Quantum mechanics (/QFT)

probabilistic theory where state of a physical system is represented by a vector in a Hilbert space

why?

vector - set of (complex) numbers

$|v\rangle$

$|v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|v\rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $|v\rangle = \begin{pmatrix} \pi \\ 17 \\ e^{i\pi} \\ (1+i)/\sqrt{2} \end{pmatrix}$

vector space

set of vectors that can be added & multiplied by scalars

- $|v_1 + v_2\rangle = |v_1\rangle + |v_2\rangle$
- $|cv_1\rangle = c|v_1\rangle$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

basis vector

if V is n-dimensional, than any n linearly independent vectors form a basis $\{|e_1\rangle, \dots, |e_n\rangle\}$

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} \right\}$$

3D vector space

$$|v\rangle = \sum_{i=1}^{N} v^i |e_i\rangle$$

basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$

$$|v\rangle = \sum_{i=1}^{N} v^i |e_i\rangle$$

<u>example</u> – 2D vector space

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= v_1 |e_1\rangle + v_2 |e_2\rangle$$

N-dim

$$|v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \sum_{i=1}^{N} v^i |e_i\rangle$$

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad |e_n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

Hilbert space

vector space with inner product

- function w/ inputs: two vectors
  outputs: one scalar
  complex

$|v\rangle, |u\rangle \in \mathcal{H}$

$$\text{inner product}(u,v) = \langle u|v\rangle$$

$$= (u^1)^* v^1 + \dots + (u^N)^* v^N$$

$$= (u^1, \dots, u^N)^* \begin{pmatrix} v^1 \\ \vdots \\ v^N \end{pmatrix}$$

$$\langle v|u\rangle = \langle u|v\rangle^*$$

$$\langle v|v\rangle = \sum_i |v_i|^2$$

example

$$|e_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |e_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\langle e_1|e_1\rangle = (1,0)\begin{pmatrix}1\\0\end{pmatrix} = \overset{1}{e_1^1}\, e_1^1 + \cancel{\overset{0}{e_1^2}\, e_1^2} = 1$$

$$\langle e_2|e_2\rangle = (0,1)\begin{pmatrix}0\\1\end{pmatrix} = 1$$

$|e_i\rangle$ are normalized (norm $=1$)

$$\langle e_1|e_2\rangle = (1,0)\begin{pmatrix}0\\1\end{pmatrix} = 0$$

$$\langle e_2|e_1\rangle = \langle e_1|e_2\rangle^* = 0$$

$|e_1\rangle$ and $|e_2\rangle$ are orthogonal

orthonormal

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\langle e_i | e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

we can always normalize "by hand"

$$|v\rangle \in \mathcal{H}$$

$$|\psi\rangle = \left(\frac{1}{\sqrt{\langle v|v\rangle}}\right)|v\rangle$$

$$\langle \psi | \psi \rangle = \langle v| \left(\frac{1}{\sqrt{\langle v|v\rangle}}\right)\left(\frac{1}{\sqrt{\langle v|v\rangle}}\right)|v\rangle$$

$$= \left(\frac{1}{\langle v|v\rangle}\right)\langle v|v\rangle$$

$$= 1$$

Quantum mechanics

let $|\psi\rangle$ be a normalized state
(vector in Hilbert space)

then $|\langle\psi_f|\psi_i\rangle|^2 = \overset{P}{\text{probability}}$
that $\psi_i \Rightarrow \psi_f$

state prepared in $|\psi_i\rangle$ will be
measured to be in $|\psi_f\rangle$

note $0 \le P \le 1$

$$\boxed{\begin{aligned} &\vec{u}\cdot\vec{v} = |u||v|\cos\theta \\ &\Rightarrow \frac{\vec{u}\cdot\vec{v}}{|u||v|} \le 1 \end{aligned}}$$

$$|\langle\psi_f|\psi_i\rangle|^2 \le \langle\psi_f|\psi_f\rangle\langle\psi_i|\psi_i\rangle$$
$$= 1$$

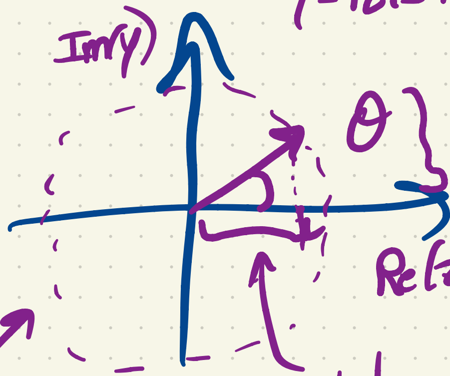
triangle inequality

phases

$\theta \in [0, 2\pi)$

$$z = x + iy = |z| e^{i\theta}$$

$x \in \mathbb{R}$ $\quad$ $y \in \mathbb{R}$



overall phase of state does not affect probabilities

$$|\psi'\rangle = e^{i\theta}|\psi\rangle$$

$= \text{Prob}(v \to \psi')$

$$\Rightarrow \text{Prob}(\psi' \to v) = |\langle v|\psi'\rangle|^2 = |\langle v| e^{i\theta}\psi\rangle|^2$$

$$= |e^{i\theta}\langle v|\psi\rangle|^2 = \cancel{|e^{i\theta}|}\,|\langle v|\psi\rangle|^2$$

$$= |\langle v|\psi\rangle|^2 = \text{Prob}(\psi \to v)$$

relative phases <u>do</u> affect things

$$|w\rangle = \left(|\psi_1\rangle + e^{i\theta}|\psi_2\rangle\right)\frac{1}{\sqrt{2}}$$

$$\langle w|w\rangle = \frac{1}{2}\Big(\cancelto{1}{\langle\psi_1|\psi_1\rangle} + e^{i\theta}\cancelto{0}{\langle\psi_1|\psi_2\rangle}$$

$$+ e^{-i\theta}\cancelto{0}{\langle\psi_2|\psi_1\rangle} + \cancelto{1}{\langle\psi_2|\psi_2\rangle}\Big)$$

$$= \frac{1}{2}(1+1) = 1$$

$$\text{prob}(w\to v) = |\langle v|w\rangle|^2$$

$$= |\langle v|\psi_1\rangle + e^{i\theta}\langle v|\psi_2\rangle|^2\,\frac{1}{2}$$

$$= \frac{1}{2}|\langle v|\psi_1\rangle|^2 + \frac{1}{2}|\langle v|\psi_2\rangle|^2 + \frac{1}{2}\Big(e^{i\theta}\langle v|\psi_1\rangle^*\langle v|\psi_2\rangle$$

$$+ e^{-i\theta}\langle v|\psi_1\rangle\langle v|\psi_2\rangle^*\Big]$$

$$|w\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + e^{i\theta}|\psi_2\rangle\right)$$

$$\text{prob}(w \to v) = \overbrace{\frac{1}{2}\text{prob}(\psi_1 \to v) + \frac{1}{2}\text{prob}(\psi_2 \to v)}^{\text{average prob of } \psi_1, \psi_2 \to v}$$

$$+ \underbrace{\text{Re}\left[e^{i\theta}\langle v|\psi_1\rangle^* \langle v|\psi_2\rangle\right]}_{\text{interference terms}}$$

can be + or −

depends on θ

Stay tuned for more fun

Asli