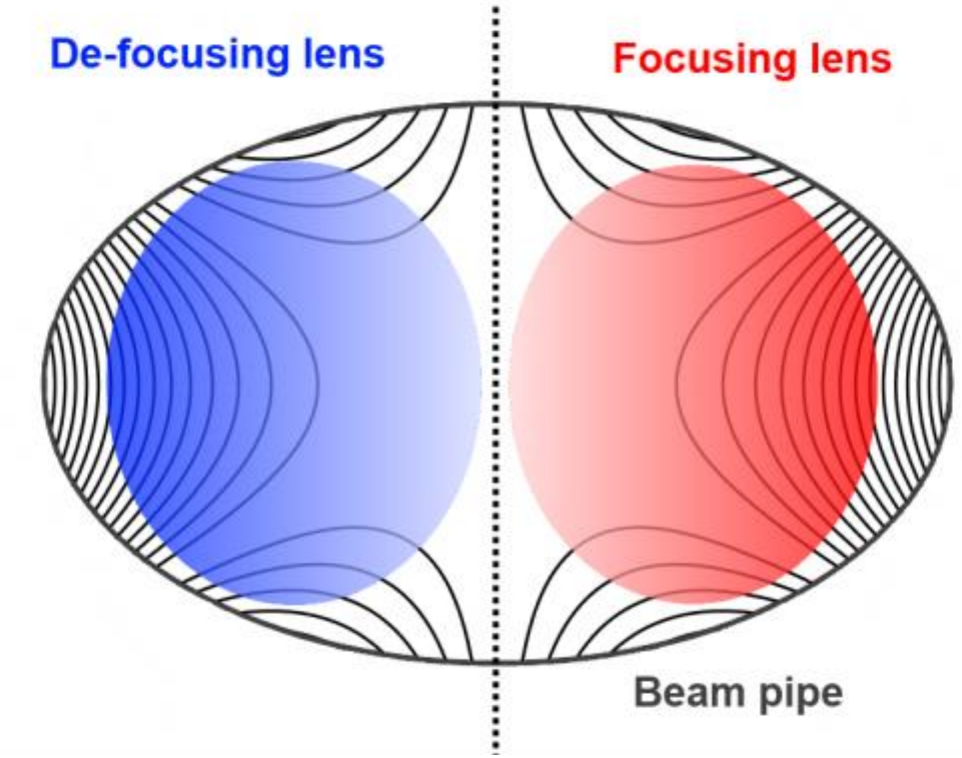
$$C_x = \frac{\partial}{\partial \delta} \Delta \nu_x = \frac{1}{4\pi} \int_0^C \beta_x \Delta K_x(s) ds$$



Barletta

Fermilab

Sextupoles & Chromaticity Correction



Dispersion is position offset dependence on momentum.

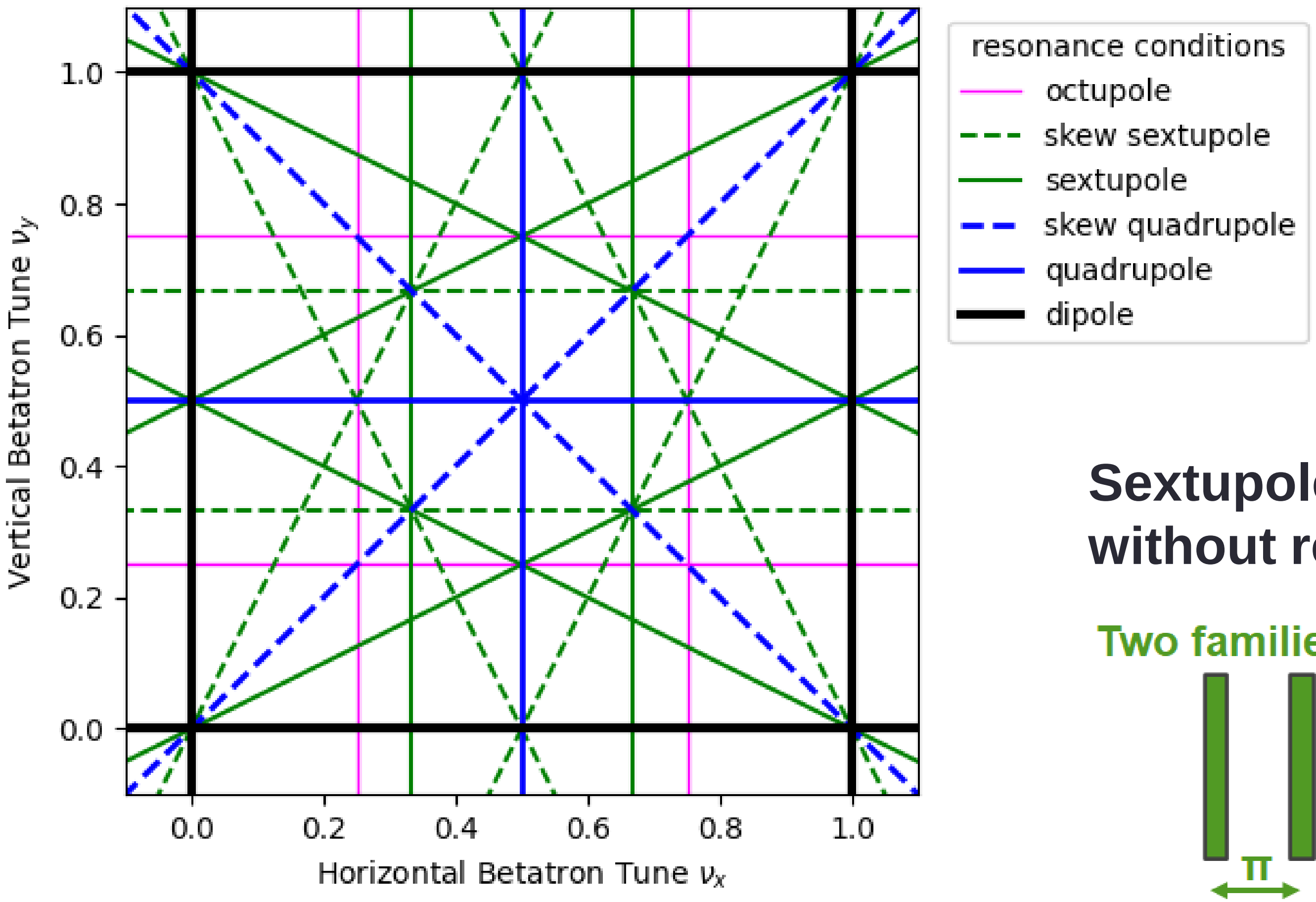
Chromaticity is tune dependence on momentum.

Sextupoles provide tune-shift depending on position offset.

$$C_x = \frac{1}{4\pi}\int_0^C \beta_x[\Delta K_x(s) + S(s)D(s)]ds$$

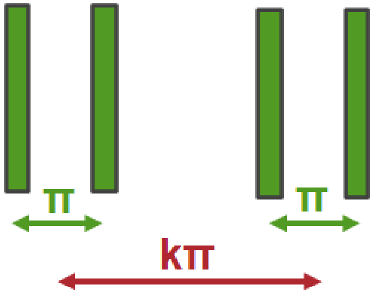
FNAL Rookie Book

Tune Diagrams (by magnet-type)



Sextupole configuration without resonances

Two families of sextupoles:



Fermilab

Summary

We’ve learned the fundamental components of particle accelerators – dipole magnets, RF cavities, quadrupole magnets.

The linear transverse dynamics of a particle accelerator are governed by Hill’s Equation, which is a time-varying harmonic oscillator.

We calculate the trajectory of individual particles through the many individual magnets of a particle accelerator using transfer matrices.

Transfer matrices are also used for the beam size and oscillation phase, which are represented by Courant-Snyder or TWISS parameters.

There are chromatic effects resonances, that complicate the process of designing and operating a particle accelerator.

Questions?